STABILITY DEGREE ANALYSIS OF LINEAR CONTROL SYSTEMS WITH DEAD TIME

By

M. HABERMAYER

Department of Automation, Technical University, Budapest

(Received May 4, 1971)

Presented by Prof. Dr. F. CSÁKI

A primary requirement in the design of control systems is the stability of the operation. At the same time the quality requirements of the technical process to be controlled must also be satisfied by the control. These quality requirements prescribe on the one hand the static state (stationary error), on the other hand the dynamic state (controlling time, overshoot, number of oscillations) of the control [1, 2].

The analysis and synthesis of linear control systems start generally in the frequency region, whereas the effective and the prescribed quality characteristics may be compared by examining the dynamic behaviour of the control system, i.e. by turning from the frequency region into the time region.

The most classic and most general way for studying the transient and the stationary states of the controlled system is to write up the differential equation of the system. For the deterministic investigation of linear systems a so-called typical test signal is generally applied to the system input. In the case of arbitrary input signal, the output signal may be determined, — in knowledge of the system's weighting function, — with the help of the convolution integral. Regarding that the setting up and solution of the differential equation for more complex systems, and the computation of the convolution integral for more complex input signals often hurts to serious difficulties, various methods have been developed for simplifying the investigation.

The computational difficulties inherent with the differential equation method and the convolution integral may be eliminated by transforming the differential equation describing the system into an algebraic equation with the help of the Fourier transformation, or of the more generally applicable Laplace transformation. The time behaviour of the system may be deduced by the inverse transformation of the result obtained in the operator region back into the time region.

The computational work is greatly reduced by the Laplace transformation, but the inverse transformation causes problems in many cases.

1. Determination of the time function of a linear control with dead time

In the case of linear controls containing dead time delays, the inverse transformation from the operator region into the time region raises no special problem in principle, but its evaluation is rather lenghty.

Let us consider e.g. the linear control with dead time shown in Fig. 1. The dependence of the controlled characteristic on the reference signal is given by the transfer function of the closed system:

$$W(s) = \frac{Y_1(s)e^{-s\tau}}{1 + Y_1(s)Y_2(s)e^{-s\tau}} = \frac{X_2(s)}{X_1(s)}\,.$$

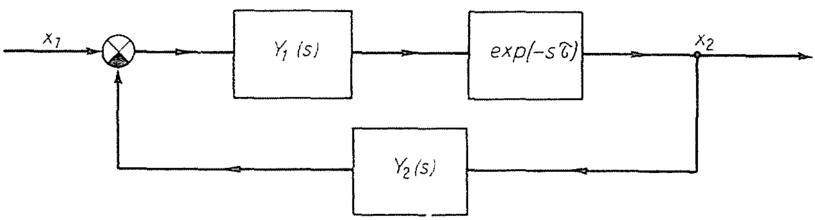

Fig. 1

The above expression of $W(s)$ may be expanded in series, if the condition

$$|\, Y_1(s)\; Y_2(s) \,| < 1$$

is satisfied, as follows:

$$\begin{aligned} W(s) = {} & Y_1(s)^{-s\tau}\{1 - Y_1(s)\; Y_2(s)e^{-s\tau} + [Y_1(s)\; Y_2(s)e^{-s\tau}]^2 - \\ & - [Y_1(s)\; Y_2(s)e^{-s\tau}]^3 + \ldots\} = \qquad (1) \\ & = Y_1(s)e^{-s\tau}\left\{\sum_{n=1}^{\infty}(-1)^{n+1}[Y_1(s)Y_2(s)e^{-s\tau}]^{n-1}\,.\right. \end{aligned}$$

Assuming unity feedback, i.e. $Y_2(s) = 1$, (1) may be simplified to

$$W(s) = \sum_{n=1}^{\infty}(-1)^{n+1}[Y_1(s)e^{-s\tau}]^n\,. \qquad (2)$$

By the inverse transformation of the infinite series obtained for $W(s)$, the weighting function of the system is obtained. The dynamic behaviour of control circuits is most often characterized by the unit step response. The unit step response of the linear control with dead time shown in Fig. 1 may be determined by utilizing (2) in the following way:

$$v(t) = L^{-1}\left\{\frac{1}{s}\sum_{n=1}^{\infty}(-1)^{n+1}[Y_1(s)e^{-s\tau}]^n\right\} = \sum_{n=1}^{\infty}(-1)^{n+1}\,1(t - n\tau)v'(t - n\tau), \quad (3)$$

where

$$v'(t - n\tau) = \left[L^{-1}\left\{\frac{Y_1^n(s)}{s}\right\}\right]_{t-n\tau}.$$

The unit step response may be produced with the expansion theorem in a form holding for multiple roots as well. For determining the unit step response, the inverse Laplace transforms of as many terms are required as there are needed up to the stationary state.

In many cases it is sufficient to determine the maximum overshoot position and value, but even in this case the inverse Laplace transforms of the first two or three terms are mostly insufficient (demanding relatively less computation work).

The determination of the closed system time function is laborious because of the lengthy evaluation of the coefficients. But with the help of the digital computer it is simple to prepare a program giving the values of the time function coefficients for a given control circuit. Of course the evaluation of the coefficients takes increasingly more computer time with the increase of the number of the terms to be transformed in inverse.

The time region and operator region tests published up to the present are difficult to apply to the investigation of the dynamic properties of linear controls with dead time. Therefore in many cases empirical relations concerning the interrelation between the frequency function and the time function can be done with establishing the relationship between the maximum value of the closed system frequency function absolute value M_m and the unit step response overshoot. The value of M_m may be established with the help of the constant $M - \alpha$ curves, or the Nichols curves [1].

The design of the control circuits is further simplified by the fact that conclusions concerning the quality characteristics may be drawn already from the knowledge of the opened circuit frequency characteristic curve. If the circle of unity radius is intersected once by the Nyquist diagram of the opened up system—in the following we shall call this type of control a normal behaviour control—the maximum overshoot of the unit step response may be deduced from the empirical relations between the phase margin φ' and the value of M_m [1].

In the following we shall study, for the case of a concrete example as well, the variation of the stability region of a linear control with dead time versus the phase margin and the time constants of the system.

Previous papers [4—7] have already dealt with the determination of the variation of the stability region for the control with second order lag and dead time [$G(s)$], with a unit feedback, compensated in the general case by a PID element [$C(s)$], as shown in Fig. 2 ($\varphi' = 0$). Further papers are published presenting diagrams of the variation of the stability region to provide any arbitrary phase margin for the control shown in Fig. 2 in case of various types of compensation.

2. Determination of equations for an arbitrary phase margin in the case of a PID compensation

The transcendent equation for the determination of the critical angular frequency, $(\omega_{cr,\varphi'} = \omega)$ to reach an arbitrary phase margin for the control shown in Fig. 2 can be written as follows:

$$-\frac{\pi}{2} - \omega\tau - \tan^{-1}\frac{2\zeta T\omega}{1 - T^2\omega^2} + \tan^{-1}\frac{\omega T_i}{1 - T_i T_d \omega^2} = -\pi + \varphi' , \quad (4)$$

where

τ — dead time,

T — time constant of the element with second order lag,

ζ — damping factor.

After simplification and standardization we have:

$$\omega\tau + \tan^{-1}\frac{2\zeta T\omega}{1 - T^2\omega^2} - \tan^{-1}\frac{\omega}{1/T_i - T_d\omega^2} = \frac{\pi}{2} - \varphi' . \quad (5)$$

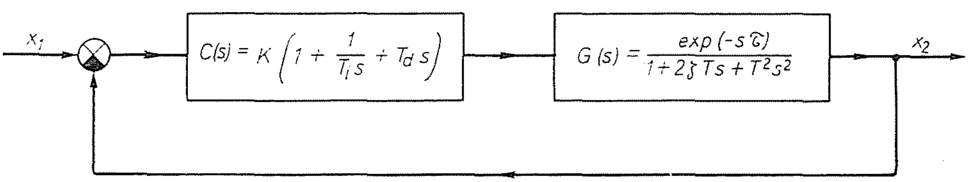

Fig. 2

With the angular frequency value obtained from the iteration of the limit position of the stability region to reach the arbitrary phase margin is:

$$K = \omega \frac{\sqrt{(1 - T^2\omega^2)^2 + (2\zeta T\omega)^2}}{\sqrt{(1/T_i - T_d\omega^2)^2 + \omega^2}} . \quad (6)$$

From the above forms of (5) and (6), the equations for P-, PI- and PD-controls, with the adequate choice of T_d and $1/T_i$ may be established:

$$\begin{array}{ll} \text{Proportional control:} & 1/T_i = 0,\ T_d = 0, \\ \text{Proportional-integral control:} & T_d = 0, \\ \text{Proportional-differential control:} & 1/T_i = 0. \end{array} \quad (7)$$

2.1. Proportional control

The transcendent equation for the calculation of the angular frequency determining the reaching of the arbitrary phase margin, by utilizing relatios (5) and (7) is:

$$\omega\tau + \tan^{-1}\frac{2\zeta T\omega}{1 - T^2\omega^2} = \frac{\pi}{2} - \varphi' . \tag{8}$$

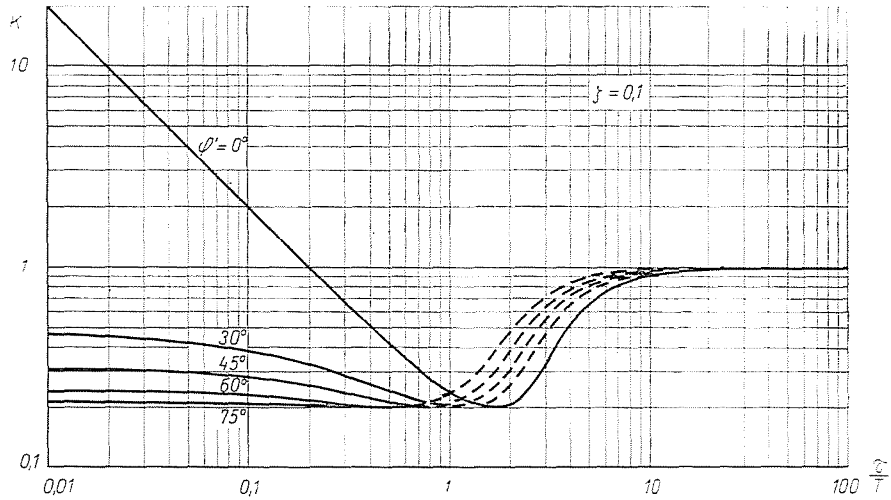

Fig. 3. Proportional control

The value of $\omega = \omega_{cr,\varphi'}$ obtained by iterating (8) substituted into relationship (6) — utilizing that $T_d = 1/T_i = 0$ — the limit position of the stability region is:

$$K = \omega\sqrt{(1 - T^2\omega^2)^2 + (2\zeta T\omega)^2} . \tag{9}$$

Figs 3 to 8 present the values of $K = K_{cr,\varphi'}$ in the region of $0.01 \leqq \tau/T \leqq 100$ for damping factor values $\zeta = 0.1, 0.3, 0.5, 0.7, 1, 2$, for phase margin $\varphi' = 30°, 45°, 60°, 75°$. The diagrams were plotted with the values of the critical loop gain for $\varphi' = 0°$, in conformity with [5].

From the figures the following conclusions may be drawn:

a) For high dead time values the loop gain tends to 1,—as expected—independently of the ζ and φ' values.

b) For low dead time values the stability region keeps decreasing with the increase of the phase margin.

c) For values of $\zeta \leqq 0.7$ in the region $0.5 \leqq \tau/T \leqq 2$ the dashed functional relationship $K = K(\tau/T)$ does not correspond to the stability region

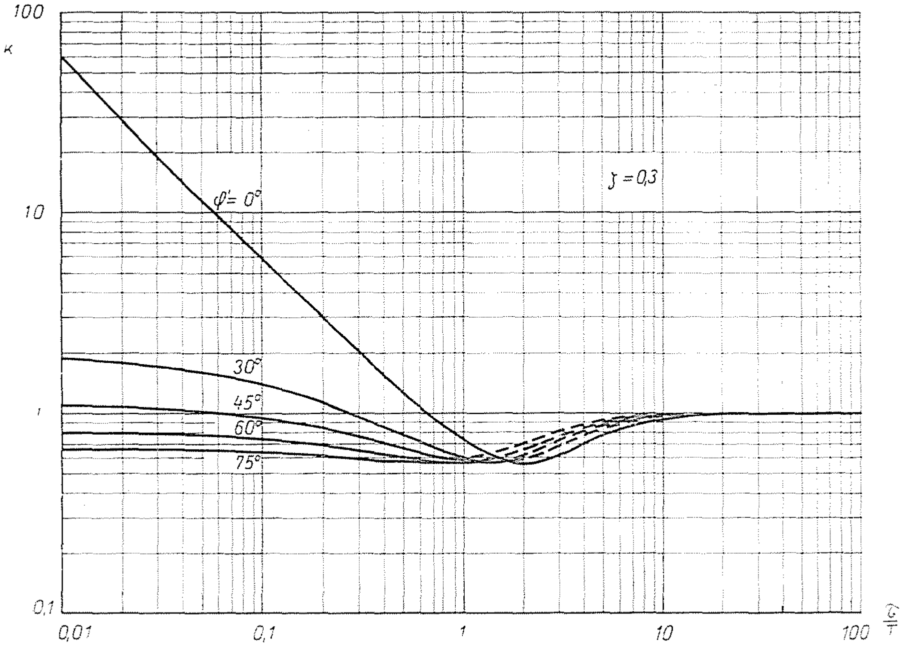

Fig. 4. Proportional control

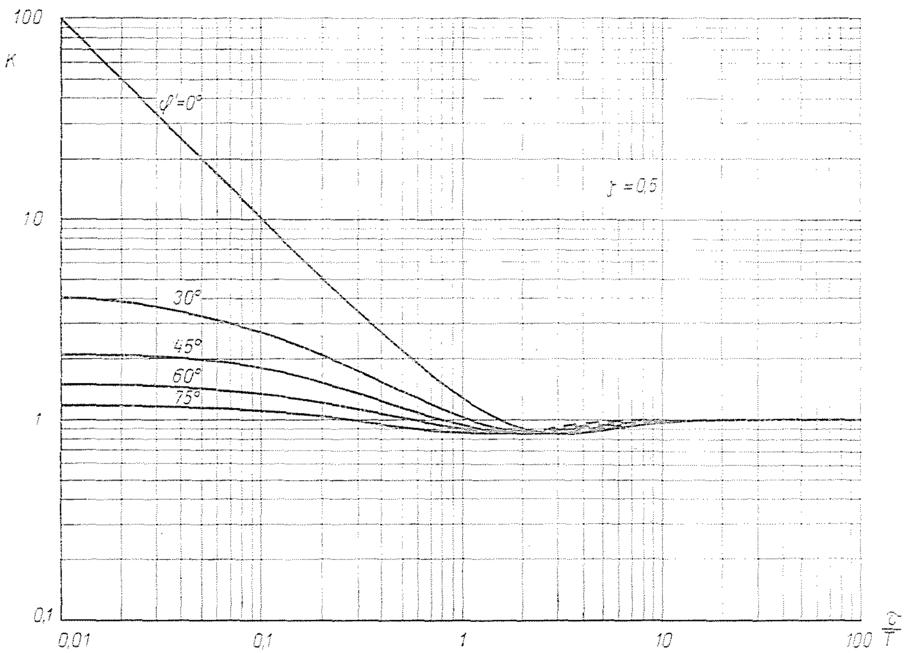

Fig. 5. Proportional control

limit to reach the phase margin $\varphi' = 30°, 45°, 60°, 75°$. For the time constant values belonging to these dashed sections, the control behaves anormally (see item 1). The loop gain satisfying the arbitrary quality requirements

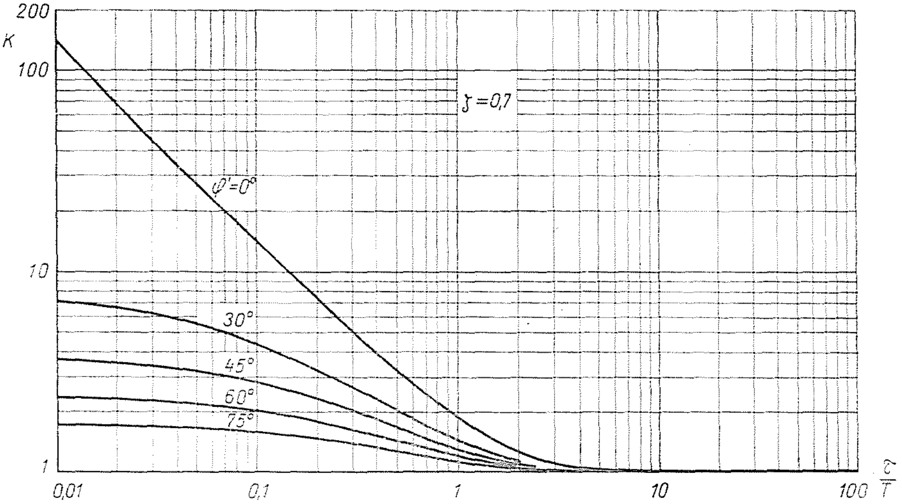

Fig. 6. Proportional control

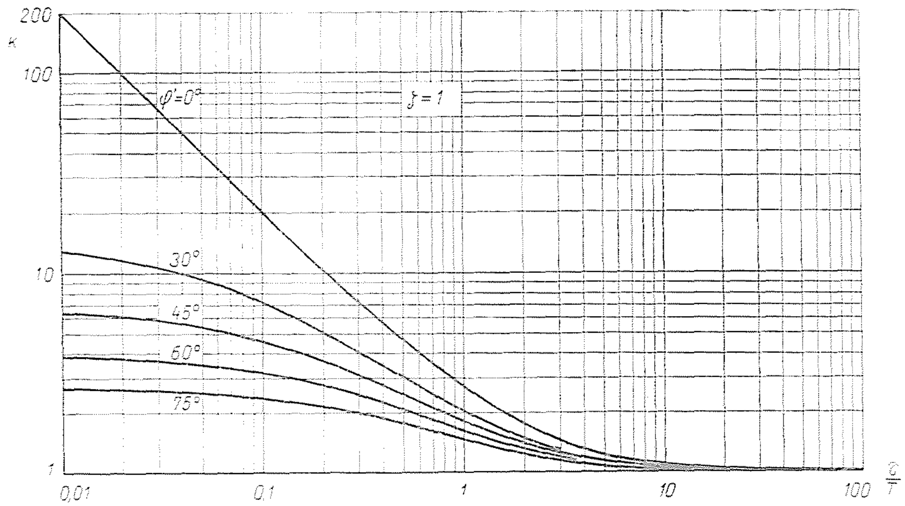

Fig. 7. Proportional control

($\varphi' > 30°$) of a control of anormal behaviour must be chosen lower than the value of the loop gain ($\varphi' = 0°$) giving the limit position of the stability region.

d) In the case of $\zeta \leq 0.7$ the loop gain shows a minimum for $\tau/T \approx 1$ values independently of the phase margin value. The position and the value of the minimum is easily determined by extreme value calculations. The obtained minimum positions are:

$$\frac{\tau}{T} = \frac{\omega\tau}{\sqrt{1-2\zeta^2}} \quad \text{and} \quad \frac{\tau}{T} = \frac{\omega\tau}{\sqrt{2(1-2\zeta^2)}},$$

respectively. The minima of the loop gain are:

$$K_{\min} = 2\zeta\sqrt{1-\zeta^2} \quad \text{and} \quad K_{\min} = 1,$$

respectively.

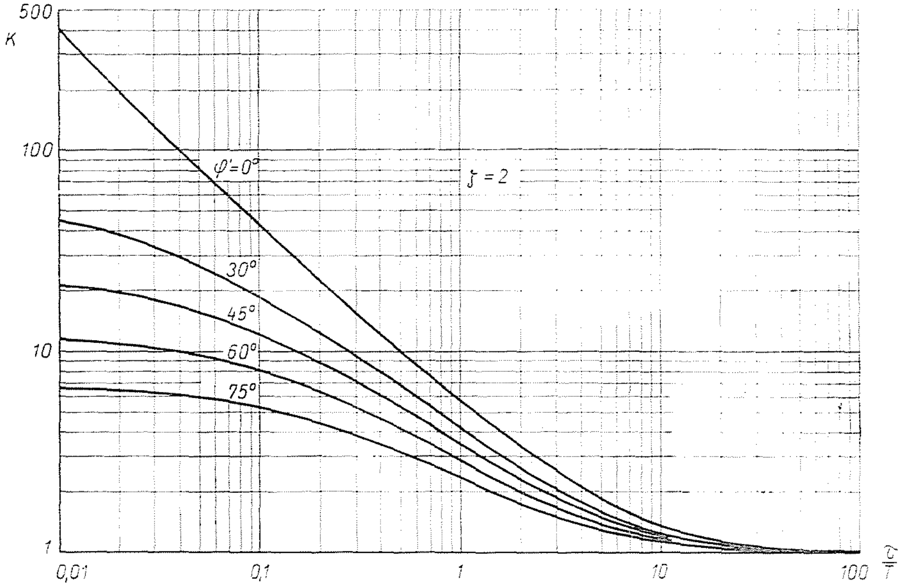

Fig. 8. Proportional control

2.2. *Integral control*

The transcendent equation used for determining the angular frequency $\omega_{cr,\varphi'} = \omega$ belonging to the phase margin of the required value is:

$$\omega\tau + \tan^{-1}\frac{2\zeta T\omega}{1-T^2\omega^2} = 90^\circ - \varphi'. \tag{10}$$

The relationship of the time constants belonging to the phase margin with the given angular frequency value obtained by iterating the equation at an arbitrary accuracy may be determined from

$$\tau/T_I = \omega\tau\sqrt{(1-T^2\omega^2)^2 + (2\zeta T\omega)^2} \tag{11}$$

where T_I is the integral time constant.

The $\tau/T_I(\tau/T; \zeta; \varphi')$ curves determined by using Eqs (10) and (11) and plotted with the help of a digital computer are shown in Figs 9 to 14.

The following properties may be read off the diagrams:

a) For $\tau/T \to 0$, the functional relationship $\tau/T_I = G(\tau/T)$ is linear, therefore the functional relationship $G = G(\zeta; \varphi')$ must be satisfied.

b) For $\tau/T \to \infty$ $\tau/T_{I,cr} \to (\pi/2 - \varphi')$, whatewer the ζ value.

c) For low values of the damping factor and dead time, the control is of anormal behaviour (see item 1).

The first two statements are easily admitted.

For low values of τ/T the dead time is negligible in comparison to the element with second order lag and the integral element. The transfer function is

$$Y(s) = \frac{1}{sT_I} \frac{1}{1 + 2\zeta Ts + T^2 s^2}.$$

After the $s = j\omega$ substitution and standardization the frequency function is:

$$Y(j\omega) = \frac{\tau}{T_I} \frac{1}{\omega T \sqrt{(1 - T^2\omega^2)^2 + (2\zeta T\omega)^2}} e^{-j\left(90^\circ + \tan^{-1}\frac{2\zeta T\omega}{1 - T^2\omega^2}\right)}. \tag{12}$$

For reaching an arbitrary phase margin, the equality

$$|Y(j\omega)| e^{j\varphi(\omega)} = 1 \cdot e^{-j(180^\circ - \varphi')} \tag{13}$$

must be satisfied.

From the equality of the phase angles we have:

$$\tan^{-1} \frac{2\zeta T\omega}{1 - T^2\omega^2} = 90^\circ - \varphi'.$$

From the resulting quadratic equation, the real solution of the angular frequency, after standardization and simplification, is:

$$\omega = \frac{1}{T} \frac{-\zeta + \sqrt{\zeta^2 + \tan^2(90^\circ - \varphi')}}{\tan(90^\circ - \varphi')} = \frac{1}{T} \cdot F(\zeta; \varphi'), \tag{14}$$

where

$$F(\zeta; \varphi') = \frac{-\zeta + \sqrt{\zeta^2 + \tan^2(90^\circ - \varphi')}}{\tan(90^\circ - \varphi')}. \tag{15}$$

On the basis of (14) we can write:

$$\omega\tau = F(\zeta; \varphi') \frac{\tau}{T}.$$

From the equality of the absolute values and utilizing (12), (13) and (15), after standardization — denoting $F = F(\zeta; \varphi')$ — we have:

$$\tau/T_I = F\sqrt{(1 - F^2)^2 + (2\zeta F)^2} \cdot \frac{\tau}{T}. \tag{16}$$

From (16) it is seen that when $\tau/T \to 0$, then

$$\tau/T_I = G(\zeta; \varphi')\frac{\tau}{T}, \tag{17}$$

which is in agreement with our statement under a).

In the case of $\tau/T \to \infty$, the controlled section may be substituted by an element with pure dead time. For reaching an arbitrary phase margin, the equation

$$Y(j\omega) = K\frac{e^{-j\omega\tau}}{j\omega T_I} = 1 \cdot e^{-j(\pi-\varphi')} \tag{18}$$

must be satisfied. From the equality of the phase angles

$$\omega\tau = \frac{\pi}{2} - \varphi'. \tag{19}$$

From the equality of the absolute values and using (19) we have, for $\tau/T \to \infty$

$$\tau/T_I = \omega\tau = \frac{\pi}{2} - \varphi', \tag{20}$$

which is in agreement with our statement under b).

It is worthwhile to note that for the evaluation of the angular frequency determining the value of τ/T_I, some numerical method must be used to solve the transcendent equation (8) producing it. This is a rather lenghty operation even with a digital computer, or it may be divergent if the initial value for the iteration process has been improperly chosen.

In the present case the determination of the initial value is greatly facilitated by (14), (17) and (19), (20), respectively. Let us determine on the basis of Fig. 15 the intersection B of the straight line $\tau/T_I = \pi/2 - \varphi'$ belonging to the high dead time values with the linear approximation belonging to the low dead time values. Let us choose for initial values for the numerical solution of the transcendent equation (8) the following ω_0 radian frequency values:

$$\omega_0 = \frac{F(\zeta; \varphi')}{T} \quad \text{for} \quad 0 \leq \tau/T \leq B,$$

and

$$\omega_0 = \frac{\pi}{2} - \varphi' \quad \text{for} \quad B \leq \tau/T \leq \infty ,$$

respectively.

By choosing an arbitrary numerical method for the solution of the transcendent equations and by substituting the values of (21), a convergent solution is obtained.

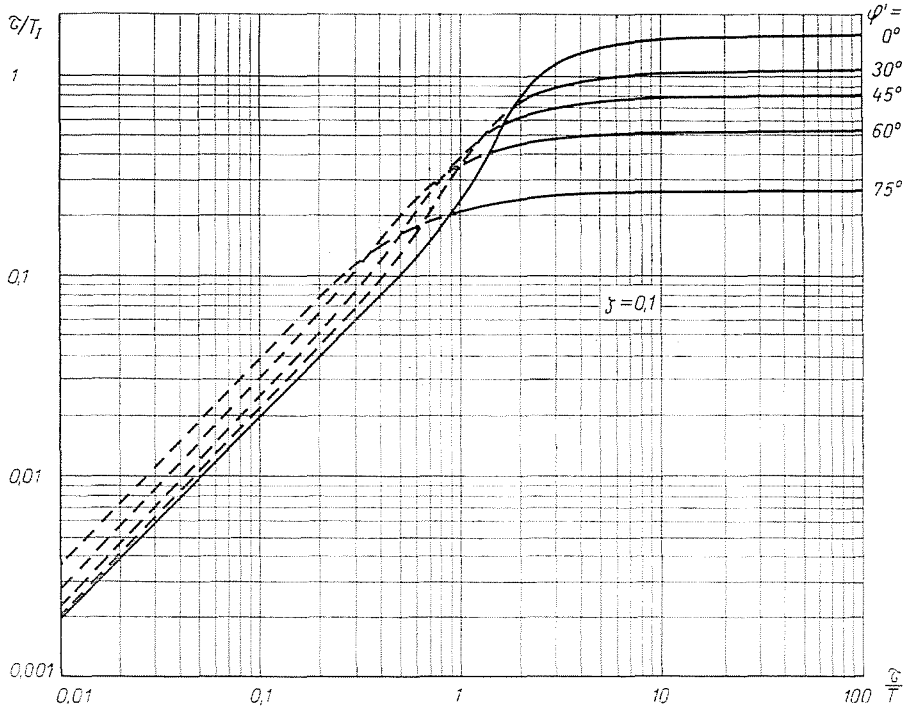

Fig. 9. Integral control

The data in Figs 9 to 14 were determined—under the above considerations—by a digital computer. For the solution of (8), the Newton—Raphson iteration formula was chosen.

The PROCEDURE written in the ALGOL program language for the determination of the stability region permitting to reach an arbitrary phase margin of a linear control with second order lag and dead time, with a unit feedback and integral compensation, is found in the Appendix. The procedure is suitable for the determination of the stability region limit position ($\varphi' = 0°$) as well. For this purpose the relationship given in [5] was utilized, according to which

$$\tau/T_I \Big|_{\omega_{cr} = \frac{1}{T}} = 2\zeta\tau/T \quad \text{for} \quad \tau/T \to 0 .$$

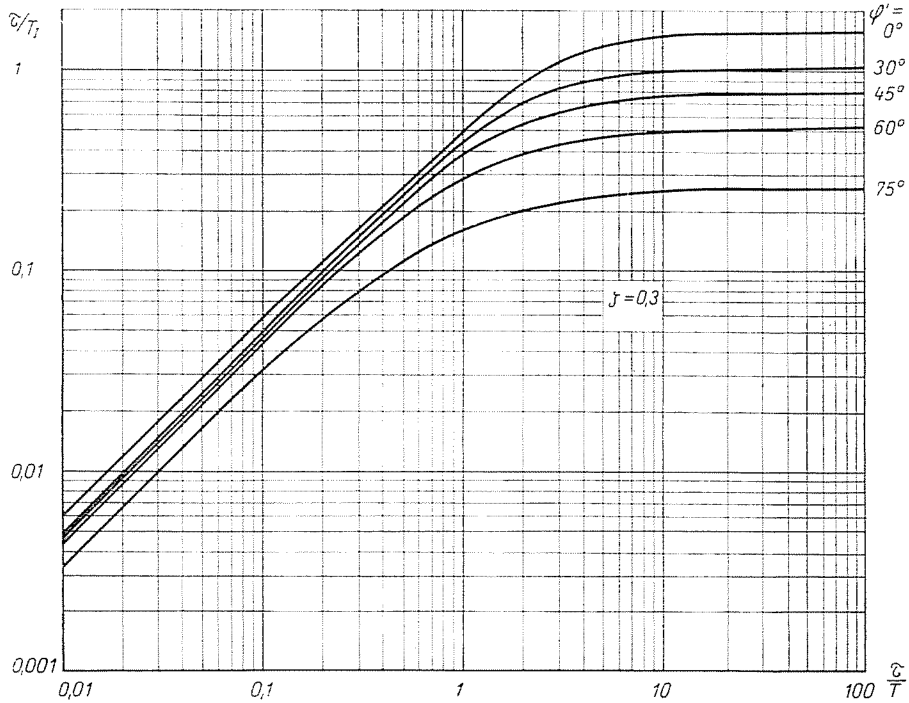

Fig. 10. Integral control

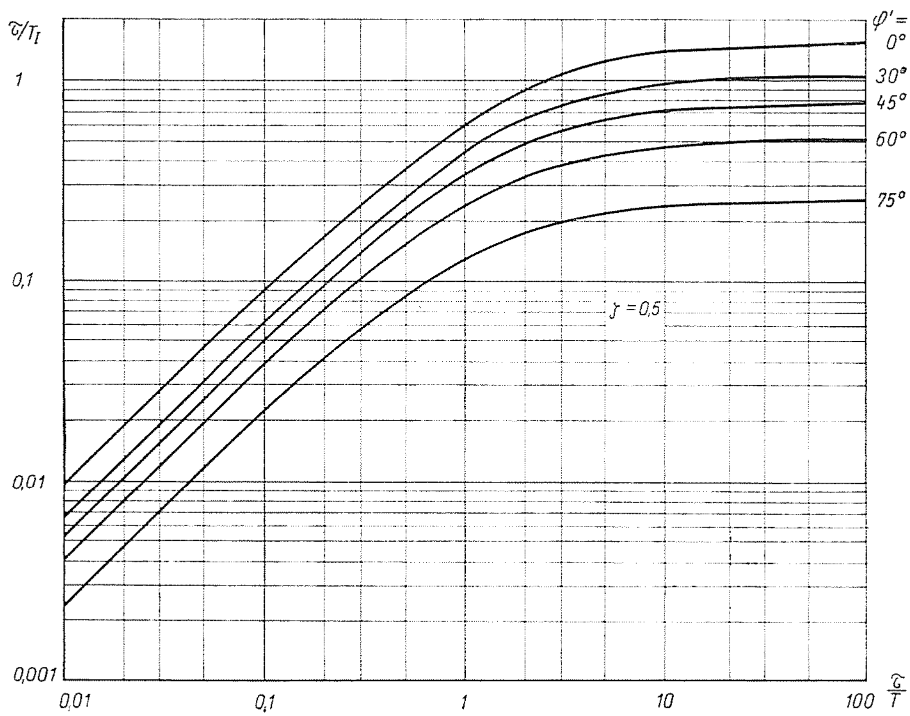

Fig. 11. Integral control

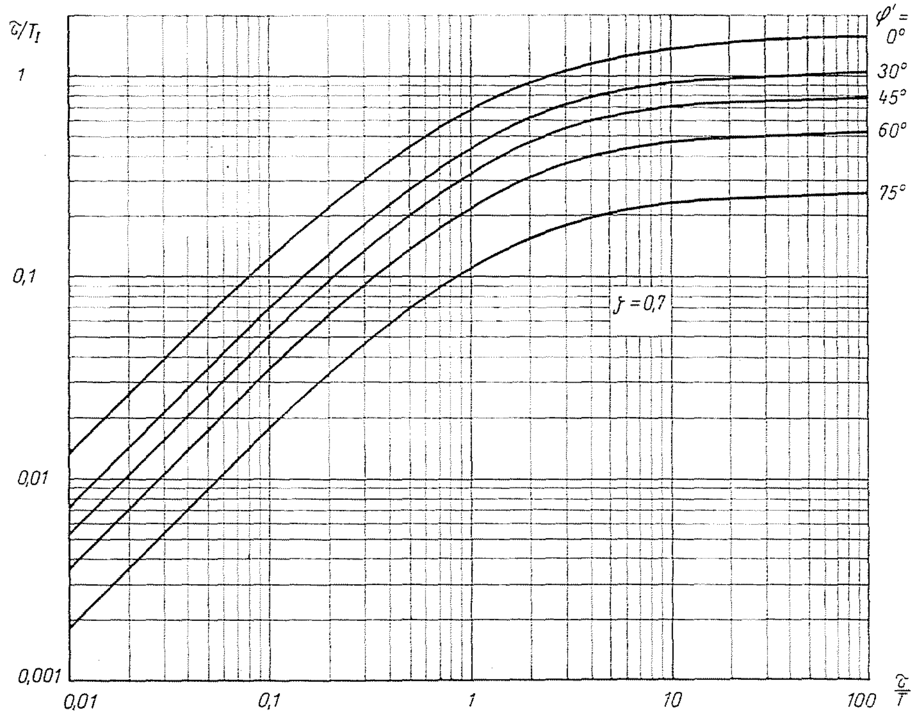

Fig. 12. Integral control

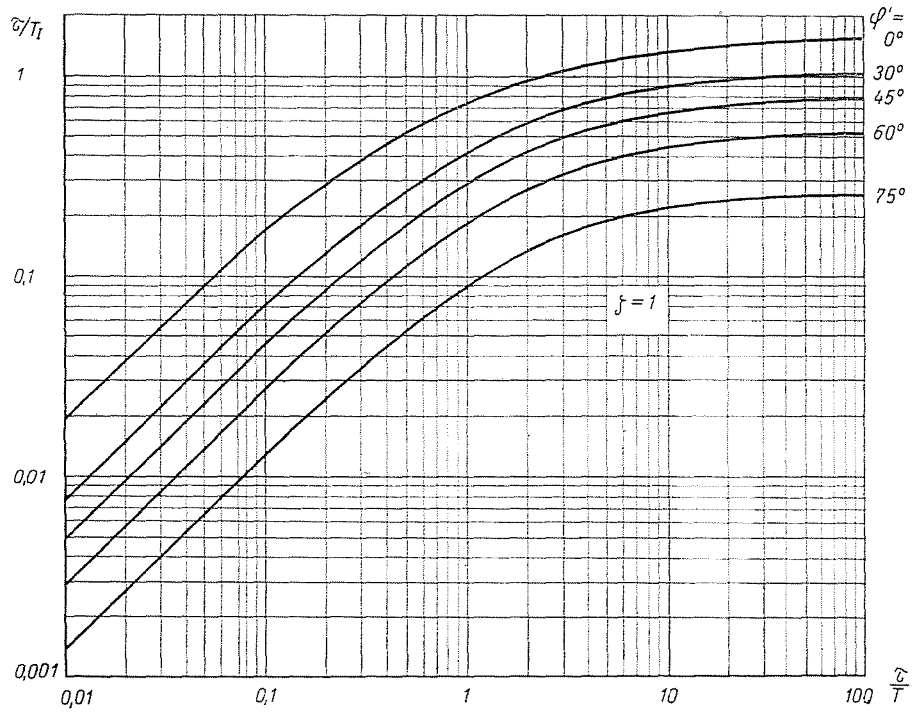

Fig. 13. Integral control

This was necessary because Eq. (14) used for the approximation of the angular frequency for low dead time values becomes meaningless for $\varphi' = 0°$.

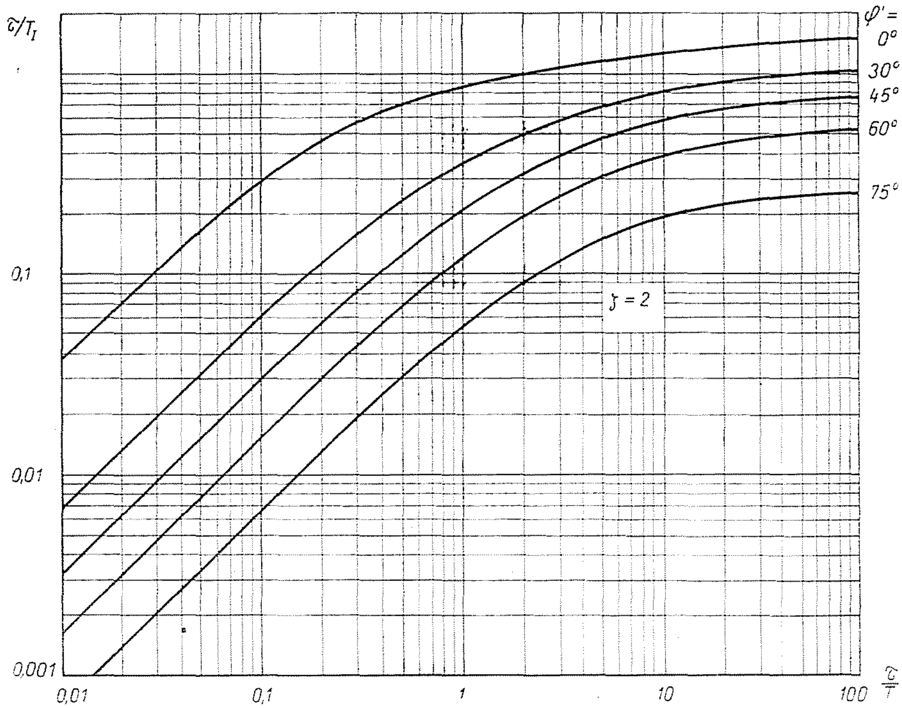

Fig. 14. Integral control

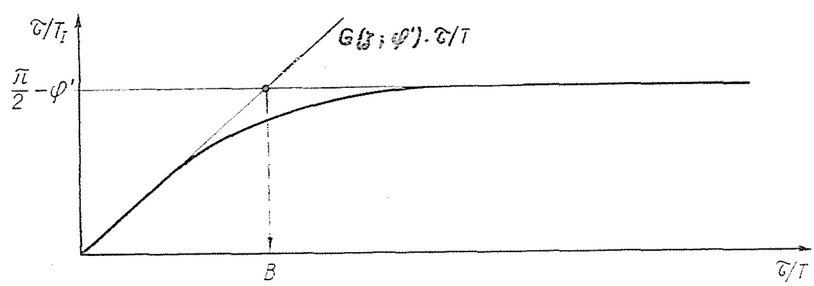

Fig. 15

Conclusions

As it has been already established in [5], in the case of a linear control with second order lag and dead time, with a unit feedback, the choice of a control compensated in series by a proportional element is advised for low dead time values (see Diagram 2 in [5] and Diagrams 3 to 8 in this paper), while a control compensated by an integral element connected in series proves to be more advantageous for high dead time values, as seen from Fig. 7 in [5] and Figs 9 to 14 in this paper.

A further advantage of the diagrams presented in this paper is that when the phase margin value is chosen higher than $\frac{\pi}{6}$, a control system satisfying also the quality requirements of the control can be constructed.

In subsequent papers we shall study the pattern of the diagrams determining stability regions which permit to reach an arbitrary phase margin when the series compensation is done by proportional-plus-integral (PI), proportional-plus-differential (PD) and proportional-plus-integral-plus-differential (PID) elements, respectively.

Appendix

Procedure in the ALGOL program language for determining the stability region of a linear control system with second order lag and dead time with a unit feedback compensated by an integral element connected in series, to reach an arbitrary phase margin.

```
PROCEDURE INTEGRAL (FIPM, ZETA, A, OMEGA, K);
VALUE FIPM, ZETA, A; REAL FIPM, ZETA, A, OMEGA, K;
  BEGIN REAL Z, Y, W, M1, M2, T, M3, B, F, M4, M5, M6, M7, FI, DFI;
    Z: = 3.141593 * (90—FIPM)/180;
    IF FIPM ≠ 0 THEN Y: = SIN(Z)/COS(Z);
                     W: = 3.141593 * (1—FIPM/180);
    M1: = 2 * ZETA; M2: = ZETA ↑ 2; T: = 1/A; M3: = M1 * T;
    IF FIPM = 0 THEN BEGIN B: = 1.570796/M1;
                           GOTO L4
                     END
              ELSE F: = (—ZETA + SQRT(M2 + Y ↑ 2))/Y;
                   B: = Z/F/SQRT((1 — F ↑ 2) ↑ 2 + (M1 * F) ↑ 2;
    L4: IF A < B THEN BEGIN IF FIPM = 0 THEN BEGIN
                                               OMEGA: = A;
                                               GOTO L3
                                               END
                                         ELSE OMEGA: = F * A
                     END
              ELSE OMEGA: = Z;
    L3:M4: = M3 * OMEGA; M5: = (T * OMEGA) ↑ 2; M6: = 1 — M5;
    IF M6 = 0 THEN BEGIN M7: = 1.570796;
                         GOTO L1
```

```
            END;
            M7: = ARCTAN(M4/M6);
            IF M7 < 0 THEN M7: = M7 + 3.141593;
                       L1:FI: = 1.570796 + OMEGA + M7;
IF ABS(FI — W) ≦ 10-6 THEN GOTO L2
                       ELSE
   DFI: = 1 + (M3 * (1 + M5))/(M6↑ 2 + M4 ↑ 2);
   OMEGA: = OMEGA — (FI — W)/DFI; GOTO L3;
L2: K: = OMEGA * SQRT(M6 ↑ 2 + M4 ↑ 2)
END INTEGRAL;
```

Summary

The variation of the stability region permitting to reach an arbitrary phase margin φ' of a linear control system with second order lag and dead time with a unit feedback and in the general case with a series PID (proportional-plus-integral-plus-differential) compensation is determined and diagrams produced with the help of a digital computer for proportional P and integral I compensations, respectively, are presented for the region $0.01 \leq \tau/T = 100$, where τ represents the dead time and T, the time constant of the second order lag. The damping factor ζ, varies between 0.1 and 2. The variations of the stability region are given by the diagrams for the values $\varphi' = 0, \pi/6, \pi/4, \pi/3, 5\pi/12$ of the phase margin.

References

1. CSÁKI, F.—BARS, R.: Automatika. Tankönyvkiadó, Budapest 1969.
2. CSÁKI, F.: Szabályozások dinamikája. Akadémiai Kiadó, Budapest 1970.
3. Серебрянский, А. Я.: Метод определения величены максимального отклонения в системах автоматического регулирования, включающих объект с запаздыванием. Приборы и системы управления, II, 1968.
4. CSÁKI, F.—HABERMAYER, M.: Stability test of linear control systems with dead time. Periodica Polytechnica, El. **12**, 311—318 (1968).
5. HABERMAYER, M.: Stability test of linear control systems with dead time by digital computer. Periodica Polytechnica El. **12**, 443—452 (1969).
6. HABERMAYER, M.: Stability test of linear control systems with dead time compensated by PI controller. Periodica Polytechnica, El. **14**, 53—59 (1970).
7. HABERMAYER, M.: Stability test of linear control systems with dead time compensated by PID controller. Periodica Polytechnica. El. **14**, 181—194 (1970).

Maria HABERMAYER, Budapest XI., Garami E. tér 3. Hungary