ON THE ANALYTICAL DESIGN OF SIMPLE CONTROLS

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1. The Graham—Lathrop standard forms

GRAHAM and LATHROP [2] carried out their investigations for the control circuit shown in Fig. 1.

The transfer function of the closed system in the general case is:

$$W(s) = \frac{C(s)}{R(s)} = \frac{c_m s^m + \ldots + c_2 s^2 + c_1 s + c_0}{r_n s^n + \ldots + r_2 s^2 + r_1 s + r_0} \tag{1}$$

The comparison of the transient processes corresponding to $W(s)$ is made possible—in spite of different time scales—by the normalization of $W(s)$ that can be performed in the following way:
We introduce in (1) the coefficients

$$q_i = \frac{r_i}{\omega_0^{n-i} r_n} \quad \text{and} \quad p_i = \frac{c_i}{\omega_0^{n-i} r_n} \tag{2}$$

where

$$\omega_0^n = \frac{r_0}{r_n} \tag{3}$$

This transformation corresponds to a time scaling according to $\tau = \omega_0 t$ and the normalized ($\lambda = s/\omega_0$) transfer functions

$$\frac{p_m \lambda^m + \ldots + p_2 \lambda^2 + p_1 \lambda + 1}{\lambda^n + \ldots + q_2 \lambda^2 + q_1 \lambda + 1} \tag{4}$$

(together with their corresponding time functions) can already be compared.

Now, if the coefficients in (4) are determined optimally according to some criterion, then the coefficients of the optimum transfer function $W(s)$ are obtained by regression according to (2) and (3).

GRAHAM and LATHROP determined—by following the denoted train of thought—the optimum coefficients of (4) according to the integral criterion of the time-weighted absolute value, i.e. they minimized the value of the integral:

$$I = \int_0^\infty t\,|e(t)|\,\mathrm{d}t \tag{5}$$

They carried out the measurements on an analogue computer and chose for the initial points of optimization the parameters of the Butterworth system of maximum bandwidth [3].

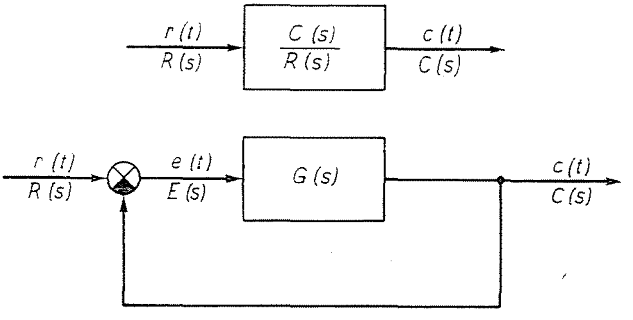

Fig. 1

In order to reduce the number of the independent variables and to define the problem mathematically, they applied the following considerations:

The behaviour of the error signal $e(t)$ of the control circuit for long t periods may be written also in the following form:

$$\underset{t\to\infty}{e(t)} = D_0 r + D_1 \frac{\mathrm{d}r}{\mathrm{d}t} + \frac{D_2}{2}\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} + \ldots \tag{6}$$

By more detailed investigations it is easily proven that

$$\begin{aligned} D_0 &= 0 \quad &\text{if} \quad c_0 &= r_0 \\ D_1 &= 0 \quad &\text{if} \quad c_1 &= r_1 \\ D_2 &= 0 \quad &\text{if} \quad c_2 &= r_2 \\ &\vdots & &\vdots \end{aligned} \tag{7}$$

By satisfying the conditions of (7) during optimization, the optimum coefficients of the (transfer function) of the closed system were given by the authors as seen in Table I. The optimum transfer functions determined in advance in this way are also called standard forms.

2. Optimum design of controls on the basis of the standard forms

In studying the analytical design of controls we shall consider one of the most simple cases, the series compensation shown in Fig. 2.

The task of the analytical design is to determine the transfer function $G_1(s)$ of the control in knowledge of a given transfer function $G_2(s)$ of the controlled section in a way that the resultant transfer function $W(s)$ of the closed system is optimum according to Table I.

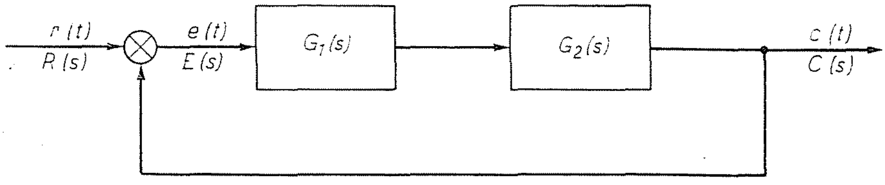

Fig. 2

This requirement can be met by the control defined by

$$G_1(s) = \frac{W(s)}{G_2(s)} \cdot \frac{1}{1 - W(s)} \tag{8}$$

and if realizability is left out of consideration then the problem is herewith solved. (We note that with the spreading of the integrated circuit operational amplifiers, the latter solution is already approached by the design.)

In our investigations we tried to solve the analytical design with the "conventional" controls and so the problem may be formulated in another way.

If the transfer function of the control is:

$$G_1(s) = \frac{M_1(s)}{N_1(s)} \tag{9}$$

and that of the controlled section:

$$G_2(s) = \frac{M_2(s)}{N_2(s)} \tag{10}$$

then it is to be investigated whether the resultant transfer function

$$W(s) = \frac{M_1(s)M_2(s)}{N_1(s)N_2(s) + M_1(s)M_2(s)} \tag{11}$$

can be brought into correspondence with Table I.

Table 1

n	a_8	a_7	a_6	a_5	a_4	a_3	a_2	a_1	a_0
$W(s) = \frac{a_0\omega_0^n}{a_n s^n + a_{n-1}\omega_0 s^{n-1} + \ldots + a_1\omega_0^{n-1}s + a_0\omega_0^n}$									
1								1.00	1.00
2							1.00	1.40	1.00
3						1.00	1.75	2.15	1.00
4					1.00	2.10	3.40	2.70	1.00
5				1.00	2.80	5.00	5.50	3.40	1.00
6			1.00	3.25	6.60	8.60	7.45	3.95	1.00
7		1.00	4.475	10.42	15.08	15.54	10.64	4.58	1.00
8	1	5.20	12.80	21.60	25.75	22.20	13.30	5.15	1.00
$W(s) = \frac{a_1\omega_0^{n-1}s + a_0\omega_0^n}{a_n s^n + a_{n-1}\omega_0 s^{n-1} + \ldots + a_1\omega_0^{n-1}s + a_0\omega_0^n}$									
2							1.00	3.20	1.00
3						1.00	1.75	3.25	1.00
4					1.00	2.41	4.93	5.14	1.00
5				1.00	2.19	6.50	6.30	5.24	1.00
6			1.00	6.12	13.42	17.16	14.14	6.76	1.00
$W(s) = \frac{a_2\omega_0^{n-2}s^2 + a_1\omega_0^{n-1}s + a_0\omega_0^n}{a_n s^n + a_{n-1}\omega_0 s^{n-1} + \ldots + a_1\omega_0^{n-1}s + a_0\omega_0^n}$									
3						1.00	2.97	4.94	1.00
4					1.00	3.71	7.88	5.93	1.00
5				1.00	3.81	9.94	13.44	7.36	1.00
6			1.00	3.93	11.68	18.56	19.30	8.06	1.00

If we wish to dimension the control parameters on the basis of the standard forms, then the following points must be considered:

1. In order to have a definite equation system—obtained by the correspondence of the coefficients—the number of the free parameters of the control must be one less than the ordinal number of the polynomial appearing in the opened circuit:

$$G(s) = G_1(s)G_2(s) = \frac{M_1(s)M_2(s)}{N_1(s)N_2(s)} = \frac{M(s)}{N(s)} \tag{12}$$

2. The constraint contained in (7) may only be satisfied if no powers of s appearing in $M(s)$ are contained by $N(s)$. So even the lowest exponent

of s in the denominator of the transfer function of the opened circuit must exceed the ordinal number of the numerator.

3. If items 1 and 2 are satisfied, then it must still be investigated whether values of the control parameters corresponding to a physically realizable system are given by the solution of the equation system obtained by comparing the coefficients (e.g. if the time constants are positive, etc.).

The above statements of general character will be applied in the following to concrete control-controlled section ensembles.

3. Diagrams to meet the standard forms

After having investigated many types of control-controlled section structures we have come to the conclusion that it is very hard to find systems satisfying completely conditions 1 and 2—with supposed "conventional" controls and controlled sections identified to have simple constructions. Systems completely satisfying the conditions 1, 2 and 3 were obtained in the following cases:

A. Be $G_1(s) = 1/sT_I$ and $G_2(s) = 1/(1 + sT)$; then

$$W(s) = \frac{1}{s^2 T_I T + sT_I + 1} = \frac{1/T_I T}{s^2 + s/T + 1/T_I T} \tag{13}$$

By comparing the coefficients of the transfer function (13) with the corresponding data of Table I, the equation system of the control parameters arises:

$$1/T = 1{,}4\omega_0 ; \qquad 1/T_I T = \omega_0^2 \tag{14}$$

whose solution gives the optimum integration period:

$$T_I = 1{,}96T \tag{15}$$

B. Be $G_1(s) = (1 + sT_1)/(1 + sT_2)$ and $G_2(s) = K/s^2$, then

$$W(s) = \frac{sKT_1 + K}{s^3 T_2 + s^2 KT_1 + K} \tag{16}$$

The solution of the equation system obtained by comparing the coefficients for the control parameters T_1 and T_2 versus the circuit amplification K is shown by the diagram in Fig. 3.

According to the above, the too strict constraints of conditions 1, 2 and 3 permit the application of the standard forms but in few cases. We studied the possibility of moderating those constraints. This could only be attained by disregarding condition 2, or by an uncomplete satisfaction of the conditions set by (7). The fundamental consideration in the following was to ensure by the control parameters at least the denominator of the optimum transfer function of the closed circuit. Though this requirement ensures the optimum

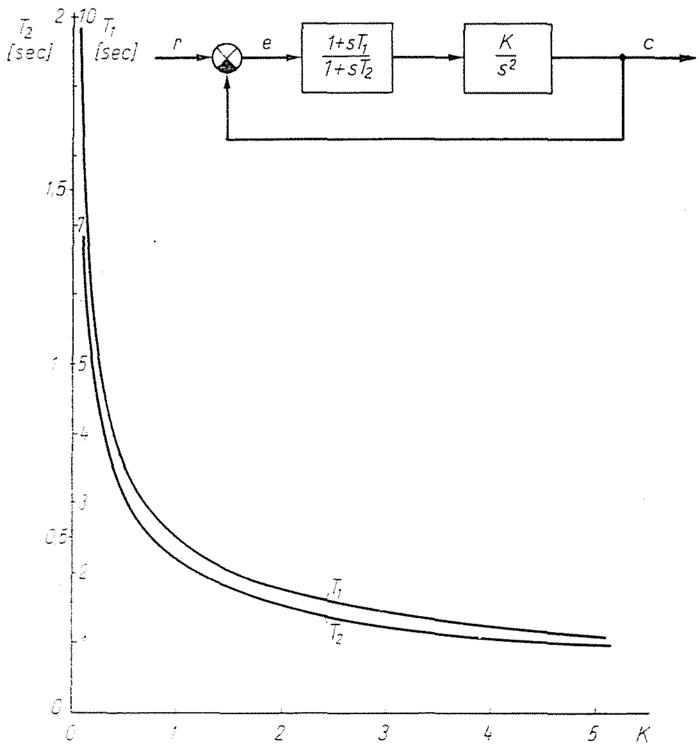

Fig. 3

behaviour of the system left alone, but it does not zero all the dynamic error coefficients (inclusive that of m-th order). Naturally it had to be investigated in each case in the time range what implications could be expected by the partial satisfaction of the conditions concerning the numerator of the transfer function of the closed system, as there is no unequivocal relationship between the zeros of the system and its response function in the general case.

These investigations showed that an optimum control according to the $G-L$ standard forms could not be adjusted to a proportional controlled section of three time lags, not even with the recent moderated constraints. (We have investigated here the P, I, PI, PID, FKS controls.)

Nevertheless, for the case of controlled sections with two time lags and of integral character controlled sections with two time lags, succesful investigations can be reported:

C. Be $G_1(s) = K(1 + 1/sT_I)$ and $G_2(s) = 1/(1 + 2\,\xi\, Ts + T^2s^2)$.

In this case

$$W(s) = \frac{sK/T^2 + K/T^2T_I}{s^3 + s^2 2\xi/T + s(K+1)/T^2 + K/T^2T_I} \tag{17}$$

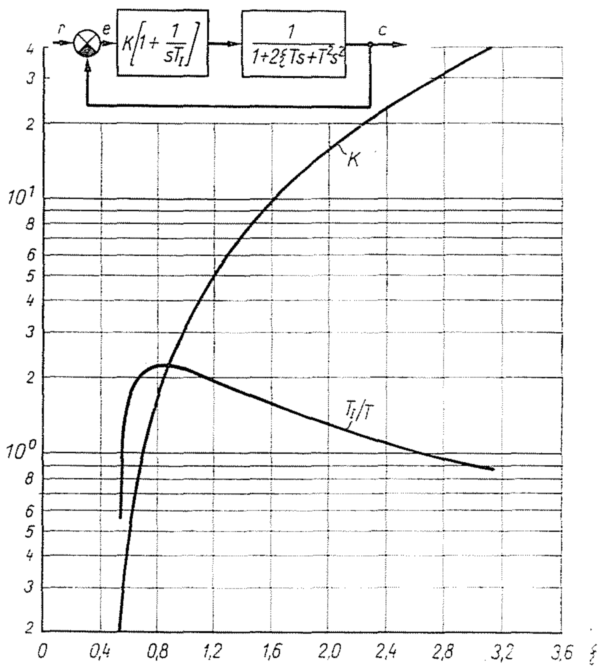

Fig. 4

Here, the coefficients were only compared for the denominator; the optimum control parameters for the obtained equation system are:

$$K = 4{,}25\xi^2 - 1$$
$$\frac{T_I}{T} = \frac{4{,}25\xi^2 - 1}{1{,}482\xi^3} \tag{18}$$

The diagram plotted for the adjustment values corresponding to (18) is shown in Fig. 4.

For the values $\xi = 1$ and $\xi = 1{,}5$ the unit step responses of the nearly optimum system are also shown in Figs 5 and 6. In the parameter ranges shown in the diagram the violation of the condition related to the numerator

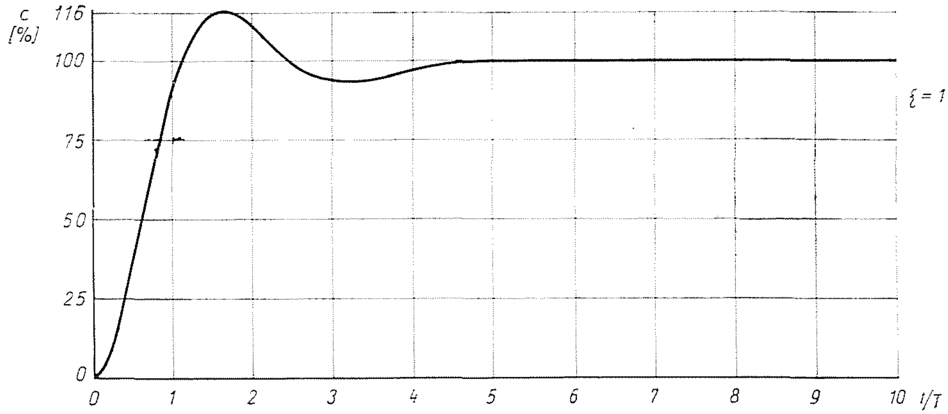

Fig. 5

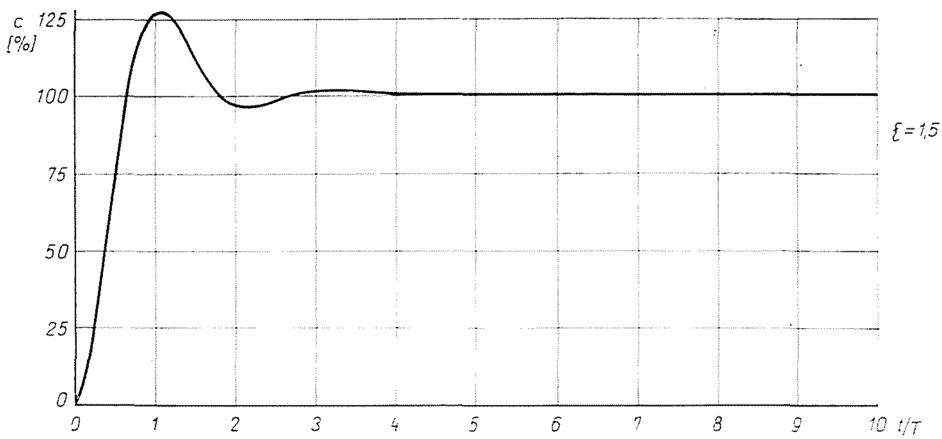

Fig. 6

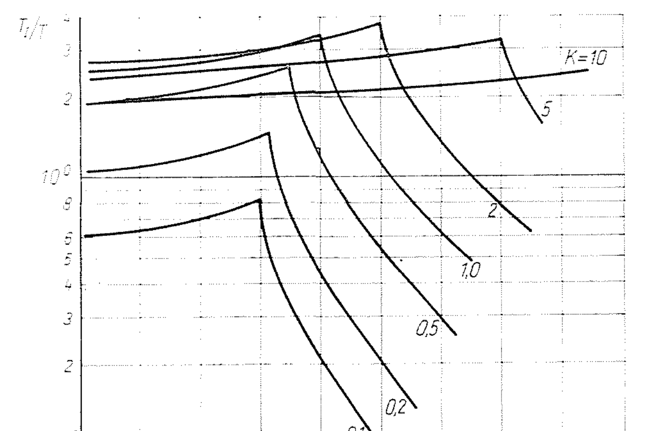

Fig. 7

did not deteriorate the quality characteristics of the time function, so the approximation is acceptable. (This is seen also by comparing the numerator and denominator of (17), as $D_0 = 0$, and $D_1 \approx 0$, for $K \gg 1$.)

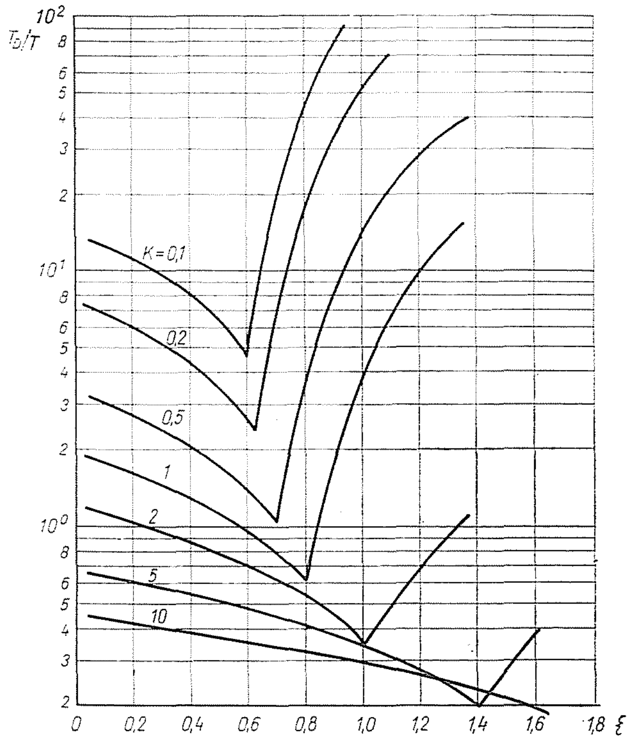

Fig. 8

We note that in the case of $\xi < 0{,}45$ there exists no physically realizable control.

In the following examples the solutions will not be detailed to this depth, only the structure of the systems and the setting diagrams will be presented with a few time functions.

D. Be $G_1(s) = K\big(1 + 1/sT_I + sT_D/(1 + sT_4)\big)$ and

$$G_2(s) = \frac{1}{1 + 2\xi Ts + T^2s^2}$$

By following the solution of example C, the diagrams obtained for the optimum control parameters by comparing the coefficients related to the denominator of $W(s)$ are shown in Figs 7, 8 and 9, parametered in K. Figs 10 and 11 show

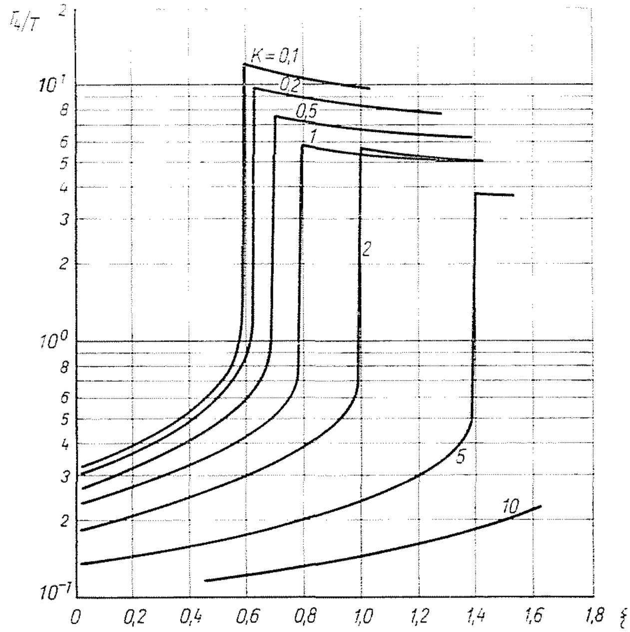

Fig. 9

also the unit step responses of the closed system for two different combinations of the parameters.

E. Be $G_1(s) = K\left(1 + sT_1\right)/\left(1 + sT_2\right)$ and

$$G_2(s) = \frac{1}{sT_I(1 + 2\xi Ts + T^2s^2)}$$

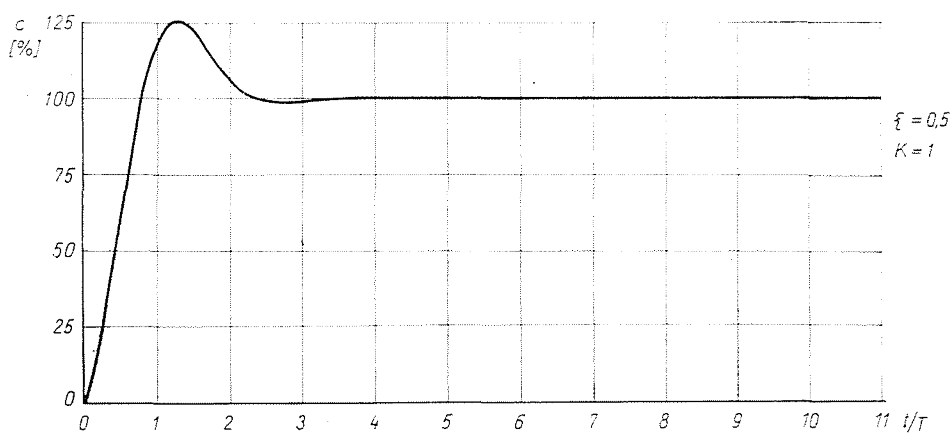

Fig. 10

The equation system ensuring the optimum denominator of the closed system had only a real solution for $\xi \leq 0{,}6$; the setting diagram of the control parameters is shown in Fig. 12. The unit step response of the nearly optimum system for $\xi = 0{,}5$ is seen in Fig. 13.

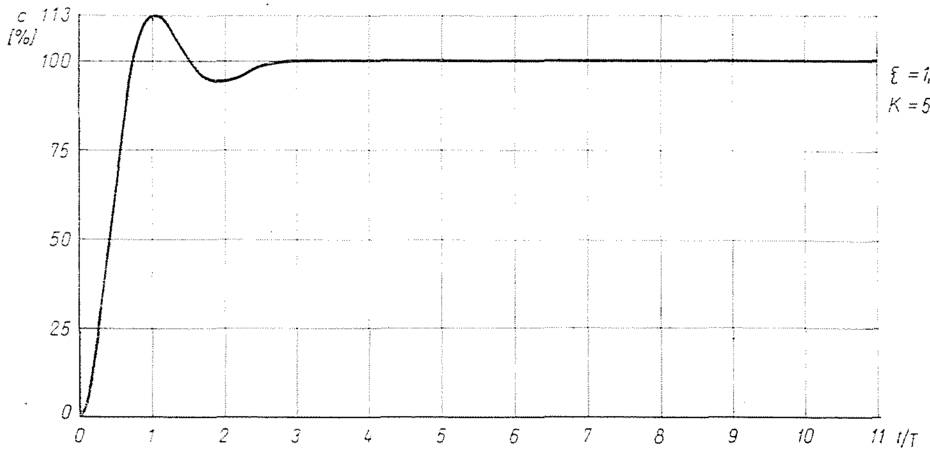

Fig. 11

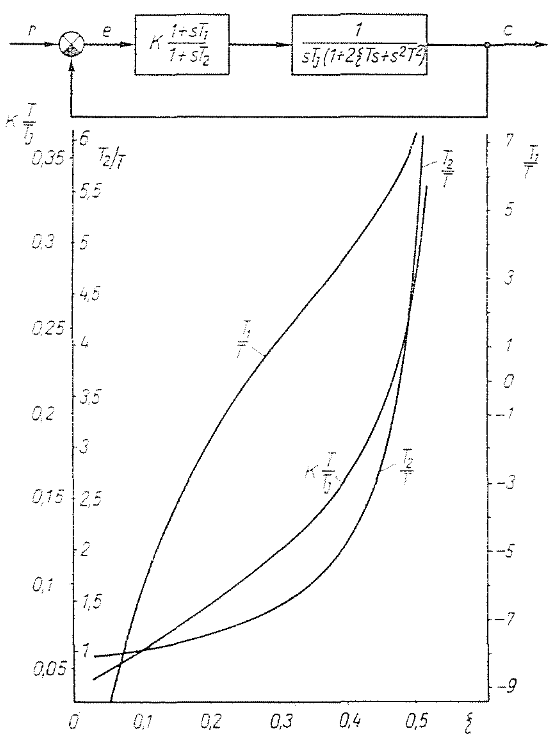

Fig. 12

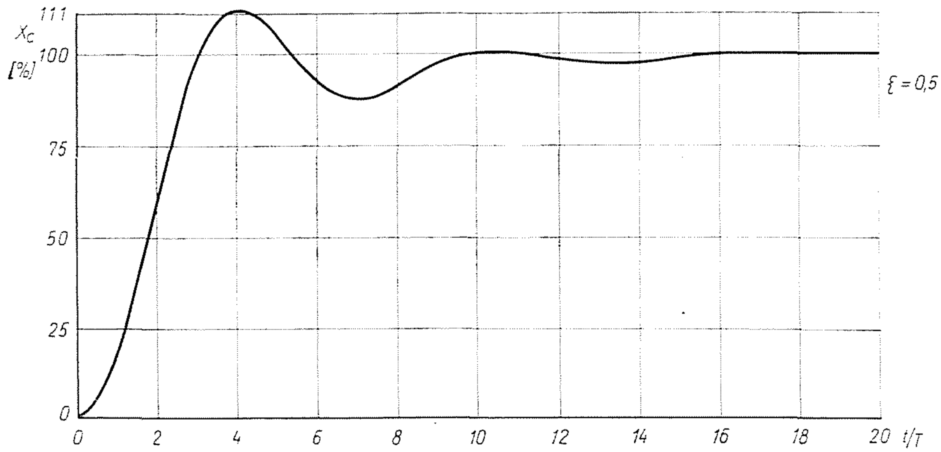

Fig. 13

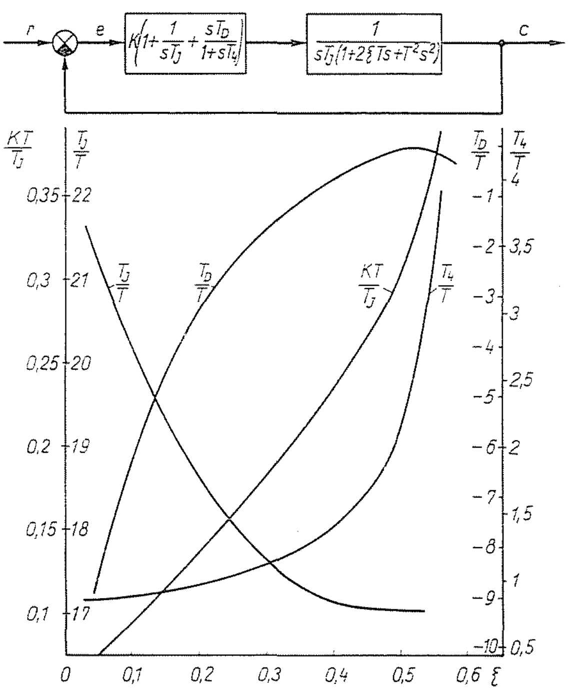

Fig. 14

F. Be $G_1(s) = K\big(1 + 1/sT_I + sT_D/(1 + sT_4)\big)$, be $G_2(s)$ as in the previous example. The control parameters ensuring the optimum denominator of $W(s)$ may be read off the diagram in Fig. 14. The curves were obtained by solving the equation system resulting from the comparison of the coefficients; no real solution existed but for $\xi \leq 0{,}625$.

Conclusions

The possibility of utilizing the GRAHAM—LATHROP standard forms for the analytical design of controls was studied. By summarizing the results it can be stated that for solving the optimum ITAE compensation with the conventional, most generally accepted (P, I, PI, PID, etc.) controls, the standard forms are unsuitable for the analytical design. (Only two structures were found where the results could be evaluated; see examples A and B.)

If the contraints defined by the standard forms are moderated (but partially satisfying the conditions $D_0 = 0, D_1 = 0, \ldots$) then the range of the structures useful for the design is extending and the diagrams resulting from the calculations may be used with advantage. The control tests performed in the time range showed that the quality characteristics of the unit step responses of the systems designed on the basis of the diagrams were also acceptable and could be regarded as good.

Based on the results, the determination of standard polynomials (forms) can be considered to be justified only in the case, when the controls capable to transform the arbitrary pole and zero will be unexpensive and widely used. Till that time the elaboration of standard forms is only worthwhile in the *a priori* knowledge of the system structure, the number of the free parameters and the relationships between the coefficients, as otherwise the problems discussed in the present paper will arise in any case.

Summary

The possibility of satisfying the GRAHAM—LATHROP standard polynomials in simple control circuits is investigated: the constraints of applying the standard forms in the cases of the conventional controls are determined: the results are presented in form of diagrams useful for the analytical design of controls.

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