SCATTERING OF ELECTROMAGNETIC PLANE WAVES ON OBSTACLES WITH LINEAR ISOTROPIC MATERIAL OF NEGATIVE CONDUCTIVITY

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Introduction

In a previous paper [1] we have dealt with the propagation and reflection of electromagnetic plane waves in the presence of isotropic substances. It has been shown that assuming a time dependence exp $(j\,\omega\,t)$, the basic parameter of all wave phenomena is the *complex refractive index*

$$\bar{n} = n' - jn'' \tag{1}$$

where $n' > 0$.

The material constant characterizing the non-magnetoactive substance is the *complex permittivity*

$$\bar{\varepsilon} = \varepsilon' - j\varepsilon'' = \bar{n}^2. \tag{2}$$

The imaginary part of the complex permittivity and the generalized conductivity are proportional. The reversal of the conductivity results in the reversal of the imaginary part of the permittivity. Taking this fact into consideration the paper compares the scattering on obstacles differing only by the sign of the conductivity.

Characteristics of the scattering of the electromagnetic plane waves

In this paper the following terminology will be used:

1. The electromagnetic plane wave induces a polarization current in the obstacle placed in the way of propagation. The secondary field generated by the polarization current is the *scattered field*, the phenomenon is the *scattering*.

2. In the case of a finite and non-zero conductivity as the result of the presence of the conduction current the scattering obstacle absorbs power from the incident wave. This phenomenon is the *absorption*.

3. The power density propagating in the wave behind the obstacle differs from that of the incident wave as a result of the scattering and absorption. This phenomenon is the *extinction*.

The connection among the above mentioned phenomena is

$$\textit{extinction} = \textit{scattering} + \textit{absorption}$$

To characterize the plane wave arriving from free space, let us set up a right-handed co-ordinate system whose unit vectors are $\mathbf{u}_r$, $\mathbf{u}_l$ and $\mathbf{u}$ in this order, and $\mathbf{u}$ points to the direction of propagation (Fig. 1). In this system the effective values of the field strengths of an arbitrarily polarized plane wave can be written as follows:

$$\overline{\mathbf{E}}_i = (\overline{E}_{li}\,\mathbf{u}_l + \overline{E}_{ri}\,\mathbf{u}_r)\,e^{-j\mathbf{k}_0\mathbf{r}} = \overline{F}_i\,e^{-j\mathbf{k}_0\mathbf{r}} \tag{3}$$

$$\overline{\mathbf{H}}_i = \frac{1}{Z_0}\,\mathbf{u} \times \overline{\mathbf{E}}_i \tag{4}$$

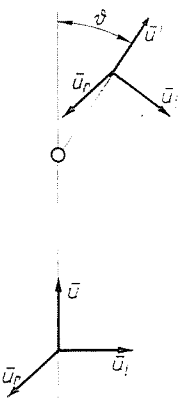

Fig. 1. Co-ordinate systems to describe the incident and scattered waves

where $k_0 = \dfrac{2\pi}{\lambda}$ is the wave number in free space and Z_0 is the wave impedance of free space.

Let the scattered field be investigated in the direction of the unit vector $\mathbf{u}'$ ($\mathbf{u}$, $\mathbf{u}_l$ and $\mathbf{u}'$ are coplanar). The components of the scattered field at a sufficiently large distance R are

$$\begin{aligned} \overline{\mathbf{E}}_{\mathrm{sca}} &= \frac{e^{-jkR}}{R}\,(\overline{E}_{l\,\mathrm{sca}}\,\mathbf{u}_l + \overline{E}_{r\,\mathrm{sca}}\,\mathbf{u}_r) = \frac{e^{-jkR}}{R}\,\overline{\mathbf{F}}_{\mathrm{sca}}\,. \\ \overline{\mathbf{H}}_{\mathrm{sca}} &= \frac{e^{-jkR}}{R}\,\frac{1}{Z_0}\,(\mathbf{u}' \times \overline{\mathbf{F}}_{\mathrm{sca}}) \end{aligned} \tag{5}$$

The connection between the twice two scalar parameters characterizing the incident and scattered fields is given by the scattering matrix

$$\begin{bmatrix} \overline{E}_{l\,\mathrm{sca}} \\ \overline{E}_{r\,\mathrm{sca}} \end{bmatrix} = \begin{bmatrix} \bar{S}_2(\vartheta,\varphi) & \bar{S}_3(\vartheta,\varphi) \\ \bar{S}_4(\vartheta,\varphi) & \bar{S}_1(\vartheta,\varphi) \end{bmatrix} \begin{bmatrix} \overline{E}_{li} \\ \overline{E}_{ri} \end{bmatrix}. \tag{6}$$

Considering linear substances, the elements of S depend only on the size, form and material of the obstacle if ϑ and φ are given. Knowing the scattering matrix, the power of the fields of different kinds can be determined.

The time-average of the power carried by the scattered field is

$$P_{\text{sca}} = Re \oint_A (\bar{\mathbf{E}}_{\text{sca}} \times \bar{\mathbf{H}}^*_{\text{sca}})\, \mathbf{dA} = \int_\Omega \frac{1}{Z_0} |F_{\text{sca}}|^2 d\Omega . \quad (7)$$

We have to integrate over a closed surface containing the scattering obstacle. Ω marks the unit sphere, $d\Omega$ is the element of the solid angle.

The absorbed power is:

$$P_{\text{abs}} = - Re \oint_A (\bar{\mathbf{E}} \times \bar{\mathbf{H}}^*)\, \mathbf{dA} = \omega\, \varepsilon_0 \int_V \varepsilon'' \, \bar{\mathbf{E}}\, \bar{\mathbf{E}}^* \, dV = 2\omega\, \varepsilon_0 \int_V n'\, n'' \, \bar{\mathbf{E}}\, \bar{\mathbf{E}}^* \, dV . \quad (8)$$

In the first integral $\bar{\mathbf{E}} = \bar{\mathbf{E}}_i + \bar{\mathbf{E}}_{\text{sca}}$ and similarly $\bar{\mathbf{H}} = \bar{\mathbf{H}}_i + \bar{\mathbf{H}}_{\text{sca}}$. The meaning of A is as before. The sign is negative, because the normal vector of the closed surface is chosen to point outward; but the absorbed power is usually considered to be positive if the energy flows into the obstacle. The second and third integrals are to be extended for the volume of the scattering obstacle and both of them are directly following from Poynting's theorem [2].

The definition of the extinction power is

$$P_{\text{ext}} = P_{\text{sca}} + P_{\text{abs}} . \quad (9)$$

Dividing the powers in Eqs (7) to (9) by the power density

$$S_i = Re(\bar{\mathbf{E}}_i \times \bar{\mathbf{H}}_i^*) = \frac{1}{Z_0} |F_i|^2 \quad (10)$$

quantities of surface dimension result. They are the *scattering cross-section*, the *absorption cross-section* and the *extinction cross-section*, respectively:

$$\sigma_{\text{sca}} = \frac{P_{\text{sca}}}{S_i} ; \qquad \sigma_{\text{abs}} = \frac{P_{\text{abs}}}{S_i} ; \qquad \sigma_{\text{ext}} = \frac{P_{\text{ext}}}{S_i} . \quad (11\text{ a, b, c})$$

There are other usual definitions to characterize the scattered power in a defined direction such as the *bistatic cross-section*

$$\sigma(\vartheta, \varphi) = \sigma(\mathbf{u}\, \mathbf{u}') = \frac{4\pi |F_{\text{sca}}(\mathbf{u}')|^2}{|F_i|^2} \quad (12)$$

and particularly in the opposite direction of the propagation of the incident wave the *monostatic* or *radar cross-section.*

$$\sigma_r = \sigma(\mathbf{u} \to -\mathbf{u}) = \frac{4\pi |\bar{F}_{\text{sca}}(-\mathbf{u})|^2}{|\bar{F}_i|^2} . \tag{13}$$

The so-called efficiency factors are obtained by dividing the cross-sections defined above by the geometric cross-section G of the obstacle perpendicular to the direction of propagation

$$Q = \frac{\sigma}{G} . \tag{14}$$

For linear media the σ and the Q are independent of the field strength. They depend only on the refraction index and the geometry.

The general properties of the efficiency factors

Theorem 1

$$Q_{\text{sca}} \geqslant 0 ; \tag{15}$$

$$Q(\vartheta, \psi) \geqslant 0 ; \tag{16}$$

$$Q_r \geqslant 0 \tag{17}$$

independently from the permittivity.

Proof

Each of the above statements directly results from the definitions (7) to (10) and (12) to (13), considering that in Eq. (11) $S_i > 0$ and in Eq. (14) $G > 0$.

Theorem 2

For a scattering object whose ε' does not reverse the sign

$$\operatorname{sign} Q_{\text{abs}} = \operatorname{sign} \varepsilon'' = \operatorname{sign} n'' . \tag{18}$$

Proof

The statement is obvious on the basis of the definition integrals in Eq. (6) considering that $n' > 0$.

Theorem 3

Let the geometry and ε' (or n') be fixed. There is such a value of ε'' (or n'') that $|Q_{\text{abs}}|$ is maximum [3].

Proof

In the case of an ideal dielectric $\varepsilon'' = n'' = 0$. Consequently on the basis of Eq. (8) $Q_{abs} = 0$. If $n \to \infty$ then $Q_{abs} \to 0$. One can approach this value either by $|\varepsilon''| \to \infty$ or by $|n''| \to \infty$. $Q_{abs}(\varepsilon'')$ or $Q_{abs}(n'')$ are non-zero functions. We can suppose that both of them are *continuous*. This statement is physically plausible. Thereafter our theorem results from Weierstrass' theorem.
Two dual theorems are

Theorem 4a

For a fixed permittivity of negative conductivity and a given geometry there is at least one frequency for which $Q_{ext} = 0$;

Theorem 4b

For a fixed permittivity of negative conductivity and a given frequency there is at least one among the bodies of similar shape and the same orientation where $Q_{ext} = 0$.

Further two dual theorems are

Theorem 5a

For a fixed permittivity of negative conductivity and a given geometry there is a frequency where $-Q_{ext}$ is maximum.

Theorem 5b

For a fixed permittivity of negative conductivity and a given frequency there is one among the bodies of similar shape and the same orientation where $-Q_{ext}$ is maximum.

Proof

Suppose that the largest linear size l of the obstacle simultaneously satisfies the conditions $k_0 l \ll 1$ and $|\bar{n}|\, k_0 l \ll 1$. This is the case of the Rayleigh scattering [4]. The change of the amplitude and the phase of the incident wave around and inside the body may be neglected. All the induced polarization and conduction currents are in phase. The whole body can be considered as a radiating short dipole. The scattering matrix is a simple diagonal one:

$$\mathbf{S} = k_0^2 \bar{\chi} \langle \cos\vartheta \;\; 1 \rangle \tag{19}$$

where

$$\bar{\chi} = (\bar{n}^2 - 1) V. \tag{20}$$

Here $\bar{n}$ is the refractive index of the obstacle and V is its volume.

Supposing that $E_{li} = E_{ri}$ and using Eqs (6) to (8), (10) to (11)

$$\sigma_{sca} = \frac{k_0^4}{6\pi} \bar{\chi}^2 \tag{21}$$

and

$$\sigma_{abs} = k_0 \, Re(j\bar{\chi}) \,. \tag{22}$$

Presuming that $\bar{\chi}$ is independent of λ, the scattering cross-section is proportional to $\lambda^{-4} V^2$ while the absorption cross-section to $\lambda^{-1} V$. When this volume is small ($V \rightarrow 0$), the absorption, if there is such, is the stronger effect.

Consequently in the case $k_0 l \ll 1$

$$Q_{ext} \cong Q_{abs} \tag{23}$$

i.e., the extinction efficiency factor of an obstacle of negative conductivity and a sufficiently small size is negative.

On the other hand, if even the smallest linear size l of the obstacle satisfies the relation $k_0 l \gg 1$ then $Q_{ext} \cong 2$, independently of the refractive index. This is the *extinction paradox* recognized by STRATTON [5].

It may be assumed that Q_{ext} is a continuous function of the relative characteristic size $k_0 l$. Stating this assumption, Theorems 4a and 4b are the direct consequences of Bolzano's theorem.

$Q_{ext} = 0$ if $k_0 l = 0$. Between this place and the smallest of the one whose existence has been proved above the function $Q_{ext}(k_0 l)$ is continuous and negative. Consequently, Weierstrass' theorem proves Theorems 5a and 5b.

Discussion of Theorems 1 through 5

The statements of Theorem 1 are obvious and express only the fact that the scattering occurs independently of the sign of conductivity.

Theorem 2 verifies the usual terminology: the concept of negative conductivity and that of negative absorption are equivalent.

Theorem 3 gives an account of an interesting "matching" mechanism. It is particularly notable that an active reflector may be constructed whose "negative" absorbed (i.e. emitted) power is maximum though its negative conductivity is finite.

To explain Theorems 4 and 5 we have to clarify the physical content of the extinction efficiency factor. Hulst ([4] p. 30) proves that this quantity is proportional to the difference between the power density of the incident wave and that of the forward propagating wave behind the scattering obstacle at a sufficiently large distance. Consequently $Q_{ext} = 0$ means that the power density behind the obstacle in the "shadow region" is the same as that of the incident wave. When Q_{ext} is negative, the power density in this direction increases.

In accordance with the proved theorems each scattering obstacle of a linear and active nature has a relative size $k_0 l$ $\left(\text{a ratio } \frac{l}{\lambda}\right)$ when the propagating power density before and behind of the obstacle is the same. What is more, there is a range of sizes where the power density behind the shadowing obstacle is greater or even maximum.

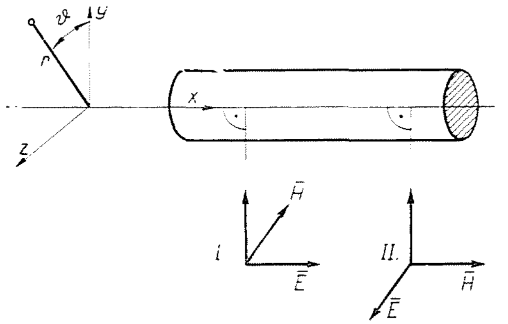

Fig. 2. Perpendicular incidence on straight circular cylinder

It is noteworthy that an active medium of very large dimensions decreases the transmitting power density according to the extinction paradox, similarly to the passive cases. This result is more surprising than the theorems of groups 4 and 5 and suggests an interference phenomenon similar to the one discussed in [1].

Scattering on an infinite straight circular cylinder

Partly to illustrate our statements, partly to investigate the reversal of the conductivity we performed detailed calculations for circular cylinders. This scattering problem was solved for a perpendicular incidence by Lord Rayleigh in 1881. For oblique incidence Wait [6] and Wilhelmsson [7] have solved the scattering problem of a dielectric cylinder. For the sake of simplicity we are dealing with the perpendicular incidence only but with complex refractive index.

The possible polarizations and the markings are shown in Fig. 2.

The solution of the vectorial Helmholtz equation can always be reduced to the determination of two properly chosen scalar functions [8], [9]. In the following we shall mark these two functions with $\bar{u}$ and $\bar{v}$ after Hulst [4]. It can be proved that for perpendicular incidence

$$\bar{E}_\vartheta = \frac{\partial \bar{v}}{\partial r} \tag{24}$$

$$\bar{E}_z = \bar{n}\, k_0\, \bar{u} \tag{25}$$

$$\bar{H}_\vartheta = \bar{n}\, \frac{\partial \bar{u}}{\partial r} \tag{26}$$

$$\bar{H}_z = \bar{n}^2 k_0^2\, \bar{v}\,. \tag{27}$$

An arbitrarily polarized incident wave may be decomposed to two independent waves on the basis of Eqs (24) through (27). With the chosen $v = 0$ the vector $\mathbf{E}_i$ of the incident wave is parallel with the axis of the cylinder (Case I), while the chosen $u = 0$ results in an $\mathbf{H}_i$ that is parallel with the axis (Case II). Using the series expansion of the incident wave by Bessel functions, a similar expansion of the scattered wave by Hankel functions of second kind and the symbol

$$\bar{F}_n = (-1)^n\, e^{jn\vartheta - j\omega t}$$

we get the following:

Case I $\bar{v} = 0$

$$\left.\begin{aligned} \bar{u} &= \sum_{n=-\infty}^{+\infty} \bar{F}n\left[J_n(k_0 r) - \bar{b}_n H_n^{(2)}(k_0 r)\right] \qquad r > a \\ \bar{u} &= \sum_{n=-\infty}^{+\infty} \bar{F}_n\, \bar{d}_n\, J_n(\bar{n}\, k_0\, r) \qquad r < a \end{aligned}\right\} \tag{28}$$

and the boundary conditions for the tangential components of the fields on the surface of the cylinder

$$\bar{n}\bar{u} \text{ and } \bar{n}\, \frac{\partial \bar{u}}{\partial r} \text{ are continuous at } r = a \tag{29}$$

Case II $\bar{u} = 0$

$$\left.\begin{aligned} \bar{v} &= \sum_{n=-\infty}^{+\infty} \bar{F}_n\left[J_n(k_0 r) \quad \bar{a}_n H_n^{(2)}(k_0 r)\right] \qquad r > a \\ \bar{v} &= \sum_{n=-\infty}^{+\infty} \bar{F}_n\, \bar{c}_n\, J_n(\bar{n}\, k_0\, r) \qquad r < a \end{aligned}\right\} \tag{30}$$

and from the boundary conditions

$$\bar{n}^2\bar{v} \text{ and are } \frac{\partial \bar{v}}{\partial r} \text{ continuous at } r = a\,. \tag{31}$$

Satisfying Eqs (29) and (31) and introducing a new marking $k_0 a = x$

$$\bar{b}_n = \frac{\bar{n} J_n'(\bar{n}x) J_n(x) - J_n(\bar{n}x) J_n'(x)}{\bar{n} J_n'(\bar{n}x) H_n(x) - J_n(\bar{n}x) H_n'(n)} \tag{32}$$

and

$$\bar{a}_n = \frac{J_n'(\bar{n}x) J_n(x) - \bar{n} J_n(\bar{n}x) J_n'(x)}{J_n'(\bar{n}x) H_n(x) - \bar{n} J_n(\bar{n}x) H_n'(x)} \tag{33}$$

where $H_n(x) = H_n^{(2)}(x) = J_n(x) - jN_n(x)$.

It is easy to accept that $b_n = b_{-n}$ and $a_n = a_{-n}$.

Using the approach of the Hankel function valid for great arguments

$$\bar{u} = \sqrt{\frac{2}{\pi k_0 r}}\, e^{-jk_0 r + j\omega t - j\frac{3\pi}{4}}\, \bar{T}_1(\vartheta) \tag{34}$$

where

$$\bar{T}_1(\vartheta) = \sum_{n=-\infty}^{+\infty} \bar{b}_n e^{jn\vartheta} = \sum_{n=0}^{\infty} \varepsilon_n \bar{b}_n \cos n\vartheta \tag{35}$$

$$\begin{aligned} \varepsilon_n &= 1 \quad \text{if} \quad n = 0 \\ &= 2 \quad \text{if} \quad n = 1, 2, \ldots \end{aligned}$$

The role of the function $T_1(\vartheta)$ is analogous to that of the element $\bar{S}_1$ of the scattering matrix.

With similar considerations for Case II we get the function

$$\bar{T}_2(\vartheta) = \sum_{n=-\infty}^{+\infty} \bar{a}_n e^{jn\vartheta} = \sum_{n=0}^{\infty} \varepsilon_n \bar{a}_n \cos n\vartheta \tag{36}$$

which is the counterpart of the matrix element $\bar{S}_2$.

In the case of perpendicular incidence the $\bar{S}_3$ and $\bar{S}_4$ have no counterparts, the character of the polarization of the scattered wave does not change.

The efficiency factors for both polarizations

$$Q_{\text{ext}} = \frac{2}{x} Re\, \bar{T}(0) \tag{37}$$

$$Q_{\text{sca}} = \frac{1}{\pi x} \int_0^{2\pi} |\bar{T}(\vartheta)|^2 \mathrm{d}\vartheta \tag{38}$$

$$Q_{\text{abs}} = Q_{\text{ext}} - Q_{\text{sca}}. \tag{39}$$

Bistatic cross-section in the plane of the incidence

$$Q(\vartheta) = \frac{2}{x} |\bar{T}(\vartheta)|^2 \tag{40}$$

and the monostatic (radar) cross-section

$$Q_r = \frac{2}{x} |\bar{T}(\pi)|^2. \tag{41}$$

The quantities defined in Eqs (37) through (41) were obtained by means of the Razdan-3 computer of the University Computer Center, Budapest. The basic data: $\bar{n} = \sqrt{2}(1 \pm j)$* and $x = 0.2(0.2)\ 10$. We introduce several results in diagrams.

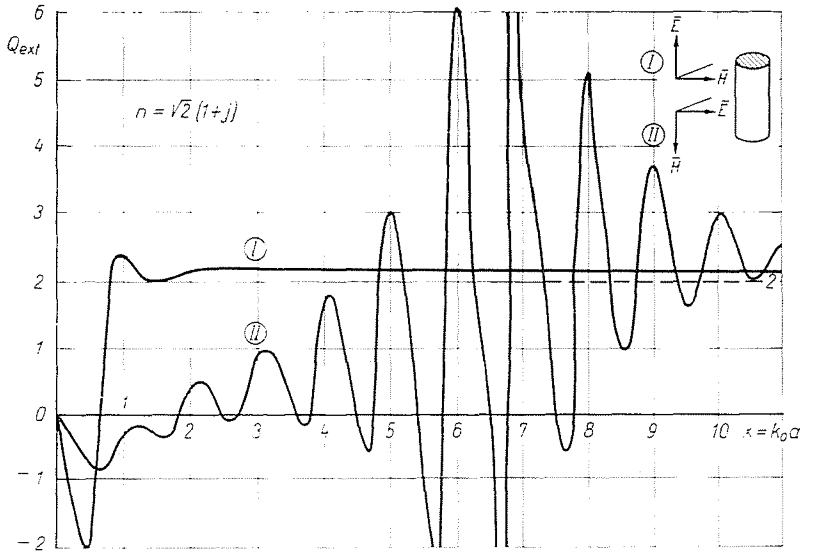

Fig. 3. Extinction efficiency factors of the cylinder of negative conductivity

In Fig. 3 the value Q_{ext} is shown as a function of x, with negative conductivity. For small x the sign of the function is negative as it was expected. In both cases of negative conductivity one can find the values connected with the maximum or with the zero value of $-Q_{ext}$. From the shape of the curves it follows that these values of x are in existence for arbitrary polarization as well.

For increasing values of x, $Q_{ext} \to 2$, in accordance with the extinction paradox.

To provide a basis for comparison with the results obtained above we give the values Q_{ext} for the appropriate positive conductivity in Fig. 4. This curve was calculated by Hulst [4] up to $x = 4$. It is interesting to note that

* This particular value was chosen to compare the results obtained here with the results of Hulst [4] by $n = \sqrt{2}(1 - j)$.

while for positive conductivity a single point satisfies the equation $Q_{\text{ext}1} = Q_{\text{ext}2}$, in the case of negative conductivity there are two such points.

It is easy to prove for small values of x that

$$Q_{\text{ext}}\Big|_{n''>0} \doteq -Q_{\text{ext}}\Big|_{n''<0} . \tag{42}$$

If we examine Eqs (32) and (33) it turns out that in Case I, b_0 is dominant among the coefficients while in Case II, a_1 and a_{-1} have the maximum

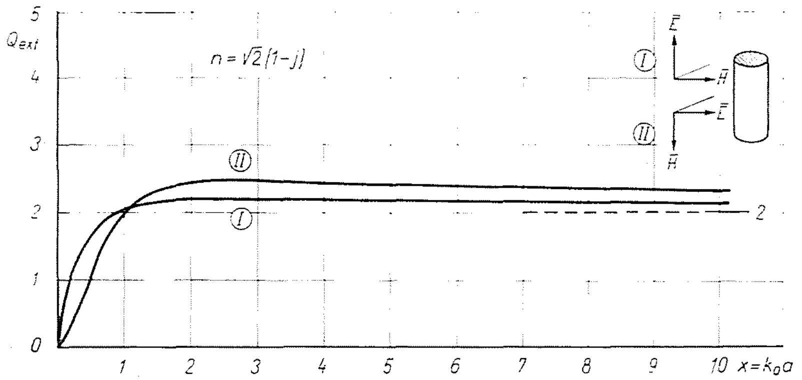

Fig. 4. Extinction efficiency factors of the cylinder of positive conductivity

value for small x. This fact is in good accordance with the physical picture: in Case I the first approximation of the currents corresponds to a monopole (line source), in Case II to a dipole. Using Hulst's results [4]

$$\bar{b}_0 \doteq \mathrm{j}\,\frac{\pi x^2}{4}\,(\bar{n}^2 - 1) \tag{43}$$

$$\bar{a}_1 \cong \mathrm{j}\,\frac{\pi x^2}{4}\,\frac{\bar{n}^2 - 1}{\bar{n}^2 + 1} \tag{44}$$

Compare these with the result

$$\bar{a}_0 = \bar{b}_1 \cong \mathrm{j}\,\frac{\pi x^4}{32}\,(\bar{n}^2 - 1) \tag{45}$$

we can accept the validity of the approximation.

In Case I

$$Q_{\text{ext}\,1} = \frac{2}{x}\,Re\,\bar{b}_0 = \pi x n' n'' = \frac{\pi x}{2}\,\varepsilon'' \tag{46}$$

while in Case II

$$Q_{\mathrm{ext}\,2} = \frac{2}{x} Re(a_1 + a_{-1}) = \pi x \frac{4n' n''}{(n'^2 - n''^2 + 1)^2 + (2n' n'')^2} = \\ = \pi x \frac{2\varepsilon''}{(\varepsilon' + 1)^2 + \varepsilon''^2} . \tag{47}$$

Obviously, for small x the sign of Q_{ext} depends only on the sign of n'' (or ε''), and is equal to it.

We have accepted that $Q_{\mathrm{ext}} \cong Q_{\mathrm{abs}}$, consequently the statement about the sign is valid for the absorption efficiency factor too, as it was expected.

In the case of Eq. (46) it is quite obvious that the defined quantity is equal to the absorption efficiency factor. For small x the amplitude of **E** can be considered as equal everywhere inside the cylinder. Consequently the power absorbed by a section of length l and radius a is

$$P_{\mathrm{abs}} = \sigma |\bar{E}|^2 \pi a^2 e . \tag{48}$$

The power density of the incident plane wave is:

$$S_i = \frac{|\bar{E}_i|^2}{Z_0} = |\bar{E}_i|^2 \frac{k_0}{\omega \varepsilon_0} . \tag{49}$$

Knowing that $G = 2al$ and using Eqs (11) and (14) we obtain directly Eq. (46) because $\varepsilon'' = \dfrac{\sigma}{\omega\, \varepsilon_0}$ and $x = k_0 a$.

The scattering and absorption efficiency factors are shown in Figs 5 and 6. The bistatic efficiency factors calculated on the basis of Eq. (40) are illustrated in Figs 7—8 for negative conductivity and in Figs 9—10 for positive conductivity as a contrast. The parameter of the curves is $x = k_0 a$. Let us consider the fact already noted: for polarization I the field has monopole character for small radii while for polarization II the field has dipole characteristics. The maxima and minima of higher order will appear only with a larger relative radius. Their successive appearance and their gradual shift to the direction of smaller angles in the case of increasing values of x are common properties of both signs of the conductivity. Incidentally this behaviour corresponds to that of the ideal dielectrics [10] (Fig. 11).

In general, the character of the bistatic curves is similar and the differences are more quantitative than qualitative. To illustrate this fact another comparative diagram is shown for $x = 3.8$ (Fig. 12).

The radar efficiency factor vs. x is given in Fig. 13. Investigating this efficiency factor and that of angle $\vartheta = \dfrac{\pi}{2}$ (Fig. 14) the most striking fact is that the values in Case II are strongly oscillating in comparison with the values of Case I. Our explanation for this fact is the following. In the change of the bistatic efficiency factor — and particularly the radar efficiency factor — the influence of the surface waves is very strong. These waves are affected

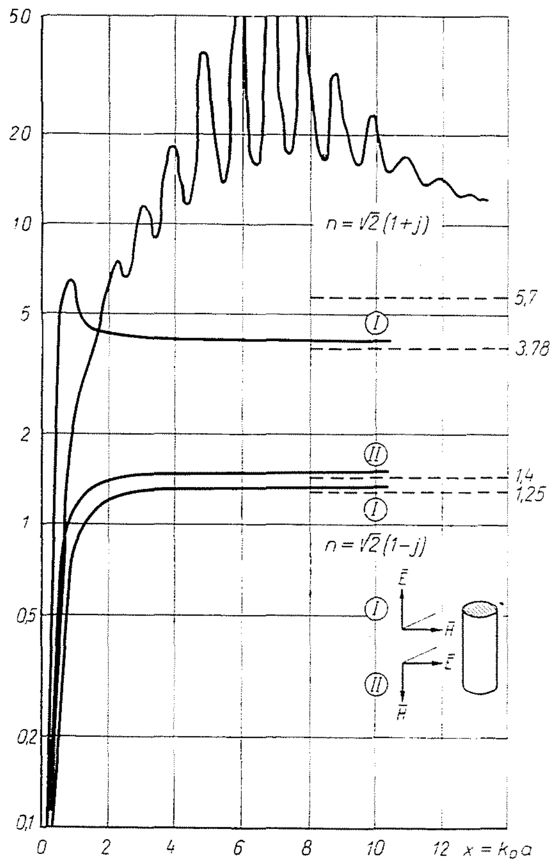

Fig. 5. Scattering efficiency factors

by the current flowing within the scattering obstacle. In the case of ideal dielectrics no damping effect takes place at all, here the interference character is very strong [10]. The reason of this phenomenon is obviously the fact that the polarization current has no component in phase with the field in an ideal dielectric. But in a lossy dielectric the current has a component in phase with the field and this component dampens the surface waves. The effect of this damping is far stronger in Case I, where the current flows parallel with the axis, than in Case II. Thus in the latter case the interference character is stronger. The phenomenon is well-known for ideal conductors [11].

To illustrate the existence of the extremum of the absorption efficiency factor we looked analytically for the extremum of Q_{abs_2} ($\cong Q_{ext_2}$) defined by Eq. (47). We found that for fixed n' the extremum could be obtained with

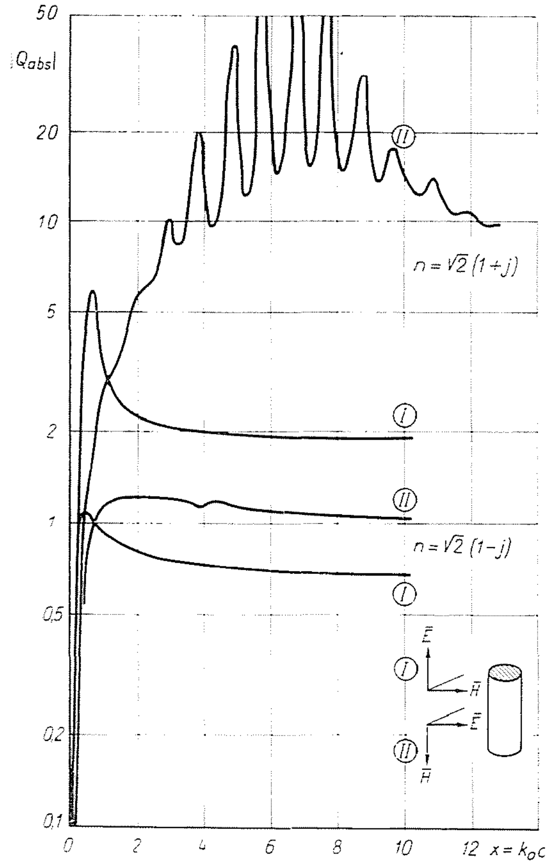

Fig. 6. Absorption efficiency factors

$$n'' = \pm \sqrt{\frac{1 - n'^2 + 2\sqrt{n'^4 - n'^2 + 1}}{3}} \tag{50}$$

while with fixed ε' the necessary value of ε'' was

$$\varepsilon'' = \pm \sqrt{\varepsilon' - 1} \tag{51}$$

On the basis of Eq. (50) $n'' = 1.05$ belongs to $n' = 1$. We give the curves of $|Q_{abs}|$ for $\bar{n} = 1 + j0.1$; $1 + j1$; and $1 + j10$ in Fig. 15. It is easy to see that around $n'' = 1$ we get a maximum of $|Q_{abs}|$ for greater values of x too.

A similar result was obtained by Bach Andersen and Majborn [12] in their investigation of the field of a circular cylinder of negative conductivity posted in a rectangular waveguide.

The high frequency approximation of the radar and scattering efficiency factors

It appears from Figs 5 and 13 that the radar or scattering efficiency factors tend to a given value if $k_0 a$ has a high value. i.e., the radius of the cylinder

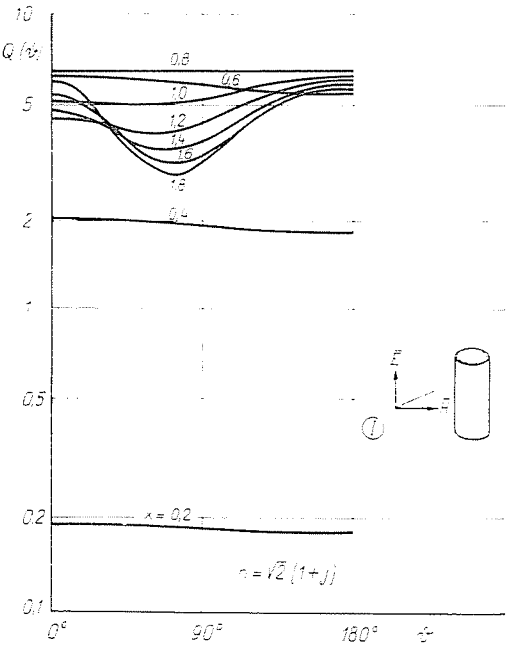

Fig. 7. Bistatic efficiency factors of the cylinder of negative conductivity (Case I)

is large in comparison with the wave-length. We shall determine this limit with the ray-optical approximation as follows (Fig. 16).

Let a plane wave arrive perpendicularly to the axis of an infinitely long straight circular cylinder. Let us divide this plane wave into small beam pencils, which attain the surface of the cylinder between the angles ϑ_0 and $\vartheta_0 + d\vartheta_0$. The cross-section of a pencil is $la \cos \vartheta_0 |d\vartheta_0|$ (for the length l). The total power carried by the pencil is $S_i la \cos \vartheta_0 |d\vartheta_0|$ where S_i is the power density of the plane wave (cf. Eq. 10).

Let the reflection coefficient on the surface be $\bar{r}$. The coefficient of the power reflection is $|\bar{r}|^2$. The reflected wave travels in the direction $\vartheta = \pi - 2\vartheta_0$ in a small sector marked by $|d\vartheta| = 2 |d\vartheta_0|$. Let the intensity of the power at a sufficiently great distance be $S(\vartheta)$. Because of the equality of powers

$$|\bar{r}|^2 S_i la \cos \vartheta_0 |d\vartheta_0| = S(\vartheta)\, lr |d\vartheta| \tag{52}$$

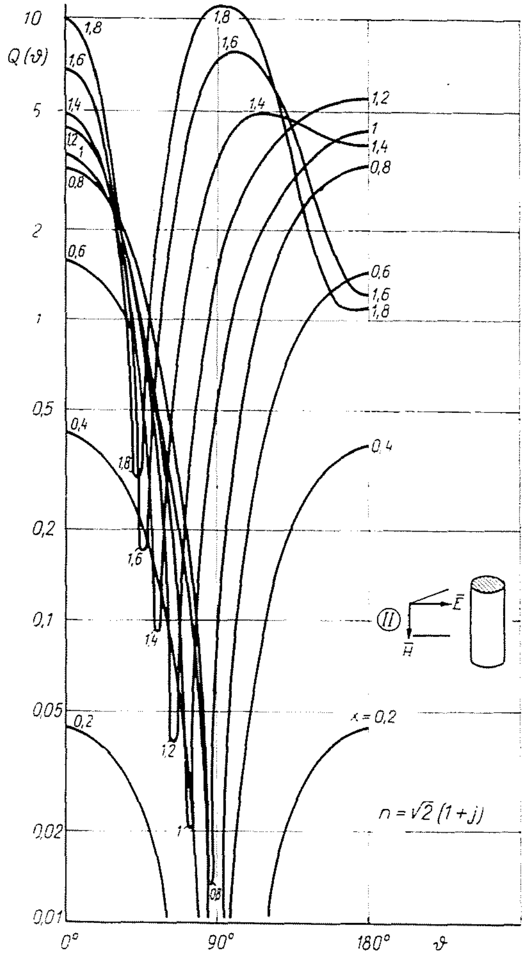

Fig. 8. Bistatic efficiency factors of the cylinder of negative conductivity (Case II)

the ratio of the power densities (using Eq. (34)) is

$$\frac{S(\vartheta)}{S_i} = |\bar{u}|^2 = \frac{2}{\pi k_0 r} |\overline{T}(\vartheta)|^2 \tag{53}$$

hence

$$|\overline{T}(\vartheta)|^2 = \frac{\pi x}{4} |r|^2 \cos \vartheta_0 \tag{54}$$

with the usual mark $x = k_0 a$. The formula is valid for both polarizations.

The ray-optical deduction given above neglects the diffraction. Thus among the bistatic efficiency factors calculated in this manner only the radar efficiency factor gives a correct result. Choosing $\vartheta = \pi$, $\vartheta_0 = 0$

$$Q_r = \frac{\pi}{2} |\bar{r}|^2 \tag{55}$$

for both polarizations.

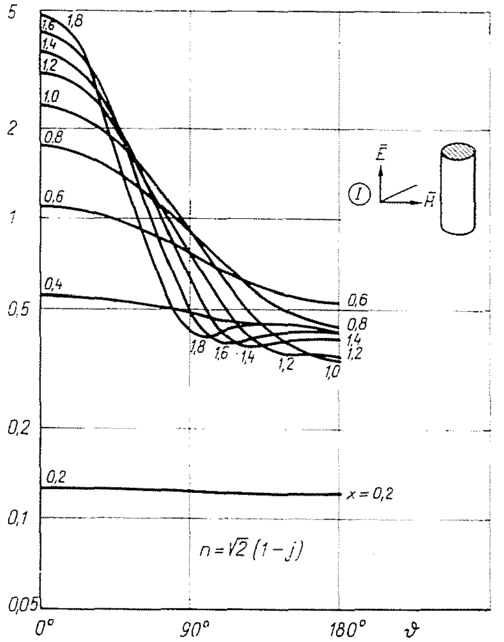

Fig. 9. Bistatic efficiency factors of the cylinder of positive conductivity (Case I)

The result is known for an ideal conductor ($|\bar{r}| = 1$) [11]. In the case of $\bar{n} = 1.41 \pm j1.41$ for both polarizations $|\bar{r}|^2 = 0.28$ and $Q_r = 0.44$. Fig. 13 indicates this value and the agreement with the value calculated from Eq. (41) is very good.

This result is obviously wrong for the case of negative conductivity ($\bar{n} = 1.41 + j1.41$) in spite of the fact that the absolute value of the Fresnel reflection coefficient is the same for both signs of conductivity.

The error we have committed here is that we have neglected the beams travelling inside the cylinder. This neglect does not lead to a mistake in the case of positive conductivity because the power propagating in the refracted beams is absorbed quickly. For negative conductivity this power density increases inside the scattering object. In [1] it has been demonstrated that on the flat surface of a half space filled with a medium of negative conductivity the reflection coefficient is

$$\bar{R}_{\infty}^{-} = \frac{1}{(\bar{r}^{+})^{*}} \tag{56}$$

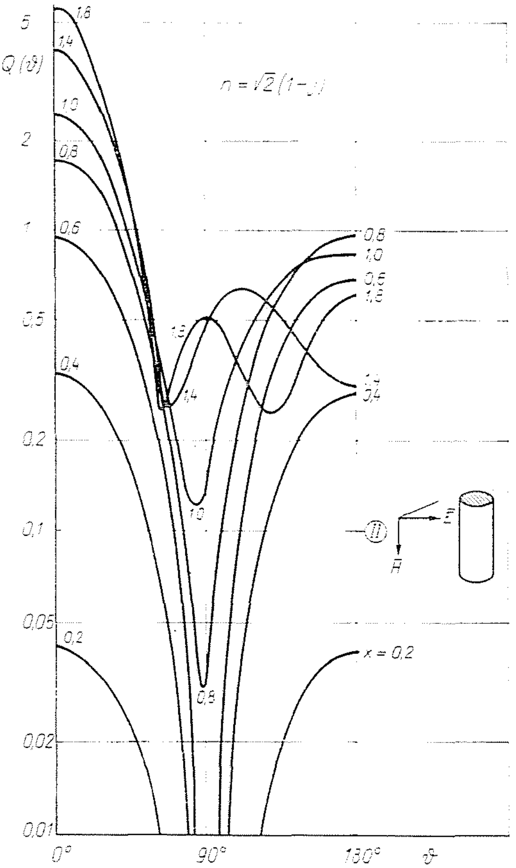

Fig. 10. Bistatic efficiency factors of the cylinder of positive conductivity (Case II)

where $\bar{r}^{+}$ is the Fresnel reflection coefficient of the medium with positive conductivity of the same absolute value. The asterisk denotes the complex conjugate. (A similar conclusion was obtained in [13].)

In the case of $\bar{n} = 1.41 + j1.41$, the value $|\bar{R}_{\infty}^{-}|^{2} = 3.57$ and thus $Q_r = 5.6$. Q_{r1} is in a good agreement with this value. The behaviour of Q_{r2} shows the same tendency but the calculated points do not give a sufficient basis for the evaluation of the curve in detail.

The calculation of the scattering efficiency factor cannot be performed by the direct substitution of Eq. (54) into Eq. (38) for the reason we have mentioned already: the ray-optical approximation does not consider the diffraction. Hulst [4] gives a formula for the sphere which takes the diffraction into consideration

$$Q_{sca} = 1 + w \tag{57}$$

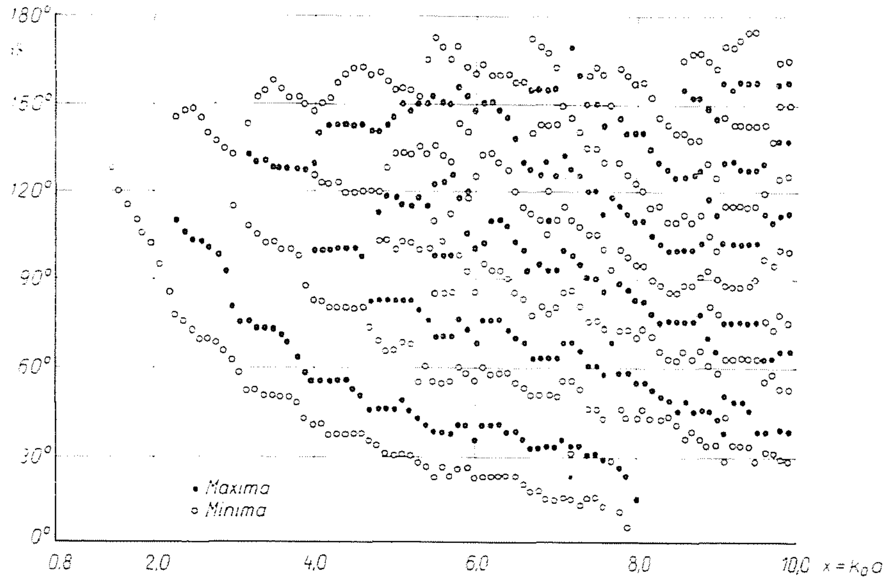

Fig. 11. Angular locations of the maxima (●) and minima (○) of the bistatic efficiency factor of the straight circular cylinder (Case I. $n = 1.46$) [10]

where the term 1 refers to the role of the diffraction and w is the part of the scattering efficiency factor obtained by the direct ray-optical calculation from the reflected and refracted waves, i.e.

$$w = \frac{1}{\pi x}\int_0^{2\pi} |\bar{T}(\vartheta)|^2 \,\mathrm{d}\vartheta = \frac{1}{2}\int_{-\pi/2}^{+\pi/2} |\bar{r}|^2 \cos\vartheta_0 \,\mathrm{d}\vartheta_0 = \int_0^1 |\bar{r}|^2 \,\mathrm{d}(\sin\vartheta_0) \tag{58}$$

and consequently

$$Q_{sca} = 1 + \int_0^1 |\bar{r}|^2 \,\mathrm{d}(\sin\vartheta_0)\,. \tag{59}$$

The expression is valid for both polarizations if we substitute the proper r.

For the refractive index $\bar{n} = 1.41 - j1.41$ $|\bar{r}_{1,2}|^2$ vs. $\sin \vartheta_0$ is given in Fig. 17. Using these curves, the limit value of the scattering efficiency factors may be obtained by numerical integration. For positive conductivity

$$Q_{1sca}^{-} = 1.25; \quad Q_{2sca}^{-} = 1.4.$$

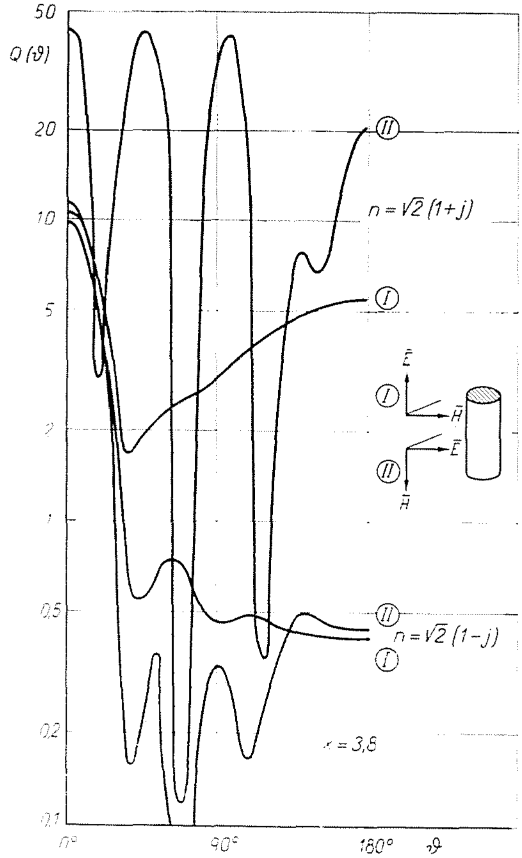

Fig. 12. Bistatic efficiency factors

Both values are good approximations as seen from Fig. 5.

For the corresponding negative conductivity we have used the $\bar{R}_{\infty}^{-}$ values to obtain the limits. They are

$$Q_{1sca}^{-} = 3.78; \quad Q_{2sca}^{-} = 5.7.$$

These limit values marked in Fig. 5 are good approximations. This fact proves on the one hand the applicability of Eq. (59), on the other the practical importance of the surface reflection coefficient defined for an obstacle of great dimensions and negative conductivity. The result above includes the statement in [1] that the power transmitted across a very thick layer of negative

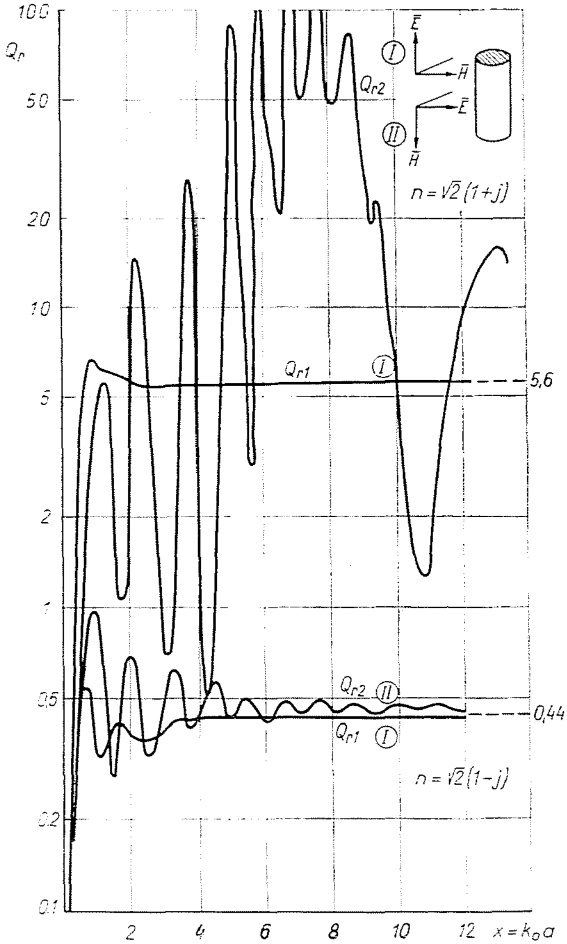

Fig. 13. Radar efficiency factors

conductivity is zero. In the deduction we consider only the reflected wave neglecting the rays after a multiple inner reflection.

Our previous two examples support the correct choice of the absolute value of $\bar{R}_\infty^-$ in [1]. Nevertheless, the arcus of this quantity has not been dealt with. On the basis of Eq. (58), it is equal to the arcus of the Fresnel coefficient on the surface of a substance differing from the previous one by the sign of the conductivity only. To prove this, we calculated the value of $\frac{2}{x}\bar{T}(0) = Q + jP$ for both polarizations. This quantity was represented by Hulst in a diagram ([4] Fig. 31) for $\bar{n} = 1.41 - j1.41$. Hulst has demonstrated with the help of

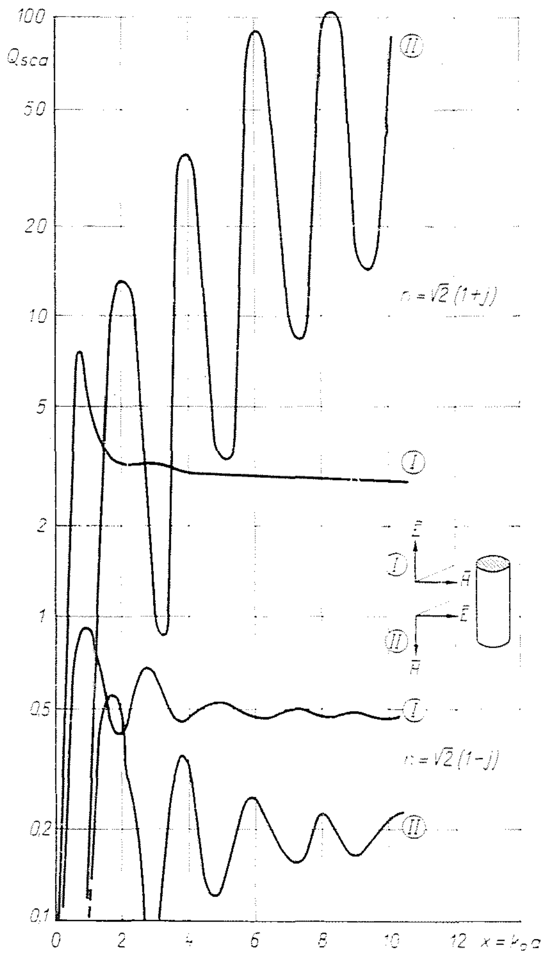

Fig. 14. Bistatic efficiency factors $\left(\vartheta = \frac{\pi}{2}\right)$

heuristic arguments that the curve approaches the point (2; 0) on the complex plain along an asymptote that includes an angle of 60° with the real axis. The consideration of the edge effects shows that the angle with the asymptote is proportional to the angle of the refraction index if the k_0a product is large

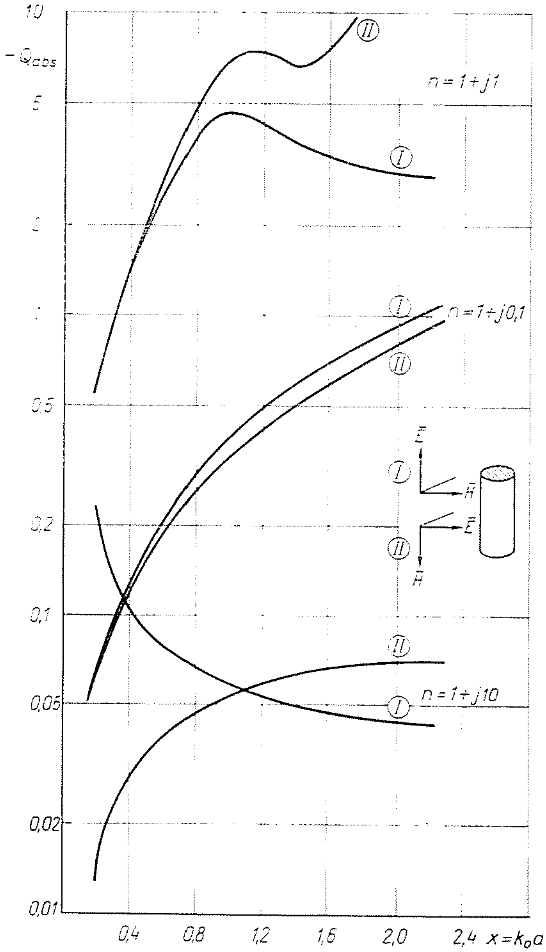

Fig. 15. Absorption efficiency factors vs. refractive index

enough. In Fig. 18, the complex diagrams of the $Q + jP$ vs. k_0a are indicated for both refractive indices $\bar{n} = 1.41 \pm j1.41$. It is apparent that the asymptotic behaviour is the same for Case I. This fact is a heuristic argument to support the adequacy of the definition in Eq. (56).

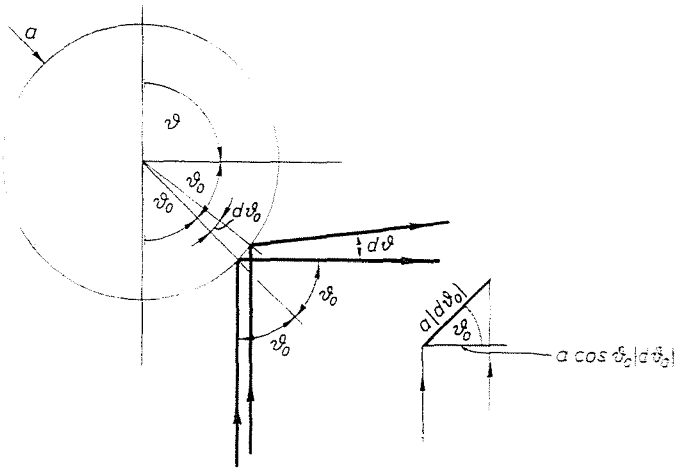

Fig. 16. Path of the ray for the ray-optical approximation

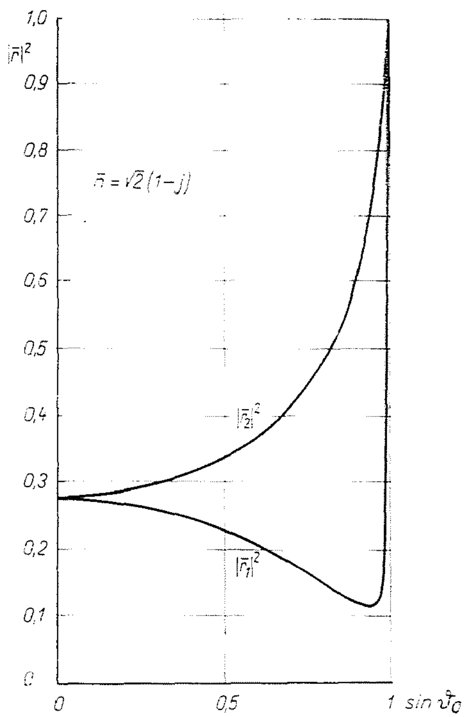

Fig. 17. Power reflection coefficient vs. the sine of the angle of incidence

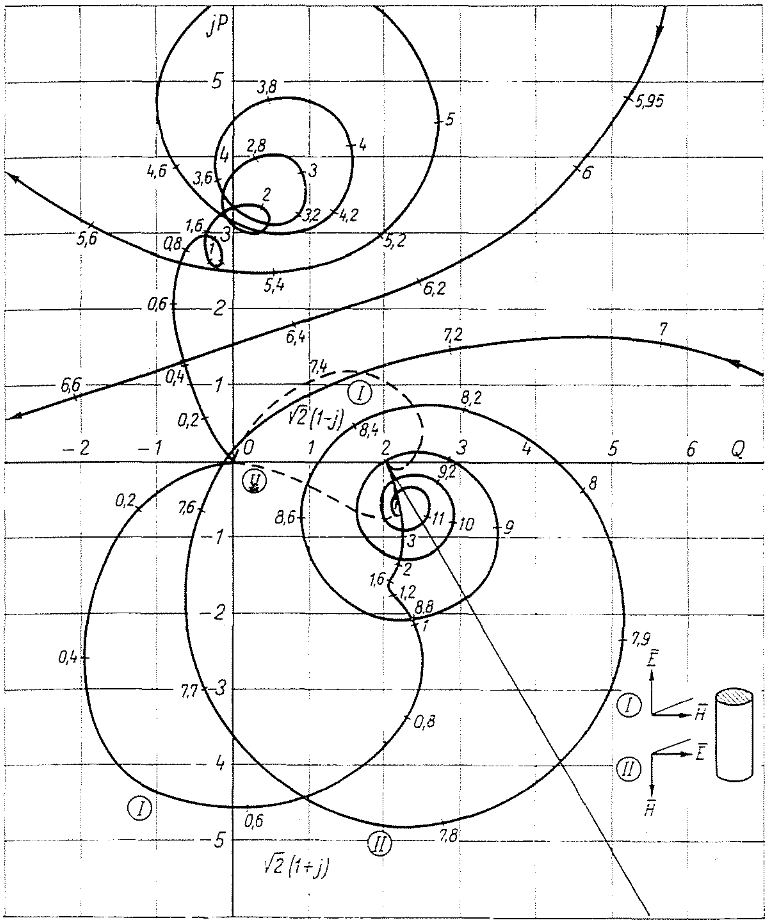

Fig. 18. Plot of $Q - jP$. The parameter is $x = k_0 a$

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Summary

A previous paper by the author has investigated the changes in the manner of the propagation and the reflection of plane waves in the case of the reversal of the conductivity. The results presented in that paper are made use of in the present paper, demonstrating the change of the characteristics of the electromagnetic scattering if the sign of the conductivity reverses. The general results are supported by the numerical results of the scattering of plane waves on a straight circular cylinder. These calculations show that the differences between the two cases are quantitative rather than qualitative.

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