ON THE ELECTROMAGNETIC FIELD OF INHOMOGENEOUSLY FILLED WAVEGUIDES

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It will be supposed that the permittivity varies in the transverse directions only and in the direction of waveguide axis it is constant. Such waveguides are important as isolators, phase shifters, attenuators and as the parts of material testing equipments.

Two important groups of the applied computational methods are

1. variational methods,
2. generalized telegraphist's equations.

The purpose of this paper is double. First new, simpler generalized telegraphist's equations are given. Second, the connection between the two methods is pointed out.

For the sake of simplicity it is supposed that the permeability is constant and the permittivity is isotropic.

1. Introduction

The essential point of the variational method is to find a functional ("variational" formula) which gives the value of the propagation factor, and which contains as function variable the electric and/or magnetic field strength.

$$\gamma = F(\mathbf{E}; \mathbf{H}) \tag{1}$$

Substituting the exact electric and magnetic field strength we obtain the exact value of the propagation factor

$$\gamma_0 = F(\mathbf{E}_0; \mathbf{H}_0) \tag{2}$$

The formula is variational or stationary, if its variation about the exact value with respect to the field strengths is zero:

$$\delta\gamma = [\delta\, F(\mathbf{E}; \mathbf{H})]_{\mathbf{E}_0;\, \mathbf{H}_0} = 0 \tag{3}$$

This property allows that approximating the field by a suitable trial function gives a good approximation of the propagation factor. (The error in propagation factor is of a higher order smaller than in the field strength.)

The field strength is often approximated as sum of functions with unknown coefficients

$$\mathbf{E} = \Sigma\, U_i\, \mathbf{e}_i \tag{4}$$

$$\mathbf{H} = \Sigma\, I_i\, \mathbf{h}_i\,. \tag{5}$$

Then coefficients U_i and I_i, giving the best approximation can be determined from the system of equations

$$\frac{\partial\gamma}{\partial U_i} = 0 \tag{6}$$

$$\frac{\partial\gamma}{\partial I_i} = 0\,. \tag{7}$$

This is the Rayleigh—Ritz method.

The variational formulae of the propagation factor have been summarized by BERK [1]. NIKOL'SKI has expanded the results for the case of anisotropic dielectrics [2] and using the empty-waveguide modes as expanding functions he has given the matrices from which — as eigenvalues — the propagation factors can be determined [3].

Let us turn to the generalized telegraphist's equations. It is well known that the field coefficients of the homogeneous waveguide satisfy telegraphist's equations. These equations can be generalized [4], [5] to take the current density and the surface excitation into consideration.

The telegraphist's equations with respect to the inhomogeneously filled waveguide — these will be seen to be special cases of those in [4] — were first derived by SCHELKUNOFF. In his paper he has not used the results of [4], but set out directly from Maxwell's equations. He has taken into account the effect of the dielectrics by the polarization current density.

Because of place shortage, instead of presenting Schelkunoff's deduction to full length, only a part from it will be quoted to show an incorrect result, which appeared in the literature. Eq. (39) in [6] referring to isotropic and non-magnetic dielectrics is

$$\frac{\mathrm{d}\,V_{(m)}}{\mathrm{d}\,z} = -j\omega\,\mu_0\,I_{(m)} + \chi_{(m)}\,V_{z,(m)} \tag{8}$$

where $V_{(m)}$ and $I_{(m)}$ are the modal voltages and modal currents, $\chi_{(m)}$ is the eigenvalue and $V_{z,(m)}$ the modal voltage belonging to the longitudinal component of the electric field.

Eq. (44) in [6] for the isotropic case:

$$V_{z(n)} \iint j\omega\varepsilon\,\chi_{(n)}\,T_{(n)}\,T_{(m)}\,\mathrm{d}S = -I_{(m)}\,. \tag{9}$$

To eliminate $V_{z,(m)}$ from (8), we must form a matrix inversion in (9). Let us mark the elements of the inverse matrix by ${}^{z}Z_{(n)(m)}$, then from (9) we obtain

$$V_{z,(n)} = - I_{(m)}\, {}^{z}Z_{(n)(m)} \tag{10}$$

Before introducing (10) to (8) an index change is needed. Schelkunoff calls the attention to it: (p. 795 in [6]) "Before substituting in Equation (8), the summation index m in (10) should be changed to avoid conflict with m in the former equations." After substitution and index change, (8) takes the form:

$$\frac{\mathrm{d}V_{(m)}}{\mathrm{d}z} = - j\omega\mu_0 I_{(m)} - \chi_{(m)}\, {}^{z}Z_{(n)(m)}\, I_{(n)} \tag{11}$$

The authors of [7] do not change the indices in (10), but they substitute m in the place of n. So they obtain Eq. (5) in [7]. So they replace the matrix in (9) by a diagonal matrix. This substitution — as can be seen from (9) — is correct only in the case of constant ε. Surely, then

$$\iint \chi_{(n)} T_{(n)} T_{(m)}\, \mathrm{d}S = \frac{1}{\chi_{(n)}}\, \delta_{nm} \tag{12}$$

where δ_{nm} is the Kronecker symbol. In the case of constant ε the incorrect Eq. (5) in (7) transforms to the correct transmission line equation

$$-\frac{\mathrm{d}V_{(m)}}{\mathrm{d}z} = \left(j\omega\mu_0 + \frac{\chi^2_{(m)}}{j\omega\varepsilon} \right) I_{(m)} \tag{13}$$

However, when ε is a function of transverse co-ordinates, then (5) in (7) is incorrect.

As (11) shows, the result of Schelkunoff is rather difficult to treat. It wants a forming of matrix inversion. This is a consequence of the fact that he has expanded the *displacement vector* **D** and the *magnetic flux vector* **B**. Our results will be simpler, because we expand the electric and magnetic *field strength* vector.

2. Recapitulation of the results for the homogeneously filled waveguide

The field of the waveguide can be obtained as follows. We solve the equation

$$\Delta_t \varphi_{(i)} + k^2_{(i)} \varphi_{(i)} = 0 \tag{14}$$

with the boundary condition $\varphi_{(i)} = 0$, and we solve the equation

$$\Delta_t \varphi_{[i]} + k^2_{[i]} \varphi_{[i]} = 0 \tag{15}$$

with the boundary condition $\frac{\partial \varphi_{[i]}}{\partial n} = 0$. Here Δ_t is the transverse Laplacian, the index in brackets means TE mode, while the index in parentheses means TM one.

We form the vectorial mode functions:

$$\mathbf{e}_{(i)} = \nabla_t \varphi_{(i)} \tag{16}$$

$$\mathbf{h}_{(i)} = \mathbf{k} \times \mathbf{e}_{(i)} \tag{17}$$

$$\mathbf{e}_{[i]} = \quad \mathbf{k} \times \nabla_t \varphi_{[i]} \tag{18}$$

$$\mathbf{h}_{[i]} = \nabla_t \varphi_{[i]} \tag{19}$$

where ∇_t is the transverse part of the gradient, and $\mathbf{k}$ is the unit vector in the z direction.

The vectorial mode functions are orthonormal:

$$\int_A \mathbf{e}_{(i)} \, \mathbf{e}_{(m)} \, \mathrm{d}A = \delta_{im} \tag{20}$$

$$\int_A \mathbf{e}_{[i]} \, \mathbf{e}_{[m]} \, \mathrm{d}A = \delta_{im} \tag{21}$$

$$\int_A \mathbf{e}_{(i)} \, \mathbf{e}_{[m]} \, \mathrm{d}A = 0 \tag{22}$$

$$\int_A \mathbf{h}_{(i)} \, \mathbf{h}_{(m)} \, \mathrm{d}A = \delta_{im} \tag{23}$$

$$\int_A \mathbf{h}_{[i]} \, \mathbf{h}_{[m]} \, \mathrm{d}A = \delta_{im} \tag{24}$$

$$\int_A \mathbf{h}_{(i)} \, \mathbf{h}_{[m]} \, \mathrm{d}A = 0 \tag{25}$$

In view of the above it follows

$$\int_A \varphi_{(i)} \, \varphi_{(m)} \, \mathrm{d}A = \frac{1}{k^2_{(i)}} \delta_{im} \tag{26}$$

$$\int_A \varphi_{[i]} \, \varphi_{[m]} \, \mathrm{d}A = \frac{1}{k^2_{[i]}} \delta_{im} \tag{27}$$

With the aid of the vectorial mode functions, the transverse field of the waveguide can be expanded:

$$\mathbf{E}_t = \sum_i U_{(i)} \mathbf{e}_{(i)} + U_{[i]} \mathbf{e}_{[i]} \tag{28}$$

$$\mathbf{H}_t = \sum_i I_{(i)} \mathbf{h}_{(i)} + I_{[i]} \mathbf{h}_{[i]} \tag{29}$$

The system $\mathbf{e}_i$, $\mathbf{h}_i$ is complete in the following sense. The necessary and sufficient conditions to expand in mean square a given transverse electric field by the series (28) is that the given field is sectionally continuous and differentiable. The necessary and sufficient condition to expand in mean square a given transverse magnetic field is that the given field is sectionally continuous and differentiable and the divergence of the given field is zero at the boundary.

U_i and I_i are the modal voltage and modal current. They satisfy the transmission line equations

$$-\frac{\partial U_{(i)}}{\partial z} = \left(j\omega\,\mu_0 + \frac{k_{(i)}^2}{j\omega\,\varepsilon_0}\right) I_{(i)} \tag{30}$$

$$-\frac{\partial U_{[i]}}{\partial z} = j\omega\,\mu_0\, U_{[i]} \tag{31}$$

$$-\frac{\partial I_{(i)}}{\partial z} = j\omega\,\varepsilon_0\, U_{(i)} \tag{32}$$

$$-\frac{\partial I_{[i]}}{\partial z} = \left(j\omega\,\varepsilon_0 + \frac{k_{[i]}^2}{j\omega\,\mu_0}\right) U_{[i]}\,. \tag{33}$$

Finally let us express E_z with the aid of the modal current.

$$E_z = -\frac{1}{j\omega\,\varepsilon} \sum_i I_{(i)}\, k_{(i)}^2\, q_{(i)}\,. \tag{34}$$

3. The excitation effect of currents flowing inside the waveguide [4, 5, 8]

If transverse current $\mathbf{J}_t$ and longitudinal current J_z flow inside the waveguide, then Eqs (30) to (33) alter to:

$$-\frac{\partial U_{(i)}}{\partial z} = \left(j\omega\,\mu_0 + \frac{k_{(i)}^2}{j\omega\,\varepsilon_0}\right) I_{(i)} + \frac{1}{j\omega\,\varepsilon_0} \int_A \nabla_t J_z \cdot \mathbf{e}_{(i)}\, \mathrm{d}A \tag{35}$$

$$-\frac{\partial U_{[i]}}{\partial z} = j\omega\,\varepsilon_0\, I_{[i]} \tag{36}$$

$$-\frac{\partial I_{(i)}}{\partial z} = j\omega\,\varepsilon_0\, U_{(i)} + \int_A \mathbf{e}_{(i)}\, \mathbf{J}_t\, \mathrm{d}A \tag{37}$$

$$-\frac{\partial I_{[i]}}{\partial z} = \left(j\omega\,\varepsilon_0 + \frac{k_{[i]}^2}{j\omega\,\mu_0}\right) U_{[i]} + \int_A \mathbf{e}_{[i]}\, \mathbf{J}_t\, \mathrm{d}A\,. \tag{38}$$

From physical point of view it is evident that no longitudinal current density generates TE mode. Surely TE mode has no closed magnetic line of force in the transverse plane. This fact can be easily verified also by vectoranalytical method. So the second term of the right-hand in (35) needs not appear in (36).

4. Generalized telegraphist's equation of waveguides filled with inhomogeneous dielectrics

The effect of inhomogeneous dielectrics is taken into consideration with the aid of the polarization current density

$$\mathbf{J} = j\omega\, \varepsilon_0(\varepsilon_r - 1)\, \mathbf{E}. \tag{39}$$

Here we expand **E** according to the empty waveguide modes. The consequence is that (generally) every mode will be self-coupled and will be mutually coupled with other modes according to Eqs (35) to (38).

Now we examine the form of Eqs (35) to (38), if the polarization current density was due to TE or to TM mode. The second term on the right hand side of Eq. (35) will be zero in the case of TE mode, because then E_z and consequently J_z is zero.

From (39) follows that J_z vanishes at the boundary. Then the second term in the right-hand side of (35) simplifies. We prove

$$\int_A \nabla_t J_z \cdot \nabla_t \varphi_{(i)}\, \mathrm{d}A = k_{(i)}^2 \int_A J_z \varphi_{(i)}\, \mathrm{d}A \tag{40}$$

where we have substituted $\mathbf{e}_{(i)}$ from (16). The starting point of the proof is the identity

$$\nabla_t (J_z \cdot \nabla_t \varphi_{(i)}) = \nabla_t J_z \cdot \nabla_t \varphi_{(i)} + J_z \Delta_t \varphi_{(i)} \tag{41}$$

(41) is valid in distributional sense, supposing J_z has countable discontinuities and $\varphi_{(i)}$ has continuous derivatives. Integrating both sides of (41) with respect to the cross-section, the Gauss theorem generalized in distributional sense [11] can be applied:

$$\oint_C J_z \frac{\partial \varphi_{(i)}}{\partial n}\, \mathrm{d}l = \int_A \nabla_t J_z \cdot \nabla_t \varphi_{(i)}\, \mathrm{d}A + \int_A J_z \Delta_t \varphi_{(i)}\, \mathrm{d}A. \tag{42}$$

The left-hand side vanishes in view of the boundary condition.

Multiplying both side of (14) with J_z and integrating over the cross-section, we obtain

$$\int_A J_z \Delta_t \varphi_{(i)}\, \mathrm{d}A + k_{(i)}^2 \int_A J_z \varphi_{(i)}\, \mathrm{d}A = 0. \tag{43}$$

Comparing (42) and (43) yields (40).

On the basis of (34), expanding J_z according to TM modes:

$$J_z = j\omega\, \varepsilon_0(\varepsilon_r - 1)\, E_z = -\sum_m I_{(m)}\, k_{(m)}^2 \frac{\varepsilon_r - 1}{\varepsilon_r} \varphi_{(m)}. \tag{44}$$

Utilizing (40), the second term in the right-hand side of (35) becomes

$$-\frac{k_{(i)}^2}{j\omega\,\varepsilon_0}\int\limits_A \sum_m I_{(m)}\,k_{(m)}^2\,\frac{\varepsilon_r-1}{\varepsilon_r}\,\varphi_{(m)}\,\varphi_{(i)}\,\mathrm{d}A. \tag{45}$$

The series in the integral is convergent in mean square, therefore it converges almost everywhere in absolute sense. With the aid of Lebesgue theorem [12] we can interchange the integration and the summation

$$\begin{aligned}&\frac{k_{(i)}^2}{j\omega\,\varepsilon_0}\sum_m I_{(m)}\,k_{(m)}^2\int\limits_A \frac{\varepsilon_r-1}{\varepsilon_r}\,\varphi_{(m)}\,\varphi_{(i)}\,\mathrm{d}A=\\&=-\frac{k_{(i)}^2}{j\omega\,\varepsilon_0}\sum_m I_{(m)}\,k_{(m)}^2\left[\int\limits_A \varphi_{(m)}\,\varphi_{(i)}\,\mathrm{d}A-\int\limits_A\frac{1}{\varepsilon_r}\,\varphi_{(m)}\,\varphi_{(i)}\,\mathrm{d}A\right].\end{aligned} \tag{46}$$

The first term in the brackets equals $\dfrac{\delta_{mi}}{k_{(m)}^2}$ so the final form of (35) is

$$-\frac{\partial U_{(i)}}{\partial z}=j\omega\,\mu_0\,I_{(i)}+\frac{k_{(i)}^2}{j\omega\,\varepsilon_0}\sum_m I_{(m)}k_{(m)}^2\int\limits_A\frac{1}{\varepsilon_r}\,\varphi_{(m)}\,\varphi_{(i)}\,\mathrm{d}A \tag{47}$$

Let us examine (37). With the aid of (39) and (28)

$$\mathbf{J}_t=j\omega\,\varepsilon_0(\varepsilon_r-1)\sum_m\left(U_{(m)}\mathbf{e}_{(m)}+U_{[m]}\mathbf{e}_{[m]}\right).$$

After substituting this into the second term of the right-hand side of (37) and interchanging the integration and the summation again, we obtain

$$\begin{aligned}&j\omega\,\varepsilon_0\sum_m U_{(m)}\Big[\int\limits_A\varepsilon_r\,\mathbf{e}_{(i)}\,\mathbf{e}_{(m)}\,\mathrm{d}A-\int\limits_A\mathbf{e}_{(i)}\,\mathbf{e}_{(m)}\,\mathrm{d}A\Big]+\\&+j\omega\,\varepsilon_0\sum_m U_{[m]}\Big[\int\limits_A\varepsilon_r\,\mathbf{e}_{(i)}\,\mathbf{e}_{[m]}\,\mathrm{d}A-\int\limits_A\mathbf{e}_{(i)}\,\mathbf{e}_{[m]}\,\mathrm{d}A\Big].\end{aligned} \tag{48}$$

The second term in the first bracket is δ_{im}, so it eliminates the first term on the right-hand side of (37). The second term in the second bracket always vanishes according to (22). The final form of (37) is

$$-\frac{\partial I_{(i)}}{\partial z}=j\omega\,\varepsilon_0\sum_m U_{(m)}\int\limits_A\varepsilon_r\,\mathbf{e}_{(i)}\,\mathbf{e}_{(m)}\,\mathrm{d}A+j\omega\,\varepsilon_0\sum_m U_{[m]}\int\limits_A\varepsilon_r\mathbf{e}_{(i)}\mathbf{e}_{[m]}\,\mathrm{d}A\,. \tag{49}$$

The transformation of Eq. (38) is identical with that of (37).

For the sake of survey we write once more the coupled transmission line equations

$$-\frac{\partial U_{(i)}}{\partial z} = j\omega\,\mu_0\, I_{(i)} + \frac{k_{(i)}^2}{j\omega\,\varepsilon_0}\sum_m I_{(m)} k_{(m)}^2 \int_A \frac{1}{\varepsilon_r}\varphi_{(i)}\,\varphi_{(m)}\,\mathrm{d}A \tag{50}$$

$$-\frac{\partial U_{[i]}}{\partial z} = j\omega\,\mu_0\, I_{[i]} \tag{51}$$

$$-\frac{\partial I_{(i)}}{\partial z} = j\omega\,\varepsilon_0 \sum_m U_{(m)} \int_A \varepsilon_r\,\mathbf{e}_{(i)}\,\mathbf{e}_{(m)}\,\mathrm{d}A + j\omega\,\varepsilon_0 \sum_m U_{[m]} \int_A \varepsilon_r\,\mathbf{e}_{(i)}\,\mathbf{e}_{[m]}\,\mathrm{d}A \tag{52}$$

$$-\frac{\partial I_{[i]}}{\partial z} = \frac{k_{[i]}^2}{j\omega\,\mu_0} U_{[i]} + j\omega\,\varepsilon_0 \sum_m U_{[m]} \int_A \varepsilon_r\,\mathbf{e}_{[i]}\,\mathbf{e}_{[m]}\,\mathrm{d}A + j\omega\,\varepsilon_0 \sum_m U_{(m)} \int_A \varepsilon_r\,\mathbf{e}_{[i]}\,\mathbf{e}_{(m)}\,\mathrm{d}A\,. \tag{53}$$

If the permittivity is homogeneous, then ε_r is factorizable out of the integral sign. Then, on account of the orthogonal relations, the set of (50) to (53) transforms to the set of (35) to (38). Eqs (51) to (53) are identical to the equations of Schelkunoff for the isotropic case. But Eq. (50) does not need a matrix inversion, so it is more practical than Schelkunoff's one.

In the case of multiply connected region — restricted to the self-coupling of TEM mode — the set of (50) to (53) yields

$$-\frac{\partial U_{(i)}}{\partial z} = j\omega\,\mu_0\, I_{(i)} \tag{54}$$

$$-\frac{\partial I_{(i)}}{\partial z} = j\omega\,\varepsilon_0 \int_A \varepsilon_r\,\mathbf{e}_{(i)}^2\, dA\,. \tag{55}$$

So we obtain a method frequently occurring in the literature, the so-called "method of static effective permittivity". From (55) it is evident that the static effective permittivity equals

$$\varepsilon_{eff} = \int_A \varepsilon_r\,\mathbf{e}_{(i)}^2\,\mathrm{d}A. \tag{56}$$

So the method of static effective permittivity can be considered the first approximation of the set (50) to (53).

5. Connection of the method of coupled transmission lines and the Rayleigh—Ritz method

Let us start from the stationary formula for the propagation factor [1]

$$\gamma = j\omega \frac{\int_A (\mu_0\,\mathbf{H}^2 + \varepsilon\,\mathbf{E}^2)\,\mathrm{d}A}{2\int_A (\mathbf{E}\times\mathbf{H})\,\mathbf{k}\,\mathrm{d}A}\,. \tag{57}$$

The condition of stationarity for $\mathbf{E}$ is to satisfy the boundary conditions on the waveguide's wall.

Expand $\mathbf{H}$ according to the mode functions of the empty waveguide.

$$\mathbf{H}^2 = \mathbf{H}_t^2 + \mathbf{H}_z^2 = \sum_i \sum_m I_i I_m \mathbf{h}_i \mathbf{h}_m + \frac{1}{(j\omega\mu_0)^2} \sum_i \sum_m k_{[i]}^2 k_{[m]}^2 U_{[i]} U_{[m]} \varphi_{[i]} \varphi_{[m]} \tag{58}$$

using the relationship

$$H_z = -\frac{1}{j\omega\mu_0} \sum_i k_{[i]}^2 U_{[i]} \varphi_{[i]} \tag{59}$$

and marking the modes TE and TM equally without parentheses.

Taking into consideration the orthogonality we obtain that the expression

$$\int_A j\omega\mu_0 \mathbf{H}^2 \, \mathrm{d}A$$

becomes after substitution

$$\tilde{\mathbf{I}} j\omega\mu_0 \mathbf{E} \mathbf{I} + \tilde{\mathbf{U}} \left\langle \frac{k_{[i]}^2}{j\omega\mu_0} \right\rangle \mathbf{U} \tag{60}$$

where $\mathbf{U}$ and $\mathbf{I}$ are column vectors consisting of modal voltages and currents. $\mathbf{E}$ is the unity matrix. the tilde marks transposing, the broken bracket marks diagonal matrix. Only those elements of the diagonal matrix differ from zero, which belong to TE modes.

Let us expand $\mathbf{E}$ according to the empty waveguide modes. Then

$$\int_A j\omega\varepsilon_0\varepsilon_r \mathbf{E}^2 \mathrm{d}A = \int_A j\omega\varepsilon_0\varepsilon_r \mathbf{E}_t^2 \,\mathrm{d}A + \int_A j\omega\varepsilon_0\varepsilon_r E_z^2 \,\mathrm{dA} =$$

$$= j\omega\varepsilon_0 \int_A \varepsilon_r \sum_i \sum_m U_i U_m \mathbf{e}_i \mathbf{e}_m \,\mathrm{d}A + \int_A j\omega\varepsilon_0\varepsilon_r \frac{1}{(j\omega\varepsilon_0\varepsilon_r)^2} \sum_i \sum_m I_{(i)} I_{(m)} k_{(i)}^2 k_{(m)}^2 \varphi_{(i)}\varphi_{(m)} \,\mathrm{d}A \tag{61}$$

Let us introduce matrices

$$y_{im} = j\omega\varepsilon_0 \int_A \varepsilon_r \mathbf{e}_i \mathbf{e}_m \,\mathrm{d}A \tag{62}$$

$$z_{im} = \frac{1}{j\omega\varepsilon_0} k_{(i)}^2 k_{(m)}^2 \int_A \frac{1}{\varepsilon_r} \varphi_{(i)} \varphi_{(m)} \,\mathrm{d}A \tag{63}$$

the elements of latter differ from zero only for TM—TM coupling. (61) becomes

$$\int_A j\omega\varepsilon_0\varepsilon_r \mathbf{E}^2 \,\mathrm{dA} = \tilde{\mathbf{U}} \mathbf{y} \mathbf{U} + \tilde{\mathbf{I}} \mathbf{z} \mathbf{I} \tag{64}$$

Finally it is well known [8]

$$2\int_A (\mathbf{E}\times\mathbf{H})\,\mathbf{k}\,\mathrm{d}A = 2\tilde{\mathbf{U}}\,\mathbf{E}\,\mathbf{I} \tag{65}$$

Using (60), (64) and (65), (57) becomes

$$\gamma = \frac{\tilde{\mathbf{I}}\,j\omega\mu_0\,\mathbf{E}\,\mathbf{I} + \tilde{\mathbf{U}}\left\langle\frac{k_{[i]}^2}{j\omega\,\mu_0}\right\rangle\mathbf{U} + \tilde{\mathbf{U}}\,\mathbf{y}\,\mathbf{U} + \tilde{\mathbf{I}}\,\mathbf{z}\,\mathbf{I}}{2\,\tilde{\mathbf{U}}\,\mathbf{E}\,\mathbf{I}} \tag{66}$$

After reducing the numerator:

$$\gamma = \frac{\tilde{\mathbf{U}}\,\mathbf{Y}\,\mathbf{U} + \tilde{\mathbf{I}}\,\mathbf{Z}\,\mathbf{I}}{\tilde{\mathbf{U}}\,\mathbf{E}\,\mathbf{I}} \tag{67}$$

Z and **Y** appear to be identical with the matrices standing on the right hand side of (50) to (53).

Let us use the shortened form of (67)

$$\gamma = \frac{M}{N} \tag{68}$$

Minimizing (68) according to the Rayleigh—Ritz method leads to the following conditions

$$\frac{\partial}{\partial\tilde{U}_i}(M - \gamma N) = 0 \tag{69}$$

$$\frac{\partial}{\partial\tilde{I}_i}(M - \gamma N) = 0 \tag{70}$$

These equations lead to the system of equations

$$\mathbf{Y}\,\mathbf{U} - \gamma\mathbf{I} = 0 \tag{71}$$

$$\mathbf{Z}\,\mathbf{I} - \gamma\,\mathbf{U} = 0 \tag{72}$$

Eq. (71) is identical with (52)—(53), while (72) is identical with (50)—(51) supposing we examine a solution of the form $e^{-\gamma z}$ in (50)—(53).

Thus we proved that the Rayleigh—Ritz method and the coupled transmission line method are the same if the empty waveguide modes are used as the trial fields.

Summary

To calculate the electromagnetic field of waveguides filled with inhomogeneous, isotropic dielectrics, Schelkunoff has derived coupled transmission line equations. It is shown that these equations can easily be derived from the Marcuvitz—Schwinger equations. One of Schelkunoff's equations contains the inverse of an infinite matrix: instead of it a simpler equation is given. A faulty interpretation of this equation, which occurred in the literature is pointed out. Finally, it is shown that applying the Rayleigh Ritz method in the variational formula of the propagation constant — using the empty-waveguide modes as co-ordinate functions — the result is identical with the coupled transmission line equations.

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