DETERMINATION OF THE CHARACTERISTIC MATRICES OF NETWORKS CONSISTING OF TRANSMISSION LINES

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Introduction

Networks composed of transmission lines are frequently applied in ultra-short wave and microwave engineering, e.g. for filters, distributors, hybrides, matching elements. A method for calculating transmission line networks with the aid of the graph theory is given in [1], discussing primarily energy distribution networks. In the present paper a developed version of the above mentioned procedure is presented, with the calculation methods for the admittance matrix **Y**, impedance matrix **Z**, and reflection matrix **S** of the transmission line network. The results can be used above all in telecommunication engineering calculations.

The graph of the transmission line network

The examined network consists of passive transmission line sections. The ends of the individual sections form the connection points of the network. At these points the examined network is connected to other networks. Let n denote the number of these. Accordingly the examined network is an $n \times 2$-pole (or n-port). The ends of the transmission line sections are named vertices. Consequently, the individual sections join by vertices. Two nodes belong to a vertex. The calculation is performed for the case when impedances, such as capacity, short circuit, break, can be connected to the transmission line sections at the vertices. The number of vertices is denoted by c, that of the sections by k.

A graph is ordered to the network. A branch of the graph corresponds to a transmission line section, and a vertex of the graph to a vertex of the network. The graph ordered to the network shown in Fig. 1*a* is seen in Fig. 1*b*.

Let us arbitrarily indicate the directions of the branches of the graph and give them order numbers. Vertices are also designated by order numbers. To this end vertices are classified in three groups. To the first group belong those vertices, which are also the connection points of the network. The number of these is n. The second group contains the vertices not included in the first group and the termination of which is not a short circuit, i.e. the connected impedance is not zero. The third group contains those vertices which are terminated by a short circuit. Vertices are numbered by considering the respective groups, i.e. by $1, 2, \ldots, n; n+1, \ldots$; first group, and the subsequent order numbers to the vertices of the second and third group respectively. E.g. among the vertices of the graph indicated in Fig. 1b, belonging to the network shown in Fig. 1a, those having the order number 1, 2, 3 belong to the first group, the order numbers 6, 7, 8 to the second group, finally 4 and 5 to the third group.

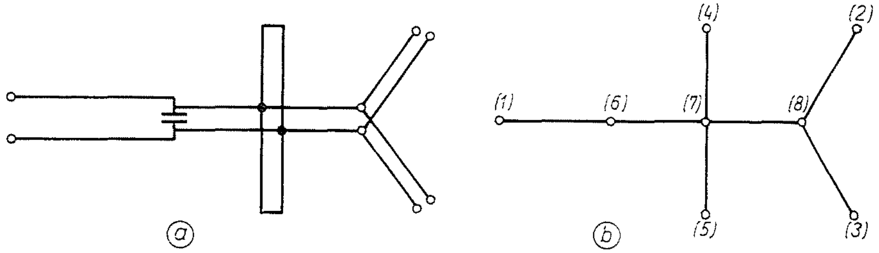

Fig. 1

The graph of the examined network contains also terminal elements. For characterizing a graph containing terminal elements the vertex matrix **A** is best suited. In this a vertex corresponds to each row in the order of the numbering of the vertices, and a branch corresponds to each column in the order of numbering of the branches. The j-th element in the i-th row of the vertex matrix is a_{ij}. If the i-th vertex is matched to the j-th branch, then $a_{ij} = 1$, if it is not matched, then $a_{ij} = 0$. The value of two elements in each column is 1, while the others are 0, since each branch is matched to two vertices.

For writing the equations of the network, the individual branches are given a direction. For characterizing the directional graph, the directional vertex matrix $\mathbf{A}_i$ can be used. The elements of this are 1, —1 and 0. Namely $a_{ij} = 1$, if the i-th vertex is matched to the j-th branch and the direction of the j-th branch is away from the i-th vertex. $a_{ij} = -1$, if the i-th vertex and the j-th branch are matched and the direction of the j-th branch is towards the i-th vertex. In the individual columns one element is 1, another —1, while the rest is 0.

In the followings we shall need also the matrices $\frac{1}{2}(\mathbf{A} + \mathbf{A}_i)$, and $\frac{1}{2}(\mathbf{A} - \mathbf{A}_i)$. In matrix $\frac{1}{2}(\mathbf{A} + \mathbf{A}_i)$ $a_{ij} = 1$, if the j-th branch is matched to the i-th vertex, and its direction is away from that, otherwise $a_{ij} = 0$. In matrix $\frac{1}{2}$ $(\mathbf{A} - \mathbf{A}_i)$ $a_{ij} = 1$, if the j-th branch is matched to the i-th vertex, and its direction is towards the i-th vertex, otherwise $a_{ij} = 0$. In each column of these last mentioned two matrices, one element is 1, while the others are 0.

Characterization of the elements of the network

In our problem the branches of the graph correspond to two-port, differently from the known graph theory calculation method of networks [2].

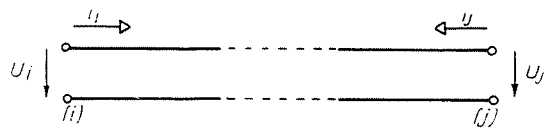

Fig. 2

For characterizing such a branch the correlation between two voltages (u_i, u_j) and two currents (i_i, i_j) should be given (Fig. 2). Considering that the direction of the m-th branch is from the i-th vertex towards the j-th, we find that

$$\begin{bmatrix} i_i \\ i_j \end{bmatrix} = \mathbf{Y}_m \begin{bmatrix} u_i \\ u_j \end{bmatrix} = \begin{bmatrix} p_m & q_m \\ r_m & t_m \end{bmatrix} \begin{bmatrix} u_i \\ u_j \end{bmatrix}. \tag{1}$$

The matrix $\mathbf{Y}_m$ of a non-reciprocal transmission line section should be determined by considering non-reciprocity. If the m-th transmission line section is a symmetrical reciprocal two-port, in the knowledge of the wave admittance Y_{om}, of the propagation coefficient γ_m, and of the length l_m of the transmission line section, $\mathbf{Y}_m$ can be written as follows.

$$\mathbf{Y}_m = Y_{om} \begin{bmatrix} \operatorname{cth} \gamma_m l_m & -\dfrac{1}{\operatorname{sh} \gamma_m l_m} \\ -\dfrac{1}{\operatorname{sh} \gamma_m l_m} & \operatorname{cth} \gamma_m l_m \end{bmatrix}. \tag{2}$$

Thus, in this case

$$\begin{aligned} p_m = t_m &= Y_{om} \operatorname{cth} \gamma_m l_m \\ q_m = r_m &= -Y_{om}/\operatorname{sh} \gamma_m l_m . \end{aligned} \tag{3}$$

Let us form diagonal matrices from each of the values p_m, q_m, r_m, t_m characterizing the individual branches.

$$\begin{aligned}\mathbf{P} &= \langle p_1 \quad p_2 \quad \ldots \quad p_k\rangle \\ \mathbf{Q} &= \langle q_1 \quad q_2 \quad \ldots \quad q_k\rangle \\ \mathbf{R} &= \langle r_1 \quad r_2 \quad \ldots \quad r_k\rangle \\ \mathbf{T} &= \langle t_1 \quad t_2 \quad \ldots \quad t_k\rangle\end{aligned} \tag{4}$$

In the followings the matrices $\mathbf{P}, \mathbf{Q}, \mathbf{R}, \mathbf{T}$ will be used for characterizing the transmission line sections of the network. In the case of a symmetrical reciprocal network $\mathbf{P} = \mathbf{T}$ and $\mathbf{Q} = \mathbf{R}$.

Let us form a diagonal matrix from the Z_{bi} values of the impedances connected to the vertices, in the order of numbering of the vertices.

$$\mathbf{Z}_b = \langle Z_{b1} \quad Z_{b2} \quad \ldots \quad Z_{bc}\rangle \tag{5}$$

Partition $\mathbf{Z}_b$ in such a way that the Z_{bi} values corresponding to the vertices belonging to a single group should form a block.

$$\mathbf{Z}_b = \langle \mathbf{Z}_{b1} \quad \mathbf{Z}_{b2} \quad \mathbf{Z}_{b3}\rangle \tag{6}$$

In $\mathbf{Z}_{b1}$ each element in the main diagonal is ∞, in $\mathbf{Z}_{b2}$ the elements in the main diagonal are ∞ or have a finite value different from zero. All the elements of $\mathbf{Z}_{b3}$ are 0.

We shall need the reciprocal of matrix $\mathbf{Z}_b$ as well. The partitioned form of this is

$$\mathbf{Y}_b = \mathbf{Z}_b^{-1} = \langle \mathbf{Y}_{b1} \quad \mathbf{Y}_{b2} \quad \mathbf{Y}_{b3}\rangle \tag{7}$$

Here all elements of $\mathbf{Y}_{b1}$ are zero, the elements in the main diagonal of $\mathbf{Y}_{b2}$ have finite values, while all the elements in the main diagonal of $\mathbf{Y}_{b3}$ are ∞.

Circuit equations

In the followings current generators of known source current are connected to each connection point of the examined network (Fig. 3) and the corresponding Kirchhoff equations are written for the network.

Circuit equations are written in the following in such a way that voltage equations should be automatically satisfied, and consequently only a number of c independent node equations are to be written. In the equations the voltage at the vertices also of a number c is the unknown value. Thus, the voltage of the vertices can be determined from the node equations.

From the voltage of the vertices the currents flowing at the ends of the transmission line sections can be determined, and so can be the currents in the impedances at the vertices.

The currents flowing out of or into one of the nodes of some of the vertices are written as the sum of three groups. To the first group belong those currents, the reference direction of which is identical with the direction of the respective branch. To the second group belong those the reference direction of which is contrary to the direction of the respective branch. Finally, the third group includes the currents flowing through the generator or impedance between two nodes of the vertex.

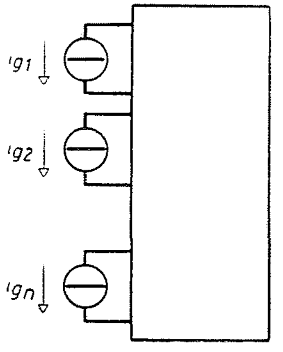

Fig. 3

In the knowledge of the currents of the three groups the node equation is written for one of the nodes of each vertex of the network.

If the m-th branch is matched to the i-th and j-th vertex and its direction is from the i-th towards the j-th, then the current of the branch belonging to the first group is given by

$$i_{mi} = p_m u_i + q_m u_j . \tag{8}$$

Similar equations can be written for all branches. The system of equations obtained in this way can be summed up in the following matrix equation.

$$\boldsymbol{I}' = \frac{1}{2}\mathbf{P}(\mathbf{A} + \mathbf{A}_l)^* \boldsymbol{U} + \frac{1}{2}\mathbf{Q}(\mathbf{A} - \mathbf{A}_l)^* \boldsymbol{U} . \tag{9}$$

(The asterisk * denotes the transpose of the matrix.)

For the m-th branch the current belonging to the second group is found to be

$$i_{mj} = r_m u_i + t_m u_j . \tag{10}$$

Such equations can be written for each branch. This system of equations is the following:

$$\boldsymbol{I}'' = \frac{1}{2}\mathbf{R}(\mathbf{A} + \mathbf{A}_l)^* \boldsymbol{U} + \frac{1}{2}\mathbf{T}(\mathbf{A} - \mathbf{A}_l)^* \boldsymbol{U}. \tag{11}$$

The current flowing in the vertices is written as the sum of two currents. One is the current of the current generators, while the other the current flowing in the passive elements. Thus the current of the i-th vertex is found to be

$$i_{ci} = i_{gi} - Y_{bi}\, u_i \tag{12}$$

where i_{gi} is the source current of the current generator of the i-th vertex, and u_i the voltage between the nodes of the i-th vertex. If no generator is connected to the vertices, then $i_{gi} = 0$ and $i_{ci} = -Y_{bi} u_i$. At the connection points $Y_{bi} = 0$ and thus $i_{ci} = i_{gi}$. By writing Eq. (12) for all the vertices. these can be summed up in matrix equation

$$\boldsymbol{I}_c = \boldsymbol{I}_g - \mathbf{Y}_b\, \boldsymbol{U} \tag{13}$$

where $\boldsymbol{I}_g$ is the column vector formed of the source current of the generators in the vertices, and $\boldsymbol{U}$ is the column vector formed of the vertex voltages. $\boldsymbol{I}_g$ can be partitioned according to the three groups of vertices.

$$\boldsymbol{I}_g = \begin{bmatrix} \boldsymbol{I}_{g1} \\ \boldsymbol{O} \\ \boldsymbol{O} \end{bmatrix}. \tag{14}$$

The currents have to satisfy the node equation. The currents written in $\boldsymbol{I}'$ are flowing away from one of the nodes of the vertices. Form of these the sum of those belonging to the individual vertices and denote the column matrix formed of these by $\boldsymbol{I}'_c$.

$$\boldsymbol{I}'_c = \frac{1}{2}(\mathbf{A} + \mathbf{A}_i)\, \boldsymbol{I}'. \tag{15}$$

Branch currents forming $\boldsymbol{I}''$ are flowing out of one of the nodes of the vertices. Form out of the sum of currents flowing out of a vertex the column matrix $\boldsymbol{I}''_c$.

$$\boldsymbol{I}''_c = \frac{1}{2}(\mathbf{A} - \mathbf{A}_i)\, \boldsymbol{I}''. \tag{16}$$

Currents figuring in $\boldsymbol{I}_c$ are flowing towards that node of the vertex, from which the corresponding current of $\boldsymbol{I}'_c$ and $\boldsymbol{I}''_c$ is flowing out. Thus the matrix form of the node rule, by using (13), (15), (16), further (9) and (11), is found to be

$$\begin{aligned} \boldsymbol{I}'_c + \boldsymbol{I}''_c - \boldsymbol{I}_c = \frac{1}{4}\{(\mathbf{A} + \mathbf{A}_i)\,[\mathbf{P}(\mathbf{A} + \mathbf{A}_i)^* + \mathbf{Q}(\mathbf{A} - \mathbf{A}_i)^*] + \\ + (\mathbf{A} - \mathbf{A}_i)\,[\mathbf{R}(\mathbf{A} + \mathbf{A}_i)^* + \mathbf{T}(\mathbf{A} - \mathbf{A}_i)^*]\}\, \boldsymbol{U} + \mathbf{Y}_b\, \boldsymbol{U} - \boldsymbol{I}_g = \boldsymbol{O}. \end{aligned} \tag{17}$$

Introduce the designation

$$\mathbf{Y}_c = \frac{1}{4}\{(\mathbf{A}+\mathbf{A}_i)\left[\mathbf{P}(\mathbf{A}+\mathbf{A}_i)^* + \mathbf{Q}(\mathbf{A}-\mathbf{A}_i)^*\right] + \\ + (\mathbf{A}-\mathbf{A}_i)\left[\mathbf{R}(\mathbf{A}+\mathbf{A}_i)^* + \mathbf{T}(\mathbf{A}-\mathbf{A}_i)^*\right]\}. \tag{18}$$

By employing this, from (17) we have

$$(\mathbf{Y}_c + \mathbf{Y}_b)\,\boldsymbol{U} = \boldsymbol{I}_g. \tag{19}$$

It should be noted that if all transmission line sections of the network are reciprocal and symmetrical, then $\mathbf{P} = \mathbf{T}$ and $\mathbf{Q} = \mathbf{R}$, and thus

$$\mathbf{Y}_c = \frac{1}{2}\,\mathbf{A}(\mathbf{P}+\mathbf{Q})\,\mathbf{A}^* + \frac{1}{2}\,\mathbf{A}_i(\mathbf{P}-\mathbf{Q})\,\mathbf{A}_i^*. \tag{20}$$

Partition equation (19) according to the grouping of the vertices:

$$\begin{bmatrix} \mathbf{Y}_{11} & \mathbf{Y}_{12} & \mathbf{Y}_{13} \\ \mathbf{Y}_{21} & \mathbf{Y}_{22} & \mathbf{Y}_{23} \\ \mathbf{Y}_{31} & \mathbf{Y}_{32} & \mathbf{Y}_{33} \end{bmatrix} \begin{bmatrix} \boldsymbol{U}_1 \\ \boldsymbol{U}_2 \\ \boldsymbol{U}_3 \end{bmatrix} + \begin{bmatrix} \boldsymbol{O} \cdot \boldsymbol{U}_1 \\ \mathbf{Y}_{b2}\,\boldsymbol{U}_2 \\ \mathbf{Y}_{b3}\,\boldsymbol{U}_3 \end{bmatrix} = \begin{bmatrix} \boldsymbol{I}_{g1} \\ \boldsymbol{O} \\ \boldsymbol{O} \end{bmatrix} \tag{21}$$

where

$$\boldsymbol{U}_3 = \boldsymbol{O} \tag{22}$$

From equation (21):

$$\begin{bmatrix} \mathbf{Y}_{11} & \mathbf{Y}_{12} \\ \mathbf{Y}_{21} & \mathbf{Y}_{22} + \mathbf{Y}_{b2} \end{bmatrix} \begin{bmatrix} \boldsymbol{U}_1 \\ \boldsymbol{U}_2 \end{bmatrix} = \mathbf{Y}_t \begin{bmatrix} \boldsymbol{U}_1 \\ \boldsymbol{U}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{I}_{g1} \\ \boldsymbol{O} \end{bmatrix}. \tag{23}$$

Determination of the matrices characterizing the network

From the equation (23):

$$\begin{bmatrix} \boldsymbol{U}_1 \\ \boldsymbol{U}_1 \end{bmatrix} = \mathbf{Y}_t^{-1} \begin{bmatrix} \boldsymbol{I}_{g1} \\ \boldsymbol{O} \end{bmatrix}. \tag{24}$$

Partition $\mathbf{Y}_t^{-1}$.

$$\mathbf{Y}_t^{-1} = \begin{bmatrix} \mathbf{Z}_{11} & \mathbf{Z}_{12} \\ \mathbf{Z}_{21} & \mathbf{Z}_{22} + \mathbf{Z}_{b2} \end{bmatrix}. \tag{25}$$

Thus

$$\boldsymbol{U}_1 = \mathbf{Z}_{11}\,\boldsymbol{I}_{g1}. \tag{26}$$

$\mathbf{Z}_{11}$ is the impedance matrix of the network,

$$\mathbf{Z} = \mathbf{Z}_{11}, \tag{27}$$

the reciprocal of which is the admittance matrix

$$\mathbf{Y} = \mathbf{Z}^{-1} = \mathbf{Z}_{11}^{-1}. \tag{28}$$

For determining the reflection matrix of the network, define the diagonal matrix which can be formed from the wave admittance of the branches matching to the connection places.

$$\mathbf{Y}_o = \langle Y_{o1} \quad Y_{o2} \ldots Y_{on} \rangle \tag{29}$$

The order number of wave admittances is identical with the order number of the vertex to which the branch is matching. If two or more branches are connected to a vertex, the sum of the wave admittances of the branches is written in the place corresponding the vertex in the above matrix.

Decompose column vectors $\boldsymbol{U}_1$ and $\boldsymbol{I}_1 = \mathbf{Y}\,\boldsymbol{U}_1$ to the sum of column vectors describing the incident and reflected waves, respectively.

$$\begin{aligned} \boldsymbol{U}_1 &= \boldsymbol{U}^{(+)} + \boldsymbol{U}^{(-)} \\ \boldsymbol{I}_1 &= \boldsymbol{I}^{(+)} + \boldsymbol{I}^{(-)} \end{aligned} \tag{30}$$

where

$$\begin{aligned} \boldsymbol{I}^{(+)} &= \mathbf{Y}_0\,\boldsymbol{U}^{(+)} \\ \boldsymbol{I}^{(-)} &= -\mathbf{Y}_0\,\boldsymbol{U}^{(-)} \end{aligned} \tag{31}$$

that is

$$\boldsymbol{I}_1 = \mathbf{Y}_0(\boldsymbol{U}^{(+)} - \boldsymbol{U}^{(-)}) = \mathbf{Y}(\boldsymbol{U}^{(+)} + \boldsymbol{U}^{(-)}) \tag{32}$$

hence

$$(\mathbf{Y}_0 - \mathbf{Y})\,\boldsymbol{U}^{(+)} = (\mathbf{Y}_0 + \mathbf{Y})\,\boldsymbol{U} \tag{33}$$

and thus

$$\boldsymbol{U}^{(-)} = (\mathbf{Y}_0 + \mathbf{Y})^{-1}(\mathbf{Y}_0 - \mathbf{Y})\,\boldsymbol{U}^{(+)}. \tag{34}$$

The definition of the reflection matrix $\mathbf{S}$ is given by the equation

$$\boldsymbol{U}^{(-)} = \mathbf{S}\,\boldsymbol{U}^{(+)} \tag{35}$$

that is, by comparing with (34), the reflection matrix

$$\mathbf{S} = (\mathbf{Y}_0 + \mathbf{Y})^{-1}(\mathbf{Y}_0 - \mathbf{Y}) = (\mathbf{Z}\,\mathbf{Y}_0 + \mathbf{E})^{-1}(\mathbf{Z}\,\mathbf{Y}_0 - \mathbf{E}) \tag{36}$$

where $\mathbf{E}$ is the unit matrix.

Summary

Transmission line systems are calculated by the help of the graph theory. Voltages and currents arising at the vertices are expressed from matrix equations. By using this result a general method for determining the matrices characterizing the network, such as the admittance matrix **Y**, the impedance matrix **Z**, and the scattering matrix **S**, is presented.

References

1. VÁGÓ, I.: The calculation of transmission networks with the aid of graph theory. Periodica Polytechnica El. **13**, 155–168 (1969).
2. SESHU, S.—REED, M. B.: Linear graphs and electrical networks. Addison—Werley, London 1961.
3. CARLIN, H. J.: The scattering matrix in network theory. IRE Transaction on Circuit Theory p. 88–96 (1956).

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