SYNTHESIS OF RC LADDER NETWORKS WITH A MINIMUM OF CAPACITANCES*

PART I

By

A. MAGOS

Department of Theoretical Electricity, Technical University, Budapest

(Received May 2, 1969)

Presented by Prof. Dr. K. SIMONYI

Introduction

It is known that every transfer voltage-ratio function with negative poles and nonpositive zeros is realizable by means of RC ladder networks. The transfer function can be expressed in terms of the parameters of the two-port:

$$G = \frac{U_2}{U_1} = \frac{z_{21}}{z_{11}} = -\frac{y_{21}}{y_{22}} . \tag{1}$$

Accordingly, for the first step of the synthesis the parameter z_{11} or y_{22} is chosen so that its numerator agrees with the denominator of the transfer function to be realized. The denominator is chosen so that a driving-point function realizable by an RC network arises. This driving-point function has to be synthetised and meanwhile the zeros of the transfer function realized. The synthesis can be executed by the method of the removal of poles. Here an outline is given of this procedure in order to point out the difficulties emerging in the synthesis. The zero shifting is the first step to realize every zero. Hence, either a "shifting" impedance (Z_t) is removed in the series arm so that the remaining impedance should have a zero at the zero (s_i) to be realized (Fig. 1a), or the shifting impedance is removed in the shunt arm so that the remaining impedanc should have a pole at the point s_i (Fig. 1b). Consequently in the first case

$$Z_i'(s_i) = 0 \quad \text{and} \quad Z_t(s_i) = Z_i(s_i) \tag{2}$$

and in the second case

$$Z_i'(s_i) = \infty \quad \text{and} \quad Z_t(s_i) = Z_i(s_i) . \tag{3}$$

* This paper is abridged from a Doctor's Thesis presented at the Faculty of Electrical Engineering of the Technical University, Budapest.

The following decomposition can be performed according to (2):

$$\frac{1}{Z_i'(s)} = \frac{as}{s-s_i} + \frac{1}{Z_{i+1}(s)} \tag{4}$$

where

$$a = \lim_{s \to s_i} \frac{s - s_i}{Z_i'(s)}$$

Thus the shunt arm contains an R and a C in series according to the decomposition; this ensures that the transfer function has a zero at s_i, and the remaining driving-point function (Z_{i+1}) is by one order less than Z_i (Fig. 1a). After

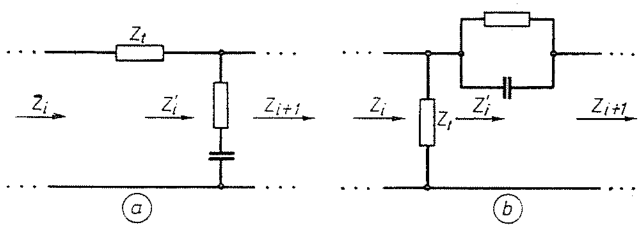

Fig. 1a, 1b

this the following zero can be realized. In the case of Fig. 1b a series arm containing an R and a C in parallel realizes the zero according to the following decomposition:

$$Z_i'(s) = \frac{a}{s-s_i} + Z_{i+1}(s) \tag{5}$$

where

$$a = \lim_{s \to s_i} (s - s_i)\, Z_i'(s)$$

If $Z_i(s_i) = 0$ or $Z_i(s_i) = \infty$, the zero shifting is omitted. Therefore the denominator of z_{11} and y_{22} is chosen so that as many of its zeros as possible coincide with the zeros of the transfer function.

Naturally both $Z_t(s)$ and $Z_i(s)$ must be drivingpoint impedances realizable by RC networks. In the case of a shifting impedance in the series arm this means that $Z_t(s)$ may have poles only where $Z_i(s)$ has too, and the following relationships must be fulfilled:

$$0 < \operatorname*{Res}_{s=s_j} Z_t(s) < \operatorname*{Res}_{s=s_j} Z_i(s) \tag{6a}$$

$$0 \leq \lim_{s \to \infty} Z_t(s) \leq \lim_{s \to \infty} Z_i(s) \tag{6b}$$

where s_j represents the poles of $Z_t(s)$. The corresponding residues of $Z_t(s)$ and $Z_i(s)$ cannot be equal because then $Z_i(s)$ would have no pole at s_j and thus

s_j would appear as an undesirable zero in the transfer function. In the case of a shifting impedance in the shunt arm the function $Y_t(s) = 1/Z_t(s)$ may have poles only where the function $Y_i(s) = 1/Z_i(s)$ has too, and the following relationships must be fulfilled:

$$0 > \operatorname*{Res}_{s=s_j} Y_t(s) > \operatorname*{Res}_{s=s_j} Y_i(s) \tag{7a}$$

$$0 \leq Y_t(0) \leq Y_i(0) \tag{7b}$$

where s_j represents the poles of $Y_i(s)$. It can be proved that in every case a driving-point impedance is found which satisfies the conditions (2) and (6), and another which satisfies the conditions (3) and (7), that is the zero shifting can be performed either in series or in shunt arm. But the form of function $Z_t(s)$ is of importance. The simpler the form, the better, and the best case is where the shifting impedance is a single resistance. In such a case, on the basis of (2) and (3) we have

$$R_t = Z_i(s_i). \tag{8}$$

According to this and to (6b) or to (7b) the zero shifting can be performed with a single resistance if

$$0 \leq Z_i(s_i) \leq \lim_{s\to\infty} Z_i(s) \quad \text{or} \quad Z_i(s) \geq Z_i(0) \tag{9}$$

Since $\lim_{s\to\infty} Z(s) < Z_i(0)$ and $Z_i(s_i)$ may assume a negative value, it may occur that neither of the above-mentioned inequalities is fulfilled. In such a case a shifting impedance of several elements must be used, which contains also capacitances. This has some disadvantages. First of all, more network elements are needed, the network becomes more complicated. Second, as the values of the elements of the built network cannot be exactly equal to the computed values, parasitic zeros and poles appear in the transfer function. Thirdly, the calculation work becomes more complicated. Among others one or more roots of the denominator or numerator of $Z_i(s)$ must be computed for determining $Z_t(s)$. The inaccuracy grows with the complexity of the calculation, badly affecting the result of the synthesis.

Whether it is possible to realize all zeros only by shifting resistance depends mainly on how the denominators of z_{11} or y_{22} are chosen, and in which order the zeros are realized. There is no generally valid method for this; but only for particular cases. The first par tof this paper is dealing with such particular cases. First, high-pass and low-pass RC ladder networks will be discussed, based to a certain extent on a paper by FUJISAWA [2]. To generalize these two particular cases, a third case will be considered. In the second part of the paper, based on the results of the first part, a modified version of the method detailed above will be presented, where the zero shifting can always be performed with a single shifting resistance. As a result the realized network contains a minimal number of capacitances, exactly the same number as the degree (n) of the denominator of the transfer function. The number of the employed resistances is $2n$ at most, but generally it is less. The pole and zero distribution of a transfer function with a denominator of degree n being characterized by $2n$ data, we must state it is a fairly good economy to use not more than $3n$ elements for the synthesis. The method has other advantages, too, which will be mentioned later, e.g. it can be easily programmed for computers. So we have made a program which performed the synthesis of RC ladder networks on the basis of this method.

High-pass and low-pass RC ladder networks

First two special classes of networks will be dealt with, the high-pass and low-pass ladder networks in Figs 2a and *b*, respectively. The first contains capacitances only in the series arms and shifting resistances in the shunt arms and the second inversely. The transfer function to be realized is of the following form:

$$G(s) = K \frac{(s+\alpha_1)(s+\alpha_2)\ldots(s+\alpha_m)}{(s+\beta_1)(s+\beta_2)\ldots(s+\beta_n)} \tag{10}$$

where

$$0 \leq \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_m \quad \text{and}$$
$$0 < \beta_1 < \beta_2 < \ldots < \beta_n .$$

The following can be stated of the transfer functions realizable by the above two classes of networks.

Theorem 1. The transfer function in (10) is realizable by the high-pass ladder network in Fig. 2a if and only if

$$m = n \tag{11a}$$

$$\alpha_i < \beta_i \qquad (i = 1, 2, \ldots n) \tag{11b}$$

$$K = 1 \tag{11c}$$

Theorem 2. The transfer function in (10) is realizable by the low-pass ladder network in Fig. 2b if and only if

$$m \leq n \tag{12a}$$

$$\alpha_i > \beta_i \qquad (i = 1, 2, \ldots m) \tag{12b}$$

$$K = \frac{\beta_1 \beta_2 \ldots \beta_n}{\alpha_1 \alpha_2 \ldots \alpha_m} . \tag{12c}$$

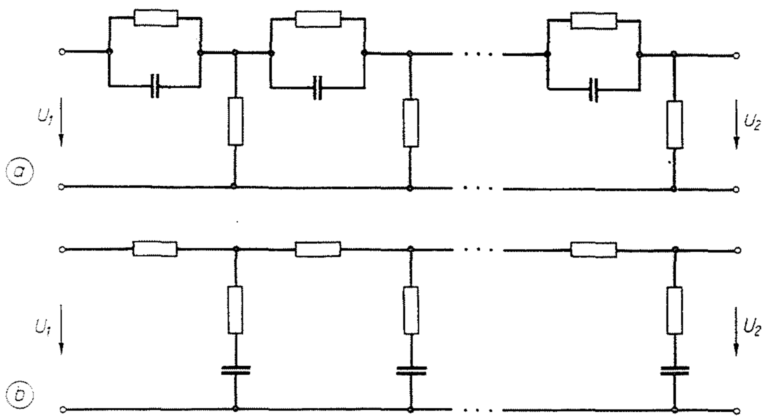

Fig. 2a, b

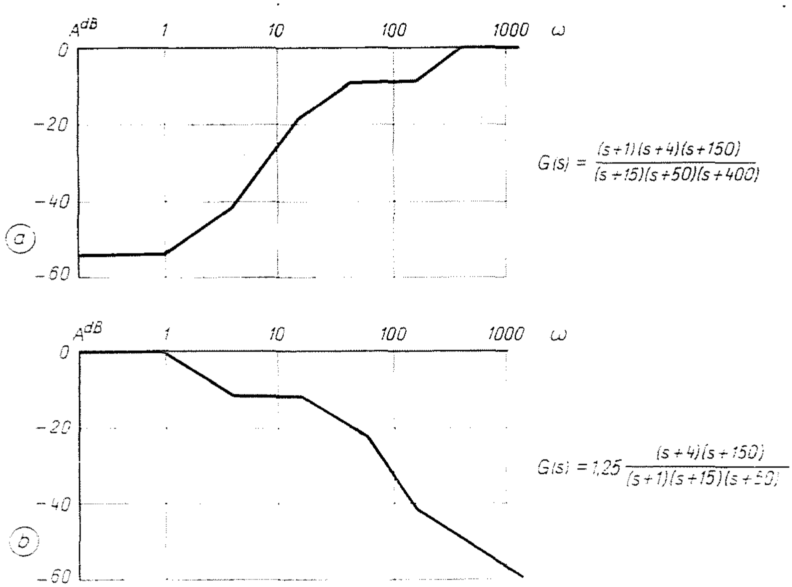

Fig. 3

Before proving the theorems, the example in Fig. 3a shows the straight line segment approximation of the logarithmic magnitude curve of a transfer function which satisfies the conditions given in (11) and that in Fig. 3b the same for a transfer function which satisfies the conditions given in (12).

First we prove Theorem 1. Let the input impedance of the network be of the following form:

$$Z_{11}(s) = H \frac{(s+\beta_1)(s+\beta_2)\dots(s+\beta_n)}{(s+\gamma_1)(s+\gamma_2)\dots(s+\gamma_n)} \tag{13}$$

where

$$0 \leq \gamma_1 < \beta_1 < \gamma_2 < \beta_2 < \ldots < \beta_n .$$

To find the relationship between quantities α_i and γ_i, let us build the network from the output terminal-pair toward the input terminal-pair and examine how the poles of the driving-point impedance migrate meanwhile. The poles of the impedances of the parallel RC configurations in the series arms mean the zeros of the transfer functions. Upon connecting such an RC configuration in series with the part already considered of the network, the poles of the driving-point impedance remain unchanged, but a new pole appears, which is the zero of the transfer function realized by the RC configuration examined. The driving-point admittance of an RC network increases monotonously as a function of the real values of s, except for the singular points. That is why connecting a resistance in parallel with the part already considered of the network, i.e. adding a positive constant to the driving-point admittance, its zeros, i.e. the poles of the driving-point impedance are shifted to the left along the real axis, so that their absolute values increase. Thus it can be stated: For an arbitrary point on the negative real axis no more poles of the input impedance can lie to the right of this point, than are zeros of the transfer function to the right of that point. Here every zero must be counted according to its multiplicity. This condition can be expressed as:

$$\alpha_i \leq \gamma_i \qquad (i = 1, 2, \ldots n) \tag{14}$$

leading directly to conditions (11a) and (11b), these being at the same time sufficient conditions. Namely their satisfaction permits to choose the input impedance $z_{11}(s)$ so that (14) is satisfied. In the course of the synthesis that zero is always realized to which the greatest shifting resistance belongs, that is. the value of the shifting resistance is given by the formula:

$$R_i = \max_j \left[Z_i(-\alpha_j') \right] \tag{15}$$

where $Z_i(s)$ represents the driving-point impedance to be realized at the given moment of synthesis and the α_j' represent the absolute values of the zeros not yet realized. If a value $Z_i(-\alpha_j')$ is not finite, obviously this zero is directly realized without shifting resistance. This method, after realizing a zero, leads to a driving-point impedance, the poles of which, together with the not yet realized zeros of the transfer function satisfy condition (14). This is obviously from the following consideration: Removing a parallel resistance from the driving-point impedance and reducing its magnitude gradually from a very great value, the poles of the driving-point impedance migrate gradually to the right. Namely the driving-point admittance increases monotonously as a function of the real s values, except for the singular points, and subtracting an increasing conductance shifts its zeros to the right. The formula (15) gives exactly that value of resistance for which a pole of the driving-point impedance first arrives to a not yet realized zero of the transfer function. That is why the condition (14) keeps satisfied after the removal of the shifting resistance. This is not altered by removing a parallel RC configuration in the series arm and effacing the same value among both the poles of the driving-point impedance and the zeros to be realized. On the other hand, if condition (14) is satisfied, one zero, more precisely the one with the smallest absolute value, can be realized by a single shifting resistance. Namely, the driving-point impedance has no pole to the right of this zero ($-\alpha_1'$) and so

$$R_i = Z_i(-\alpha_1') \geq Z_i(0)\,, \tag{16}$$

hence a shifting resistance can be removed in the shunt branch.

The network in Fig. 2a does not attenuate at infinite frequency, as expressed by Equation (11c).

The proof of Theorem 2 will be presented only briefly because it is similar to the previous one. The synthesis of the network in Fig. 2b must be begun with the output terminal-pair, hence from the parameter y_{22}. Let be $y_{22}(s)$ of the following form:

$$y_{22}(s) = H\,\frac{(s+\beta_1)(s+\beta_2)\,\ldots\,(s+\beta_n)}{(s+\gamma_1)(s+\gamma_2)\,\ldots\,(s+\gamma_n)} \tag{17}$$

where $0 < \beta_1 < \gamma_1 < \beta_2 < \gamma_2 < \ldots < \gamma_n$

It can easily be proved that the following condition must be satisfied:

$$\alpha_i \geq \gamma_i \qquad (i = 1, 2, \ldots m) \tag{18}$$

Condition (12b) follows from this. (Condition (12a) is self-evident.)

If (12b) is satisfied, $y_{22}(s)$ can be chosen so that (18) too is satisfied. In the course of the synthesis the zero to which the smallest shifting resistance belongs is to be realized throughout, i.e. the value of the shifting resistance is always given by the following formula:

$$\frac{1}{R_i} = G_i = \max_j\,[Y_i(-\alpha_j')] \tag{19}$$

where $Y_i(s)$ represents the driving-point admittance to be realized and the α_j' represent the absolute values of the zeros not yet realized. If the zeros to be realized include infinity then the value $\lim_{s\to\infty} Y_i(s)$, too, is to be considered when choosing the maximum. The network in Fig. 2b does not attenuate at zero frequency, as expressed by condition (12c).

Notice the advantage of the network that all capacitances are grounded, a special advantage for integrated circuits. On the other hand, it is a disadvantage that there is no resistance connected in parallel with the output terminal-pairs to represent the load.

Example 1

Let us realize the following transfer function:

$$G(s) = \frac{(s+1)(s+3)(s+6)^2}{(s+4)(s+8)(s+10)(s+13)} .$$

In our case

$$\alpha_1 = 1 \;\; \alpha_2 = 3 \;\; \alpha_3 = 6 \;\; \alpha_4 = 6 \;\; \text{and} \;\; \beta_1 = 4 \;\; \beta_2 = 8 \;\; \beta_3 = 10 \;\; \beta_4 = 13$$

This means that the transfer function satisfies condition (11), hence it is realizable by the class of network in Fig. 2a. The input impedance is assumed in the following form:

$$z_{11}(s) = \frac{(s+4)(s+8)(s+10)(s+13)}{(s+1)(s+6)(s+9)(s+11)} .$$

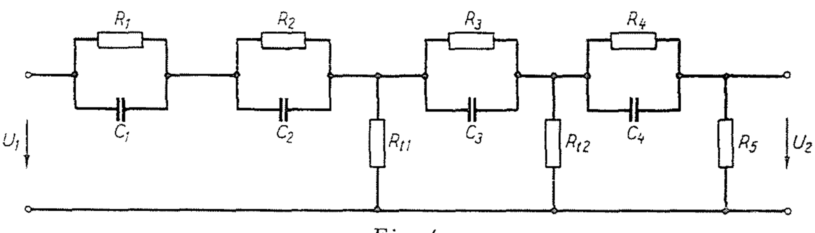

Fig. 4

Zeros $s = -1$ and $s = -6$ can be realized directly, without shifting resistance. For this purpose decompose $z_{11}(s)$:

$$z_{11}(s) = \frac{a}{s+1} + \frac{b}{s+6} + Z_2(s) .$$

$$\text{Here } a = 5.67, \; b = 1.493 \text{ and } Z_2(s) = \frac{s^2 + 20.837\, s + 107.363}{s^2 + 20\, s + 99} .$$

It follows from this that (Fig. 4)

$$C_1 = \frac{1}{a} = 0.176 \quad R_1 = 5.67 \quad C_2 = \frac{1}{b} = 0.670 \quad R_2 = 0.249 \ .$$

Zeros $s_1 = -3$ and $s_2 = -6$ remain still to be realized. The values of $Z_2(s)$ at these points are:

$$Z_2(-3) = 1.164 \text{ and } Z_2(-6) = 1.223$$

From this obviously the zero $s_2 = -6$ is to be realized first. The value of the shifting resistance is

$$R_{t1} = 1.223$$

The driving-point impedance reamaining after the removal of the shifting resistance is decomposed in the following way:

$$Z_2(s) = \frac{1}{\frac{1}{Z_2} - \frac{1}{R_{t1}}} = \frac{23.67}{s+6} + \frac{s+10.527}{0.18226\,s+1.8675} \ .$$

Hence:

$$C_3 = 0.0423 \text{ and } R_2 = 3.94$$

Now, only the zero $s = -3$ keeps to be realized. The detailed calculation gives the following values:

$$R_{t2} = 5.698 \quad C_4 = 0.00271 \quad R_4 = 123 \quad R_5 = 147$$

The complete network is seen in Fig. 4.

The synthesis of the general transfer function

These results can be generalized at a little effort. Let the transfer function be again of the form:

$$G(s) = K \frac{(s+\alpha_1)(s+\alpha_2)\,..\,(s+\alpha_m)}{(s+\beta_1)(s+\beta_2)\,...\,(s+\beta_n)} \tag{20}$$

where $m \leq n$

$$0 \leq \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_m$$

$$0 < \beta_1 < \beta_2 < \ldots < \beta_n$$

Let some zeros and poles of the transfer function which have the smallest absolute values and the number of which is r, satisfy the condition stated at the treatment of the high-pass ladder networks and let the other zeros and poles satisfy the condition stated at the treatment of the low-pass ladder networks. That is

$$\alpha_i < \beta_i \qquad \text{if } i = 1, 2, \ldots r \tag{21a}$$

and $$\alpha_i > \beta_i \qquad \text{if } i = r + 1, \ r + 2, \ldots m \tag{21b}$$

where $0 \leq r \leq m$

The extreme cases $r = 0$ and $r = m = n$ correspond to the already treated low-pass and high-pass transfer functions respectively. If r differs from these values, the transfer function has a bandpass character.

The realization of transfer functions satisfying condition (21) consists in the following steps. By means of parallel RC configurations in series arms the r zeros with the smallest absolute values are realized and the other zeros by means of series RC configurations in shunt arms. Accordingly, $z_{11}(s)$ has to be chosen carefully so, that r of its poles with the smallest absolute values satisfy the condition (14), that is, with the symbols used in (13),

$$\alpha_i \leq \gamma_i \qquad \text{for} \qquad i = 1, 2, \ \ldots r. \tag{22}$$

In principle, it ought to be cared that $(n - r)$ zeros of $z_{11}(s)$ with the greatest absolute values should satisfy a condition analogous to (18), but this is satisfied automatically because of (21b). Before each step of the synthesis it can be decided whether the zero to be realized is in the shunt or in the series arm. If it is in the series arm, then from the possible zeros (for these $i \leq r$) the one is chosen for realization to which the greatest shifting resistance belongs. If this zero is in the shunt arm, from the possible zeros (for these $i \geq r + 1$) the one with the smallest shifting resistance is chosen. It is easy to prove with a simple reasoning, similar to the preceding one but somewhat longer, that this method is expedient, i.e. the transfer function can be realized by using only resistances in the zero shifting.

If the transfer function does not satisfy condition (21), then no rule for choosing $z_{11}(s)$ [or $y_{22}(s)$] and for the order to realize the zeros can be established to ensure the possibility of zero shifting in all cases by means of a resistance. Let us reconsiderate the two groups of zeros of the transfer function. One of these contained the zeros then realized in shunt arms, and the other contained the zeros then realized in series arms. Factorize now the transfer function to the product of two factors, a low-pass and a high-pass type satisfying condition (12) and (11), respectively. It is easy to prove that this factorization can always be performed in several ways. The two respective transfer functions are realized separately, then the two networks connected in cascade so that the impedance level of the second one is chosen much greater than is that of the first. It is obvious that the transfer function of the total network nearly corresponds to the transfer function to be realized. The second part of this paper will deal with the problem of how the synthesis can be performed if the impedance levels of the two parts of the network are not to be chosen so different and the transfer function is to be realized accurately.

Example 2

Let us realize the following transfer function:

$$G(s) = K \frac{s(s+1)(s+12)}{(s+2)(s+4)(s+7)(s+10)} .$$

The transfer function is seen to satisfy condition (21) and $r = 2$. The zeros $s_1 = 0$ and $s_2 = -1$ are to be realized in series arms, and the zero $s_3 = -12$ and the zero at infinity in shunt arms. Let us choose the input impedance in the following form:

$$z_{11}(s) = \frac{(s+2)(s+4)(s+7)(s+10)}{(s+1)(s+3)(s+5)(s+8)} .$$

Of course, the zero $s_2 = -1$ in series arm is realized first, this needing no shifting resistance. The input impedance is decomposed in the following way:

$$z_{11}(s) = \frac{a}{s+1} + Z_2 s = \frac{2.893}{s+1} + \frac{s^3+19.11s^2+114.61s+212.81}{s^3+16s^2+19s+120} .$$

From this $R_1 = 2.893$ and $C_1 = 0.346$ (Fig. 5).
Let us now realize the other zero in series arm, namely the zero $s_1 = 0$. The value of the shifting resistance is:

$$R_{t1} = Z_2(0) = 1.773.$$

The driving-point impedance, which remains after the removal of the shifting resistance in the shunt arm and its decomposition are as follows:

$$Z_2'(s) = \frac{1}{\dfrac{1}{Z_2(s)} - \dfrac{1}{R_{t1}}} = \frac{s^3+19.11s^2+114.6s+212.8}{0.437s^3+5.22s^2+14.4s} =$$

$$= \frac{a}{s} + Z_3(s) = \frac{14.8}{s} + \frac{s^2+12.65s+37.4}{0.437s^2+5.22s+14.4} .$$

From this $C_2 = 0.0676$.
Now, only the zeros in shunt arms are to be realized. As

$$Z_3(-12) = 2.03 \text{ and } \lim_{s \to \infty} Z_3(s) = 2.29,$$

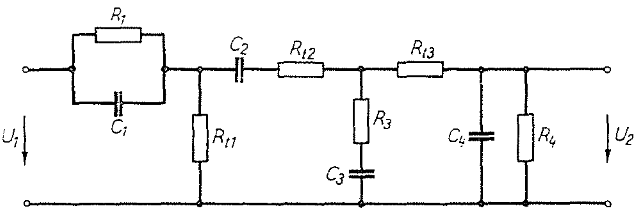

Fig. 5

the zero $s_3 = -12$ is realized first, because the shifting resistance belonging to this is smaller. The realization of the zero at infinity remains the last step. The detailed calculation gives the following values for the respective elements (Fig. 5):

$$R_{t2} = 2.03 \quad R_3 = 0.562 \quad C_3 = 0.148$$

$$R_{t3} = 0.496 \quad C_4 = 2.52 \quad R_4 = 13.1$$

Summary

The synthesis of RC ladder networks can be performed by the method of the removal of poles in such a way that the zero shifting is done only by resistances, using a minimum of capacitances for the synthesis. Two special classes of ladder network, the high-pass and low-pass networks have been dealt with.

The necessary and sufficient conditions for transfer functions to be realized by these classes of networks are presented together with the method of realization. The synthesis of a more general transfer function and opportunity for realizing entirely general transfer functions are treated, too. The second part of the paper will deliver ample details.

References

1. Fialkow, A.—Gerst, I.: The transfer function of an RC ladder network, Journal Math. and Phys. **30**, 49—72. (1951).
2. Fujisawa, T.: Realizability theorem for mid-series or mid-shunt low-pass ladders without mutual induction, IRE Trans. CT-**2**, 320—325. (1955).
3. Watanabe, H.: Synthesis of band-pass ladder networks, IRE Trans. CT-**5**, 256—264. (1958).
4. Weinberg, L.: Network analysis and synthesis, McGraw-Hill, New York, 1964.
5. Chen, W. H.: Linear network design and synthesis, McGraw-Hill, New York, 1964.

Dr. András Magos, Budapest, XI., Egry József u. 20, Hungary