ALGOL PROCEDURES FOR THE EVALUATION OF QUADRATIC INTEGRAL CRITERIONS

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The quality requirements of the transient control processes are more or less contradictory. A compromise may be realised with the help of the integral criteria (among others). A dynamic control process is called optimum, when a certain predetermined integral (functional) attains a minimum. The general form of the integral criterion is:

$$I = \int_0^\infty F[x(t), t]\, dt = \text{Min.} \tag{1}$$

F represents here a certain bivariate function of the time t and of a suitably chosen signal $x(t)$. The function to be integrated should be purposefully chosen in a way that on the one hand it appropriately characterizes the quality of the transient process (e.g. taking into consideration both the overshot and the control period), and on the other hand it is of a relatively simple form expressing its relation to the system parameters as simply as possible. The enumerated requirements are more or less contradictory. So it is not surprising to find integral criteria of the most different forms in the literature.

While by the linear criteria aperiodic processes may be evaluated in the first place, the quadratic integral criteria may be used to study both aperiodic and oscillating processes [1].

The most obvious criteria would be integral criteria relating to absolute values suitable to evaluate both aperiodic and overshooting processes [1]. Yet the quadratic criterion is generally used in mathematical analyses, because the mathematical treatment of absolute values is rather circumstantial. The analogue computer, however, much simplifies the control system's study based on the criterion of the absolute values.

When solving optimization problems it is advisable to start from the unit step response, i.e. from the time range behaviour of the control system. But attention must be drawn to the relatively close connection between the time domain and the frequency domain, so that the optimization conditions may often be determined also by the parameters of the frequency domain.

From the viewpoint of designing it is essential to know the kind of relation existing between a certain integral criterion and the parameters (e.g. coefficients or time constants, transfer factor) of the transfer function. This problem gains significance from the fact that the relations of the control system are usually given in the complex frequency (or operator) domain and however simple it is to return to the time domain in principle, practically it is laborious, often troubling to find direct relation to the system parameters.

In this paper we have described ALGOL procedures for the evaluation of the *quadratic integral criterion* of response functions under the effect of deterministic signals in single-variable linear control systems containing elements with constant, concentrated parameters and others with dead time.

The described procedures may be well utilized in designing control systems either independently or built into multivariable extreme value finding programs.

It must be emphasized that the value of this integral, similarly to the rest of integral criteria, is meaningless in itself, but if for two different parameters P_1 and P_2 the values I_1 and I_2 respectively are obtained and $I_2 < I_1$, then the transient process, response signal of the control system chosen as basis of the criterion runs off more favourably with the parameter P_2. If the integral value I versus the parameter (or parameters) P displays a minimum, then that parameter value (or values) is the most favourable, to which the minimum ensues.

The definition of the quadratic integral criterion of the time function $x(t)$ is

$$I = \int_{-\infty}^{\infty} x^2(t)\,dt. \tag{2}$$

According to the PARSEVAL-theorem [1] the following expression described in the frequency domain is equivalent to the form of integral I in the time domain:

$$I = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} X(-s)X(s)\,ds \tag{3}$$

where $s = j\omega$ and $X(s) = L\{x(t)\}$.

This latter expression may be described also in the following form:

$$I = \frac{1}{2\pi} \int_{-\infty}^{\infty} \check{X}(j\omega)X(j\omega)\,d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2\,d\omega \tag{4}$$

where $\check{X}(j\omega)$ is the conjugate complex of $X(j\omega)$.

As $|X(j\omega)|^2$ is an even function, the following relationship holds:

$$I=\frac{1}{2\pi}\int_{-\infty}^{\infty}|X(j\omega)|^2\,d\omega=\frac{1}{\pi}\int_{0}^{\infty}G^2(\omega)\,d\omega. \tag{5}$$

Evaluation of the quadratic integral criterion, when X(s) is a rational fractional function

Be

$$X(s)=\frac{C(s)}{D(s)}=\frac{\sum_{k=0}^{n-1}c_k\,s^k}{\sum_{k=0}^{n}d_k\,s^k} \tag{6}$$

where the polynomial $D(s)$ with a real coefficient has only left-side zeros. In most cases the real coefficient polynomial $C(s)$ which is at least by one degree lower than $D(s)$ has also only left-side zeros, though this is not a necessary condition for the present discussion.

Now the integral form of (3) is:

$$I=\frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\left|\frac{C(s)}{D(s)}\right|^2 ds. \tag{7}$$

The value of integral I may be expressed as a rational fractional function of the coefficients c_k and d_k and is found tabulated up to $n = 10$ in the literature [1]. The expressions obtained in this way, especially for high n values, are complicated and very inconvenient, therefore we utilized for the computer evaluation of I the following method:

The evaluation of integral I may be reduced to the solution of the linear algebraic equation system of

$$B = DA \tag{8}$$

Here

$$B=\begin{bmatrix}b_0\\ b_2\\ \vdots\\ b_{2n-2}\end{bmatrix}\qquad A=\begin{bmatrix}a_0\\ -a_1\\ \vdots\\ (-1)^{n-1}\,a_{n-1}\end{bmatrix} \tag{9}$$

where

$$2b_m=\begin{cases}\sum\limits_{k=0}^{m}(-1)^k\,c_k\,c_{m-k} & 0\le m\le n-1\\ \sum\limits_{k=m-n+1}^{n-1}(-1)^k\,c_k\,c_{m-k} & n\le m\le 2n-2\end{cases} \tag{10}$$

If n is an odd number, then

$$D = \begin{bmatrix} d_0 & 0 & 0 & \dots & 0 \\ d_2 & d_1 & d_0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ d_{n-1} & d_{n-2} & d_{n-3} & \dots & d_0 \\ 0 & d_n & d_{n-1} & \dots & d_2 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & d_{n-1} \end{bmatrix} . \tag{12}$$

If n is an even number, then

$$D = \begin{bmatrix} d_0 & 0 & 0 & \dots & 0 \\ d_2 & d_1 & d_0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ d_n & d_{n-1} & d_{n-2} & \dots & d_1 \\ 0 & d_n & d_n & \dots & d_3 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & d_{n-1} \end{bmatrix} . \tag{11}$$

By solving the above equation system for the coefficient a_{n-1}, we obtain the sought integral as

$$I = \frac{a_{n-1}}{d_n} . \tag{13}$$

The ALGOL procedure developed for this case:

procedure quadratic integral (n, y, transformer, generator, les);
value n; **integer** n; **real** y; **procedure** transformer, generator, les;
transformer (parameters, c, d);
generator (n, c, d, a);
les (n, a, x);
$y = (-1) \uparrow (n-1) * x[n-1]/d[n]$
end quadratic integral;

Output parameter : y — value of the integral. The other procedures in the program are to effect:

1) **procedure** transformer (parameters, c, d);
array parameters, c, d;
. . . . .
(preparation of the blocks c [1 : n—1] and d [1 : n] from the parameters of the system)
. . . . .
end transformer;

If the procedure is applied for a one-time evaluation of the integral criterion, then the transformer is dropped and the correspondent value at the beginning of the program is provided for by the instruction **input** (n, **array** c, d,);

```
2) procedure generator (n, c, d, a)
   value n; integer n; array c, d, a;
   begin integer i, k, l, m, li, lk; real s;
   array b [1 : n], dd [1 : n, 1 : n];
   for m: = 0, m+2 while m le n—1 do begin s: = 0;
   for k: = 0 step 1 until m do s: = s + (—1) ↑ k*c[k+1]*c[m—k+1];
   b[m/2+1]: = s/2; l: = m end;
   for m: = l + 2, m + 2 while m le 2*n—2 do begin s: = 0;
   for k: = m—n+1 step 1 until n—1 do
   s: = s + (—1) ↑ k*c [k + 1] *c[m — k + 1];
   b[m/2 + 1]: = s/2 end;
   if (n div 2 *2=n then begin li: = n/2 + 1; lk: = n/2—1 end
   else begin li: = (n + 1)/2; lk: = (n—1)/2 end;
   for k: = 1 step 1 until n do for l: = 1 step 1 until n do
   if 2*k—l ge 1 and k le li then dd[k, l] : =d [2*k—l] else
   dd[k, l]: = 0;
   do if 2*k—l + 1 ge and 2*k—l le n—1 and k le lk then
   for k: = 0 step 1 until n—1 do for l : = 0 step 1 until n—1
   dd [n—k, n—l]: = d[n—2*k+l];
   for k: = 1 step 1 until n do  begin for l: = 1 step 1 until n
   do a [k, l]: = dd [k, l]; a[k, n + 1]: = b[k] end
   end generator;
```

This procedure produces the B and D matrixes of equation system (8). The output A block contains extended matrix of the equation system.

```
3)   procedure les (n, a, x);
.....
(Supplies the solution of equation system (8) in vector X)
.....
     end les;
```

The equation systems were solved by the GAUSS—JORDAN elimination procedure.

The "quadratic integral" procedure completed to an ALGOL program was tried on numerical examples of known results.

Evaluation of the quadratic integral, with a dead time element in the control system

In this case the value of the integral criterion cannot be evaluated as simply as for the equation system (8), because $X(s)$ is not a rational fractional function. Here again we must rely on the integral of definition (3), which is convenient to evaluate in the frequency domain with the help of (5). Now the improprius integral is to be evaluated:

$$I = \frac{1}{\pi} \int_0^\infty G^2(\omega)\, d^\omega . \tag{14}$$

Here $G^2(\omega)$ is a conjugate fractional function in ω, with trigonometrical factors in both the numerator and the denominator because $X(s)$ contains elements of the form $e^{-S\tau}$ as well.

The improprius integral (14) was evaluated by the method used in the numerical LAPLACE transformation. Accordingly the interval $[0; \infty]$ was transferred into the interval $[0; 1]$ be the transformation

$$\Omega = e^{-\omega} . \tag{15}$$

So the integral to be evaluated assumes the form

$$I = \frac{1}{\pi} \int_0^\infty G^2(\omega)\, d\omega = \frac{1}{\pi} \int_0^1 \frac{G^2(-\ln\Omega)}{\Omega}\, d\Omega . \tag{16}$$

As the integrand has a discontinuity at $\Omega = 0$, therefore a low number a is chosen for the lower limit of the integral ($a \sim 10^{-3}$—10^{-6}) depending on the numerical range of the computer and on the required accuracy. Hence:

$$I = \frac{1}{\pi} \int_a^1 \frac{G^2(-\ln\Omega)}{\Omega}\, d\Omega . \tag{17}$$

The integral had been evaluated by a procedure utilizing the method of ROMBERG—STIEFEL—BAUER for the numerical integration [4, 5, 7]. A general statement of numerical analysis is namely that from among the NEWTON—COTES formulae it is the method of ROMBERG based on a trapezoidal formula, is the most convenient one because of its simplicity and rapid convergence [6].

Conversion (15) is not indispensable for the evaluation of the integral. It is to be noted that the integral obtained from either in ranges corresponding to the same ω, it is (17) which involves less iterations with identical accuracy and so this is also faster.

The program developed by using this procedure was tested succesfully in control circuits containing dead time elements with one and two time constants. This procedure will also be presented because of its convenience.

```
real procedure romberg (a, b, f, eps, ord);
value a, b, eps, ord; integer ord; real a, b, eps;
real procedure f;
begin real q, s, f0, i1, i2, x, delta;
integer j, k, p; array t[1 : ord+1];
s: = b—a; x: = a; i1: = f(x); x: = b;
t[1]: = (i1 + f(x))*s/2; i1: = 0;
for k: = 1 step 1 until ord do begin i2: = 0;
s: = s/2; p: = 2 ↑ k;
for j: = p—1 step —2 until 1 do begin x: = j/p;
x: = b + (a — b) * x; f0: = f(x); i2: = i2 + f0 end;
i2: = t[k + 1]: = t[k]/2 + i2*s; q = 1;
for j: = k step —1 until 1 do begin q: = 4*q;
i2: = t[j]: = (i2*q — t[j])/(q — 1) end;
delta: = abs((i2 — i1)/i2);
if delta le eps then goto fine; i1: = i2 end;
fine: romberg: = i2
end romberg;
```

Here **real procedure** f is that producing the function to be integrated from the system parameters, so it corresponds to the transformer **procedure** in the previous case.

With the developed procedures also the quadratic integral criterion weighted by the square of the time

$$I' = \int_{-\infty}^{\infty} t^2 x^2(t)\, dt \tag{18}$$

can be evaluated, as

$$I' = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \chi(-s)\, \chi(s)\, ds = \frac{1}{\pi} \int_{0}^{\infty} \Phi^2(\omega)\, d\omega \tag{19}$$

where

$$\chi(s) = -\frac{d}{ds} X(s)\,. \tag{20}$$

Here formula (19) corresponds fully to the previous ones, so the numerator and denominator of $\chi(s)$ are now the polynomials $C(s)$ and $D(s)$ respectively.

Thus, differentiating (be it by a computer) according to (20) in advance, the procedures may be used directly in this case too.

The described procedures may be very useful in designing control systems, as recently the computer analysis of such problems is gaining more and more ground.

These procedures may be employed also by themselves, but their best application would be to incorporate them into multivariable extreme value finding programs, as in this way the optimum parameters of the control system with a given structure could also be obtained.

Our further research is being carried out in this direction.

Summary

ALGOL procedures have been described for the evaluation of the *quadratic integral criterion* of response functions produced under the effect of deterministic signals in single-variable linear control systems containing elements with constant, concentrated parameters and others with dead time.

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