INVESTIGATION OF NONLINEAR SYSTEMS WITH STOCHASTIC SIGNALS

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The purpose of the investigation is to find a linear model approximating best the hardly manageable nonlinear system.

The various theoretical methods [2, 3, 4] developed for the linearization of nonlinear systems are mainly of theoretical significance, employable only in cases of very simple nonlinear systems.

The correlator technique to be discussed here removes this limitation and permits simple and fast linearization even in the most complicated cases.

One of the linearization methods is the statistical linearization based on the determination of the statistically equivalent gains. Such gains are the quotients of the mean values of the output and input stochastic signals, or the ratios of the corresponding standard deviations

$$\frac{m_x}{m_y} = K_M \tag{1}$$

$$\frac{\sigma_x}{\sigma_y} = K_S \tag{2}$$

By combining K_M and K_S a resultant gain may be derived [2], which satisfies the requirements best. Real control systems receive, besides the deterministic signals, also disturbances. Noise is necessarily present in every system for physical reasons. These disturbances can be taken into consideration only by their statistical characteristics, calculable on the basis of the disturbing signal probability density function.

The mean value of the signal is:

$$\overline{y(x)} = m_y = \int_{-\infty}^{\infty} y(x)\, p(x)\, dx\,. \tag{3}$$

The variance of the signal is:

$$\sigma_y^2 = \overline{y^2(x)} - \overline{y(x)}^2 = \int_{-\infty}^{\infty} \left(y(x) - \overline{y(x)}\right)^2 \, p(x)\, dx\,. \tag{4}$$

In case of random processes met with in nature mostly a Gaussian distribution is found. The probability density function of this is:

$$p(x) = \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-1/2\,[(x-m)/\sigma]^2} = \frac{1}{\sigma}\,\varphi(z)\,. \tag{5}$$

The normal distribution function is:

$$P(x) = \frac{2}{\sqrt{2\pi}} \int\limits_{0}^{(x-m)/\sigma} e^{-z^2/2}\, dz = \Phi(z)\,. \tag{6}$$

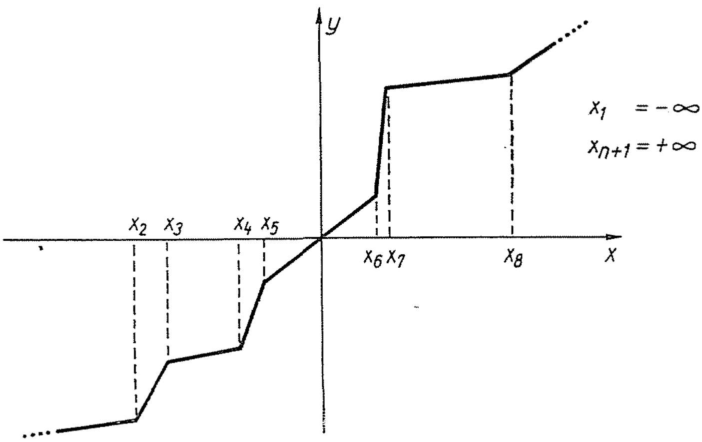

Fig. 1

Here $z = \dfrac{x-m}{\sigma}$ is the new random variable [5]. The mean value and the standard deviation of the system output signal is calculable only after the linearization of the nonlinear elements appearing in the control systems.

The calculation may be carried out in the general case as follows [3, 4]: The nonlinear characteristic is approximated by the straight-line pieces shown in Fig. 1, then the statistical characteristics of the individual sections are calculated one by one.

$$m_y = \sum_{k=1}^{n} \int\limits_{z_k}^{z_{k+1}} y(\sigma_x z + m_x)\,\varphi(z)\,dz \tag{7}$$

and

$$\sigma_y^2 = \sum_{k=1}^{n} \int\limits_{z_k}^{z_{k+1}} [\,y(\sigma_x z + m_x)]^2\,\varphi(z)\,dz - m_y^2\,. \tag{8}$$

The calculation is more complicated when the output signal of the nonlinear element depends on the time derivative of the input signal. In this case the latter may be treated as an independent input signal. E.g. be the nonlinearity the hysteresis loop with limitation shown in Fig. 2. Now the output signal may be given as a function of both input signals:

$$y(x) = y_{11}(x)y_{12}(\dot{x}) + y_{21}(x)y_{22}(\dot{x}) \tag{9}$$

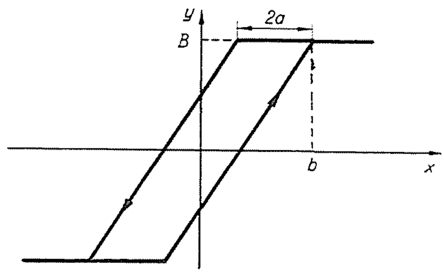

Fig. 2

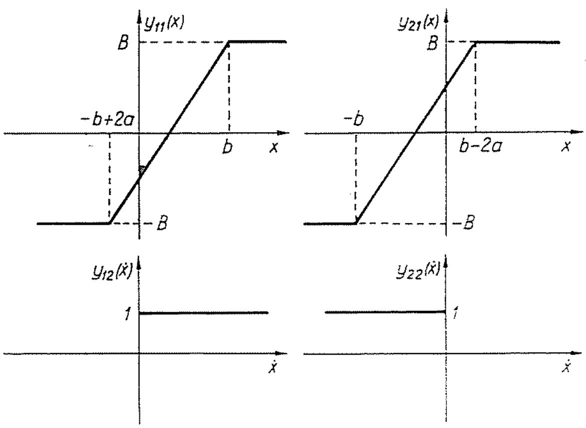

Fig. 3

where $y_{11}(x)$, $y_{12}(x)$, $y_{21}(x)$, $y_{22}(x)$ represent the auxiliary functions shown in Fig. 3 obtained on resolving the nonlinearity. The mean values and the variances of the y_{ij} signals are calculable and from these the mean value and the standard deviation of the original signal — considering that the y_{ij} signals are independent by pairs [3] — may be given:

$$m_y = m_{11}m_{12} + m_{21}m_{22} \tag{10}$$

where e.g. m_{11} is expressed as

$$m_{11} = \int_{-\infty}^{\infty} y_{11}(x)\, p_x(x)\, dx. \tag{11}$$

The variance in general is:

$$\sigma_y^2 = \sum_{i=1}^{2} \sum_{k=1}^{2} m_1^{(ik)}\, m_2^{(ik)} - m_y^2. \tag{12}$$

Here e.g. $m^{(ik)}$ is calculated as:

$$m_1^{(ik)} = \int_{-\infty}^{\infty} y_{i1}(x)\, y_{k1}(x)\, p_x(x)\, dx. \tag{13}$$

For the actual calculation it is recommended to introduce the functions $\varphi(z)$ and $\Phi(z)$, as these are available in tabulated form for different values of z. The following relations may simplify the calculations:

$$\int_{z_K}^{z_{K+1}} \varphi(z)\, dz = \frac{1}{2}\left[\Phi(z_{K+1}) + \Phi(z_K)\right] \tag{14}$$

$$\int_{z_K}^{z_{K+1}} z\varphi(z)\, dz = \varphi(z_K) - \varphi(z_{K+1}) \tag{15}$$

$$\int_{z_K}^{z_{K+1}} z^2 \varphi(z)\, dz = z_K\, \varphi(z_K) - z_{K+1}\, \varphi(z_{K+1}) + \frac{1}{2}\left[\Phi(z_{K+1}) - \Phi(z_K)\right]. \tag{16}$$

On carrying out the calculations we obtain the following results:

$$\begin{aligned} K_M = \frac{m_y}{m_x} = \frac{B}{4m_x} \Bigg\{ &\left(1 + \frac{m_x - a}{b - a}\right)\left[\Phi\left(\frac{b + m_x - 2a}{\sigma_x}\right) + \Phi\left(\frac{b + m_x}{\sigma_x}\right)\right] + \\ &+ \left(\frac{m_x + a}{b - a} - 1\right)\left[\Phi\left(\frac{b - m_x}{\sigma_x}\right) + \Phi\left(\frac{b - m_x - 2a}{\sigma_x}\right)\right] + \\ &+ \frac{2\sigma_x}{b - a}\left[\varphi\left(\frac{b + m_x}{\sigma_x}\right) + \varphi\left(\frac{m_x + b - 2a}{\sigma_x}\right) - \varphi\left(\frac{b - m_x - 2a}{\sigma_x}\right) - \varphi\left(\frac{b - m_x}{\sigma_x}\right)\right]\Bigg\}. \end{aligned} \tag{17}$$

It is seen by this expression that although the mean value of the input signal is zero, that of the output signal may differ from zero. The cause may be that the nonlinearity input signal — not regarding a great initial shift — varies between the values $\pm k (k < a)$. If the initial shift is great enough

($> \pm a$), i.e. it causes a nonzero output signal value, then $m_y \neq 0$ is obtained with $m_x = 0$, if the period is long enough. On the contrary, if $2a \rightarrow 0$, then $m_y \rightarrow 0$.

The result obtained for the hysteresis with limitation gives at the limit $2a \rightarrow 0$ the values referring to the simple static limitation, which may be calculated by a simpler method as well on the basis of (7) and (8). For the value of K_S a very complicated relation is obtained. In this case too, at the limit $2a \rightarrow 0$, the related results of the static limitation are obtained. Fig. 4 shows the curve $K_S(\sigma_x)$ calculated under the assumption of $m_x = 0$.

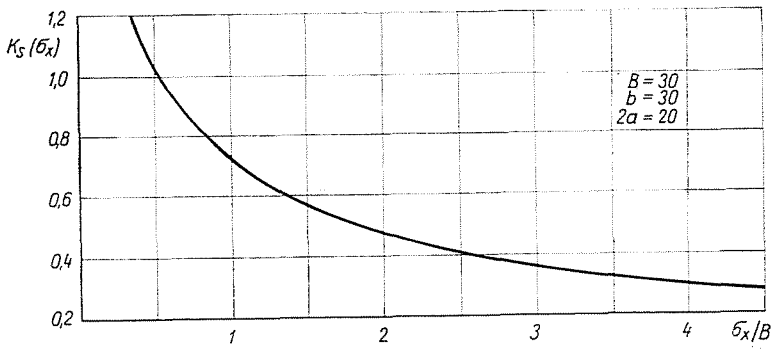

Fig. 4

The stochastic output signal of the control system may be determined, i.e. in the present case the mean value and the variance of the output signal may be calculated after the linearization of the nonlinear elements existing in the control system.

For simplicity's sake be $m_x = 0$ in the cases under investigation, so $K_M = 0$. The standard deviation of the output signal may be calculated in this case by the relation

$$\sigma_y^2 = \frac{1}{2\pi j} \int_{-\infty}^{\infty} \Phi_{yy}(j\omega)\, dj\omega \,. \tag{18}$$

Here Φyy is the power density spectrum.

We assume in the theoretical studies a white noise of a Gaussian amplitude distribution as input signal. The results obtained theoretically do not always hold in reality. One reason of this is that the assumption of the stationarity of noises and signals, i.e. of the time constancy of their statistical characteristics is not perfectly correct; another reason is that the signal assumed to be of a Gaussian amplitude distribution has no pure normal distribution. This is why the process employing correlation calculus in studying nonlinear systems gained ground, as it supplies more accurate results in a much simpler way and the accuracy of the results is not influenced by the distribution being not purely normal. The principle of the method is that the mean value and the standard deviation of the signal is determined from the plotted autocorrelation functions.

As known, the autocorrelation function of a signal is

$$\varphi_{yy}(\tau) = \int_{-\infty}^{\infty} y(t)\, y(t-\tau)\, dt. \tag{19}$$

The value assumed at $\tau = 0$ of the autocorrelation function gives the mean square value of the signal under investigation, and in case of $\tau \to \infty$ the square of the expected value of the signal is obtained. The variance is given by the difference of these values [1]. A signal of known mean value and standard deviation is fed into the input of the tested system, or if these parameters are unknown, the autocorrelation function of the input signal is plotted simultaneously with the autocorrelation function of the system output signal. The statistical characteristics may be calculated from the values obtained at $\tau = 0$ and $\tau \to \infty$ of the plotted curves.

Let us apply this method now to the linearization of a maximum second degree "0"-type or "1"-type control system containing a static limitation or hysteresis. Let us have again a white noise of Gaussian amplitude distribution for input signal. The type NOR 5004 random signal generator used to provide this signal generates a low-frequency binary noise, from which the test signal may be produced by a low pass filter. The noise obtained in this way may be used to advantage for investigations carried out with ISAC correlators.

The power density spectrum of the noise is:

$$\Phi_{xx}(\omega) = \frac{A^2}{\lambda} \cdot \frac{1}{1+\omega^2/4\lambda^2} \cdot \frac{1}{1+\omega^2 T^2} \tag{20}$$

where λ is the average number of impulses per unit time, and $T = RC$ the time constant of the filter.

The value of the autocorrelation function $\varphi_{xx}(\tau)$ at $\tau = 0$ is

$$\varphi_{xx}(0) = A^2 \frac{\alpha}{1+\alpha} \tag{21}$$

where $\alpha = \dfrac{1}{2\lambda T}$.

This constitutes also the variance of the input signal in reality, as the mean value of the signal is zero.

The analogue computer used for the measurement is a type TY 1351 Solartron Analogue Tutor equipment. The records permit to determine the standard deviations of the signals and these can be compared with the theoretical values.

First we compare the calculated values of the static limitation with those obtained on the basis of the auto-correlation functions. Fig. 5 shows the characteristic curves $K_S(\sigma_x/B)$ obtained theoretically and practically. By comparing both curves it is seen that there is a good agreement between the theoretical and the practical results. The two results obtained by different methods show a deviation of merely 5%, when $\sigma_x/B \geq 1$. The maximum deviation is found around the value $\sigma_x/B \simeq 0.5$. A deviation of about 2%

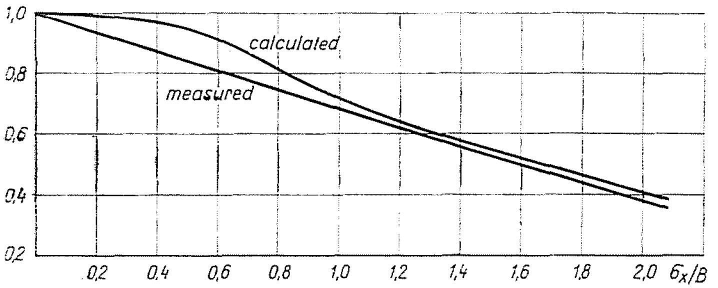

Fig. 5

is due to the fact that recording the process did not last for an infinite time, but only for about 80 sec. Also the theoretical result may be considered to be accurate only within 1—2% in consequence of the inaccuracies in the course of the calculation.

In the second case let us investigate the results obtained for the pure hysteresis. The obtained correlation functions and the results of the theoretical calculations may be illustrated by the σ_y (σ_x) curves shown in Fig. 6. There is an obviously essential, apparently constant deviation between both curves. This is mainly due to the fact — as proved by later investigations — that the analogue computer model does not reproduce the hysteresis with an adequate accuracy. If we disregard this apparently constant error, we obtain again the theoretical value. The formulae given for the calculation of the assumed transfer factor K_S are correct.

In order to establish whether K_S could be considered a gain in case of $K_M = 0$, in reality it is not sufficient to take into consideration the nonlinear element alone, but more complex systems must be investigated. Let us calculate the standard deviation of the system output signal, with the calculated gain K_S. If this corresponds to the value obtained by the correlation functions, then K_S is really the gain of the nonlinear element,

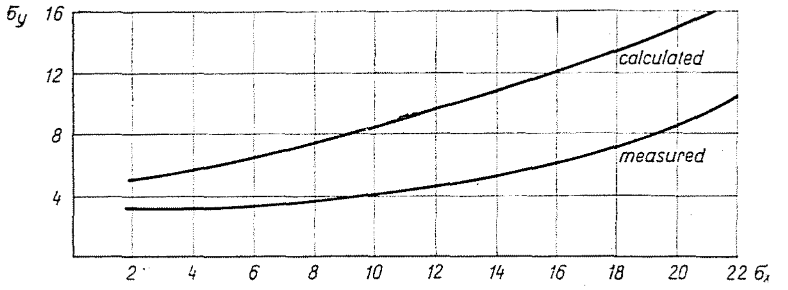

Fig. 6

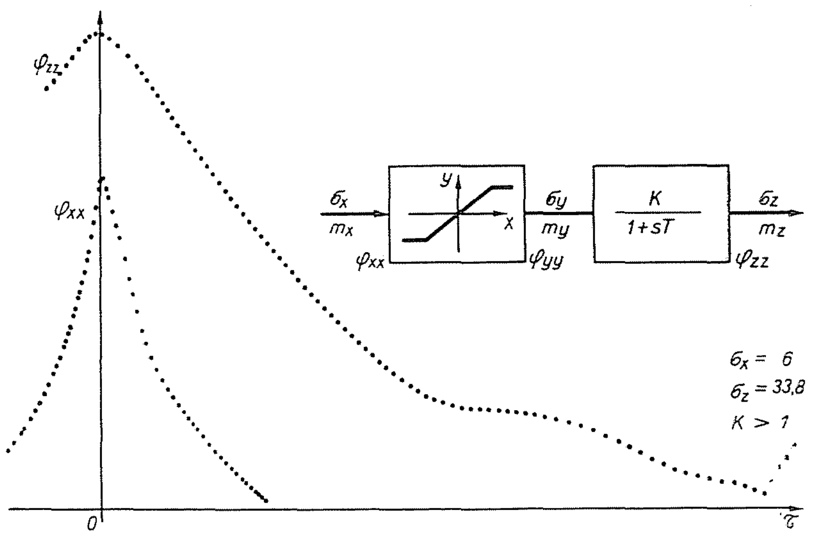

Fig. 7

inasmuch as the system is concerned. If the output signal standard deviations obtained theoretically and practically do not coincide, then the real gain is not K_S, but a more complicated function of the same. Conceivably it is sufficient to employ the approximation $(aK_S + bK_S^2)$ to calculate the output signal standard deviation. The values a and b may be determined by comparing the results.

The autocorrelation functions recorded in the individual cases are shown in Figs 7 through 10. This paper ignored the case where the mean value of the input signal was not zero. In this case, as it was seen, the calculations became much more complicated. Also the determination of the exact gain awaits solution, as it was calculated only approximately. On the other hand a general method has been demonstrated for determining the factors K_S and K_M of open, and closed systems, respectively.

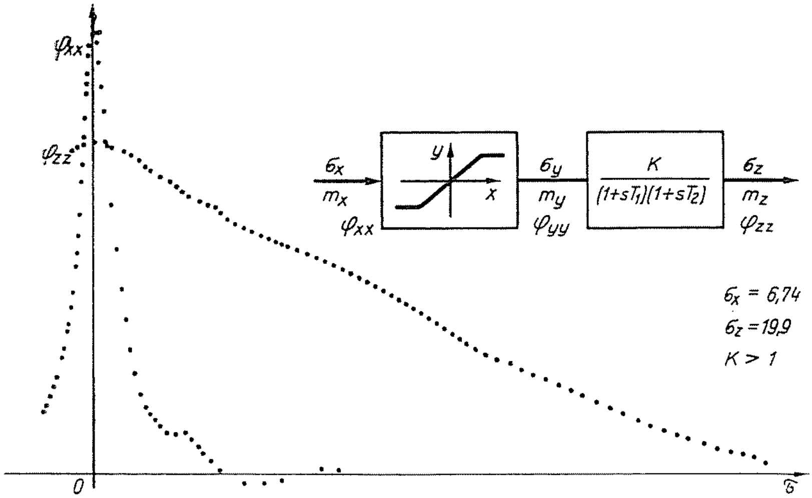

Fig. 8

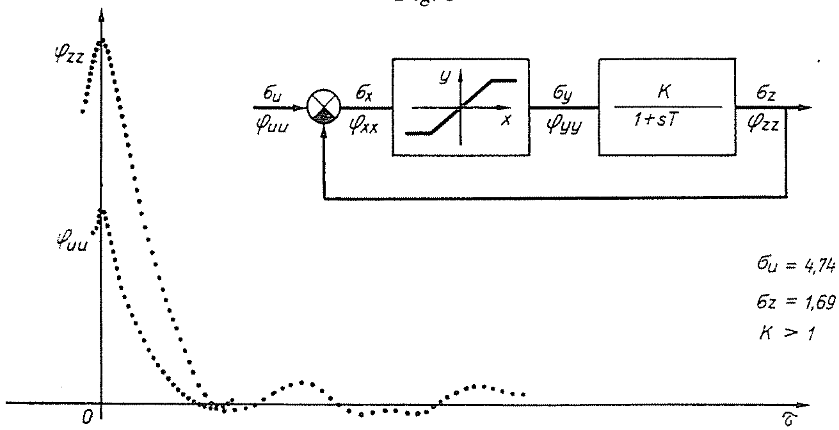

Fig. 9

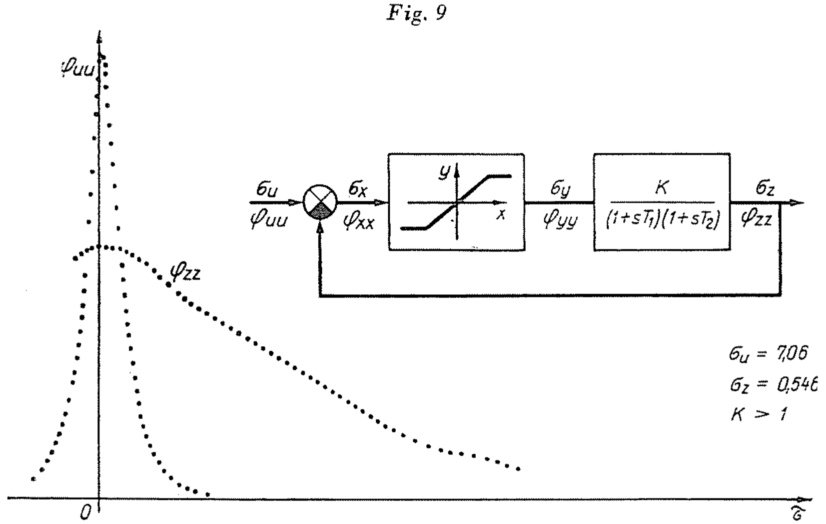

Fig. 10

Summary

This paper presented a comparison of theoretical and measured results of investigations of nonlinear systems by stochastic signals.

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