SYNTHESIS OF RC LADDER NETWORKS WITH A MINIMUM OF CAPACITANCES*

PART II

By

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Introduction

A method of synthesis of an RC ladder network with a given transfer voltage ratio by removal of poles will be presented. The zero shifting is done only by resistance, applying a minimum of capacitances for the synthesis.

The high-pass and low-pass ladder network tested in the first part of the paper is shown in Fig. 1. A short summary of the results is given here. The transfer function of an RC ladder network can be written in the following form:

$$G(s) = K \frac{(s+\alpha_1)(s+\alpha_2)\dots(s+\alpha_m)}{(s+\beta_1)(s+\beta_2)\dots(s+\beta_n)} = K \frac{A(s)}{B(s)}, \tag{1}$$

where

$$m \leq n$$
$$0 \leq \alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_m$$
$$0 < \beta_1 < \beta_2 < \dots < \beta_n.$$

The parameter z_{11} of the two-port can be expressed as:

$$z_{11}(s) = H \frac{(s+\beta_1)(s+\beta_2)\dots(s+\beta_n)}{(s+\gamma_1)(s+\gamma_2)\dots(s+\gamma_n)}, \tag{2}$$

where

$$0 \leq \gamma_1 \leq \beta_1 \leq \gamma_2 \leq \beta_2 \leq \dots < \beta_n$$

The parameter y_{22} can be expressed as:

$$y_{22}(s) = H \frac{(s+\beta_1)(s+\beta_2)\dots(s+\beta_n)}{(s+\gamma_1)(s+\gamma_2)\dots(s+\gamma_n)}, \tag{3}$$

where

$$0 < \beta_1 < \gamma_1 < \beta_2 < \gamma_2 < \dots < \gamma_n.$$

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The transfer function (1) can be realized by the high-pass network in Fig. 1*a* if and only if $m = n$ and the parameter z_{11}, given in (2), fulfills the following condition:

$$\varkappa_i \leq \gamma_i \qquad (i = 1, 2, \ldots, n) . \tag{4}$$

Hence, the condition for the transfer function:

$$\varkappa_i < \beta_i \qquad (i = 1, 2, \ldots, n) \tag{5}$$

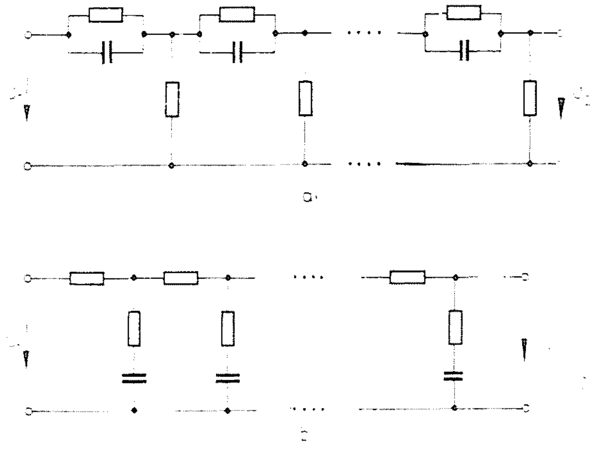

Fig. 1

The transfer function (1) can be realized by the low-pass ladder network in Fig. 1*b* if and only if the parameter y_{22} of the network, given in (2), fulfills the following condition:

$$\varkappa_i \geq \gamma_i \qquad (i = 1, 2, \ldots, m). \tag{6}$$

Hence, the condition for the transfer function:

$$\varkappa_i > \beta_i \qquad (i = 1, 2, \ldots, m). \tag{7}$$

In both cases, the method of synthesis was given for proving the above-mentioned statements.

The synthesis of the general transfer function

If the transfer function satisfies neither condition (5) nor (7) the synthesis can be done as follows. Let us factorize the transfer function to be realized as product of two factors, satisfying conditions (5) and (7), respectively:

$$G(s) = G'(s)\,G''(s)\,, \tag{8}$$

where

$$G'(s) = K\,\frac{(s+\alpha'_1)\,(s+\alpha'_2)\,\ldots\,(s+\alpha'_g)}{(s+\beta'_1)\,(s+\beta'_2)\,\ldots\,(s+\beta'_h)} = K\,\frac{A'(s)}{B'(s)}$$

$$G''(s) = \frac{(s+\alpha''_1)\,(s+\alpha''_2)\,\ldots\,(s+\alpha''_k)}{(s+\beta''_1)\,(s+\beta''_2)\,\ldots\,(s+\beta''_k)} = \frac{A''(s)}{B''(s)}\,.$$

The quantities α'_i, β'_i, α''_i and β''_i have subscripts in increasing order. According to what has been shown above:

$$\begin{aligned} \alpha'_i &> \beta'_i \qquad (i = 1, 2, \ldots g) \\ \alpha''_i &< \beta''_i \qquad (i = 1, 2, \ldots k)\,. \end{aligned} \tag{9}$$

It is easy to see that there are always several means of factorization. Realizing transfer functions $G'(s)$ and $G''(s)$, connecting the two networks in cascade so that the impedance level of the second is chosen much higher than that of the first, the transfer function of this network approximates the transfer function to be realized. If, however, a great difference between the impedance levels of the two network parts is undesirable and the transfer function is to be realized accurately, then another method of synthesis is needed.

It is easy to prove the correctness of the transfer function of the two two-ports in cascade (Fig. 2) in the following form:

$$G = \frac{U_2}{U_1} = -\,\frac{y'_{21}\,z''_{21}}{1+y'_{22}\,z''_{11}}\,. \tag{10}$$

For a low-pass ladder network N' with zeros at $-\alpha'_i$ and a high-pass ladder network N'' with zeros at $-\alpha''_i$, the parameters of the two tow-ports can be written as:

$$\begin{aligned}
y'_{21} &= -\,L'\,\frac{(s+\alpha'_1)\,(s+\alpha'_2)\,\ldots\,(s+\alpha_g)}{(s+\delta'_1)\,(s+\delta'_2)\,\ldots\,(s+\delta'_h)} = -\,L'\,\frac{A'(s)}{D'(s)} \\
y'_{22} &= H'\,\frac{(s+\varepsilon'_1)\,(s+\varepsilon'_2)\,\ldots\,(s+\varepsilon'_h)}{(s+\delta'_1)\,(s+\delta'_2)\,\ldots\,(s+\delta'_h)} = H'\,\frac{E'(s)}{D'(s)} \\
z''_{11} &= H''\,\frac{(s+\varepsilon''_1)\,(s+\varepsilon''_1)\,\ldots\,(s+\varepsilon''_k)}{(s+\delta''_1)\,(s+\delta''_2)\,\ldots\,(s+\delta''_k)} = H''\,\frac{E''(s)}{D''(s)} \\
z''_{21} &= L''\,\frac{(s+\alpha''_1)(s+\alpha''_2)\,\ldots\,(s+\alpha''_k)}{(s+\delta''_1)(s+\delta''_2)\,\ldots\,(s+\delta''_k)} = L''\,\frac{A''(s)}{D''(s)}\,.
\end{aligned} \tag{11}$$

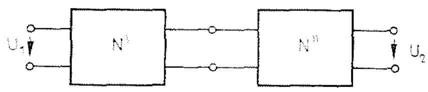

Fig. 2

The following conditions are to be fulfilled (subscripts of quantities δ_i', δ_i'', ε_i' and ε_i'' indicate increasing orders):

$$0 < \varepsilon_1' < \delta_1' < \varepsilon_2' < \delta_2' < \ldots < \delta_h' \tag{12a}$$

$$\varkappa_i' \geq \delta_i' \quad (i = 1, 2, \ldots g) \tag{12b}$$

$$0 \leq \delta_1'' < \varepsilon_1'' < \delta_2'' < \varepsilon_2'' < \ldots < \varepsilon_k'' \tag{12c}$$

$$\varkappa_i'' \leq \delta_i'' \quad (i = 1, 2, \ldots k)\,. \tag{12d}$$

Denote:

$$\begin{aligned} A(s) &= A'(s)\,A''(s) \\ D(s) &= D'(s)\,D''(s) \\ E(s) &= E'(s)\,E''(s) \end{aligned} \tag{13a}$$

$$\frac{H_1}{H_2} = H'\,H'' \quad \text{at the same time} \quad H_1 + H_2 = 1\,. \tag{13b}$$

Substituting the two-port parameters given in (11) into (10) the following expression for the transfer function arises:

$$G(s) = \frac{H_1 L' L'' A(s)}{H_1 D(s) + H_2 E(s)}\,. \tag{14}$$

Comparing (14) with (1) yields the relationship:

$$H_1 D(s) + H_2 E(s) = B(s) \tag{15}$$

(15) will be used to determine the quantities δ_i', δ_i'', ε_i' and ε_i'' as well as H_1 and H_2, needed for the synthesis. This means $(2n + 2)$ unknowns altogether, but (15) gives only $(n + 1)$ equations, hence $(n + 1)$ unknowns may be arbitrarily chosen, only that besides (15), conditions (12) must be fulfilled. This problem can be solved as follows.

The $(n + 1)$ arbitrary unknowns will be the quantities H_1, δ_i' and δ_i''. Quantities δ_i' and δ_i'' are to be chosen so as to satisfiy, besides (12b) and (12d) the following conditions:

$$\beta_1' < \delta_1' < \beta_2' < \beta_2' < \ldots < \delta_h' \tag{16a}$$

$$0 \leq \delta_1'' < \beta_1'' < \delta_2'' < \beta_2'' < \ldots \beta_k'\,. \tag{16b}$$

Such a choice is always possible because of (9). H_1 should be in the interval

$$0 < H_1 < \min \left\{ 1 ; \frac{B(0)}{D(0)} ; h_0 \right\} , \tag{17}$$

where h_0 represents the smallest positive value of h, if any, where the equation $B(s) - hD(s) = 0$ has a double root on the negative real axis.

Below it will be proved that choosing H_1 and the polynomial $D(s)$ in the way previously described, the polynomial $E(s)$ determined by (15) satisfies the conditions (12a) and (12c). In the limiting case where $H_1 = 0$, $E(s) = B(s)$ and so (12a) and (12c) are satisfied, too, because of (16). As the zeros of the polynomial $E(s)$ can migrate only continously if H_1 is changed, it is sure that for sufficient small values of H_1 the conditions (12a) and (12c) are satisfied. Let us determine the upper limit of H_1. If $B(s)$ and $D(s)$ happen to have a common zero this will be a zero of $E(s)$, too, at any value of H_1, i.e. the zero does not migrate with increasing H_1, but remains in the interval defined by (12a) and (12c), respectively. The other zeros of $E(s)$, confined by two zeros of $D(s)$, remain in the interval defined by these two zeros as long as they do not turn into complex values. Namely none of such zeros of $E(s)$ can reach a zero of $D(s)$ for a finite value of H_1, because then it would be a zero of $B(s)$, too, a special case dealt with separately. A root of $E(s)$ can have a complex value only if two of its roots have previously coincided on the real axis, requiring the condition $H_1 \leq h_0$. Prescribe that $H_1 < B(0)/D(0)$ and $H_1 \leq 1$, so that no zero of $E(s)$ should get to the positive axis through the origin and so through the infinity, respectively.

Now let us determine h_0. The equation $B(s) - hD(s) = 0$ can have a double root at a given point only if the equation $\mathrm{d}B(s)/\mathrm{d}s - h[\mathrm{d}D(s)/\mathrm{d}s] = 0$ has a root there, too. Eliminating h from the two relationships we get the following equation for the locus of the contingent double root:

$$B(s)\frac{\mathrm{d}D(s)}{\mathrm{d}s} - D(s)\frac{\mathrm{d}B(s)}{\mathrm{d}s} = 0 . \tag{18}$$

The negative roots of this equation and the pertaining h values are to be determined and the smallest of these values gives h_0. Evidently roots of (18) are of interest only in intervals defined by two adjacent zeros of $B(s)$ between which $D(s)$ has no zero, and in which intervals $B(s)$ and $D(s)$ have the same sign. This simplifies the determination of h_0. Notice that the factorization given in (8) and the choice of $D(s)$ can be always made so that $D(s)$ has a zero between any two adjacent zeros of $B(s)$, and so one need not to be concerned with determining h_0.

The particular steps of the synthesis are:

1. To factorize according to (8) the transfer function to be realized into two factors, which satisfy conditions (9).

2. Choosing polinomials $D'(s)$ and $D''(s)$ as to satisfy conditions (12b), (16a) and (12d), (16b), respectively.

3. Choosing H_1 in the interval defined by (17).

4. In knowledge of H_1 and the polynomial $D(s) = D'(s)\,D''(s)$ determination of the polynomial $E(s)$ according to (15).

5. Factorizing the polynomial $E(s)$ to the product of two polynomials ($E(s) = E'(s)\,E''(s)$) so that $E'(s)$ and $E''(s)$ satisfy conditions (12a) and (12c), respectively.

6. Choosing any of quantities H' and H'' and determining the other according to (13b).

7. Writing parameters y_{22} and z_{11} according to (11), and realization of both parts of the network by the method given in the first part of this paper.

Equivalent networks

To compare the equivalent networks on the basis of the constant K in the transfer function (1), let us determine the expression for K. From (14):

$$K = H_1 L' L''. \tag{19}$$

As the two-port does not attenuate at zero frequency, when the output terminal is open-circuited:

$$\frac{y'_{21}(0)}{y'_{22}(0)} = \frac{L'}{H'} \cdot \frac{\prod_i \alpha'_i}{\prod_i \varepsilon'_i} = 1\,. \tag{20}$$

If the shunt arms of the ladder network contain only zeros at infinity, i.e. no α'_i is defined, (20) takes the following form:

$$\frac{y'_{21}(0)}{y'_{22}(0)} = \frac{L'}{H'} \cdot \frac{1}{\prod_i \varepsilon'_i} = 1\,. \tag{20a}$$

This will affect subsequent relationships, but will not be discussed here in detail. From (20):

$$L' = H' \frac{\prod_i \varepsilon'_i}{\prod_i \alpha'_i}\,. \tag{21}$$

The two-port N'' does not attenuate at infinite frequency, when the output terminal is open-circuited, hence,

$$L'' = H''. \tag{22}$$

Substituting (21) and (22) into (19) and applying (13b):

$$K = (1 - H_1)\frac{\prod_i \varepsilon_i'}{\prod_i \alpha_i'}. \tag{23}$$

Obviously, the smaller is H_1, the greater the value of K. In limiting case where $H_1 = 0$, $\varepsilon_i' = \beta_i'$ and so

$$K = K_0 = \frac{\prod_i \beta_i'}{\prod_i \alpha_i'}. \tag{24}$$

The value of K_0 depends on the decomposition according to (8), i.e. on the grouping of α_i and β_i defined by (9). It can be shown by a somewhat lengthy argument that K_0 will be maximum for the following way of grouping: the elements of the same subscript get into similar groups, namely, for every value of i if $\alpha_i > \beta_i$, then both will belong to the group with one comma and if $\alpha_i < \beta_i$, then to the groups with two commas. With increasing H_1, K decreases monotonously, partly because of the factor $(1 - H_1)$, partly because for the above grouping the zeros of the polynomial $E(s)$ are shifted to the right along the real axis, if H_1 is increased. But choosing H_1 sufficiently small, the coefficient K may assume a value as near to the maximum K_0 belonging to the above grouping as desired.

Nevertheless, for a very low value of H_1, according to (13b) the value of the product $H'H''$ is a very high one, while if H_1 is near to unity, $H'H''$ is very small. In both cases the impedance levels of the networks N' and N'' are very different, and this is in general unfavourable. The two impedance levels will be nearly equal if $H'H'' \approx 1$, i.e. $H_1 \approx 0.5$.

Notice that if the transfer function to be realized has a zero at infinity, the order of the denominator of y_{22} may be by one less than the order of its numerator, i.e. for $D(s)$ a polynomial of order $(n - 1)$ can be chosen. The procedure remains unchanged, only (13b) changes into:

$$H'H'' = \frac{1}{H_1} \tag{25}$$

hence:

$$K = \frac{\prod_i \varepsilon_i'}{\prod_i \alpha_i}. \tag{26}$$

Finally, the same synthesis is possible if N' is a high-pass ladder network and N'' is a low-pass one. Since the steps are the same as described above, instead of details, an example will be presented.

Example 1

Let us realize the following transfer function:

$$G(s) = K \frac{(s+5)^2\,(s+9)}{(s+2)\,(s+4)\,(s+15)} \; .$$

The synthesis will be done first by the traditional method. Input impedance of the following form will be chosen:

$$z_{11}(s) = \frac{(s+2)\,(s+4)\,(s+15)}{(s+1)\,(s+3)\,(s+5)} \; .$$

To yield at once one zero at $s = -5$ without shifting resistance:

$$z_{11}(s) = \frac{a}{s+5} + Z_2(s) \; .$$

Here $a = 3.75$, thus $R_1 = 0.750$ and $C_1 = 0.267$ (see Fig. 3). Further

$$Z_2(s) = \frac{s^2+12.25\,s+21.75}{(s+1)(s+3)} \; .$$

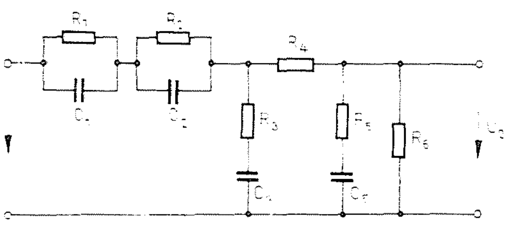

Fig. 3

The value of $Z_2(s)$ at both zeros to be realized is negative ($Z_2(-5) = -1.81$ and $Z_2(-9) = -0.156$), hence no zero shifting can be done with shifting resistance. Realize first the zero at $s = -5$ by a series RC configuration in shunt arm and for this purpose try the following decomposition of Z_2:

$$Z_2(s) = \frac{a}{s+1} + Z_2'(s).$$

Since $Z_2'(-5) = 0$, $a = 7.24$, but $\operatorname{Res}_{s=-1} Z_2(s) = 5.25$, hence $Z_2'(s)$ is not realizable. Neither the following decomposition leads to a result:

$$Z_2(s) = \frac{a}{s+3} + Z_2(s)$$

because $a = 3.62$ and $\operatorname{Res}_{s=-3} Z_2(s) = 3$. There are three possibilities:

a) Either to choose the shifting impedance in the following form:

$$Z_s(s) = \frac{a_1}{s+3} + \frac{a_2}{s-1}.$$

b) Or to try realizing the zero at $s = -5$ by a parallel RC configuration in series arm.

c) Or to try realizing the zero at $s = -9$ first. According to the last possibility Z_2 can be decomposed as:

$$Z_2(s) = \frac{1.25}{s+1} + \frac{(s+2)(s+9)}{(s+1)(s+3)}$$

and the synthesis can be continued without difficulties. The whole network is seen in Fig. 3, where $R_1 = 0.750$, $C_1 = 0.267$, $R_2 = 1.50$, $C_2 = 0.800$, $R_3 = 1.31$, $C_3 = 0.0847$. $R_4 = 3.50$. $R_5 = 0.971$. $C_5 = 0.206$. $R_6 = 0.400$. yielding $K = 0.0347$.

Now, performing the synthesis by the method shown in this paper, the quantities α_i and β_i are grouped so that K_0 should be maximum:

$$\begin{array}{ll} \alpha_1' = 5 & \alpha_2' = 5 \\ \beta_1' = 2 & \beta_2' = 4 \\ \alpha_1'' = 9 & \\ \beta_1'' = 15 & \end{array}$$

On the basis of (16) the polynomial $D(s)$ can be chosen in the following form:

$$\delta_1' = 3 \qquad \delta_2' = 5 \qquad \delta_1'' = 9$$

As $D(s)$ has a zero between any two adjacent zeros of $B(s)$, h_0 need not be determined. As $B(0)/D(0) = 0.8889$, according to (17) H_1 can be chosen in the interval:

$$0 < H_1 \leq 0.889$$

(23) yielded the value of the "multiplication factor" K for several values of H_1:

H_1	0	0.1	0.2	0.5
K	0.320	0.274	0.229	0.106

Let us present the detailed synthesis for the case $H_1 = 0.5$. Here, the impedance levels of the two parts of the network are equal. (15) yields for the polynomial $E(s)$:

$$E(s) = s^3 + 25s^2 + 109s + 105$$

Factorizing the polynomial $E(s)$ delivers the polynomials $E'(s)$ and $E''(s)$:

$$E'(s) = (s + 1.368)\,(s + 3.882) \qquad E''(s) = s + 19.75$$

On the basis of (13b):

$$H'H'' = 1.$$

Let $H' = 1$ and $H'' = 1$, so the parameters y_{22} and z_{11} are given by the following expressions:

$$y'_{22} = \frac{(s+1.368)\,(s+3.882)}{(s+3)\,(s+5)} \qquad z''_{11} = \frac{s+19.75}{s+9}\,.$$

The synthesis being easy to complete the whole network is seen in Fig. 4. The element values are:

$$\begin{array}{lll} R_1 = 1.775 & R_2 = 0.9891 & C_2 = 0.2022 \\ R_3 = 1.049 & R_4 = 2.462 & C_4 = 0.08124 \\ R_5 = 1.194 & C_5 = 0.09302 & R_6 = 1.000 \end{array}$$

For $H_1 = 0.1$ the element values are the following:

$$\begin{array}{lll} R_1 = 0.1026 & R_2 = 0.08177 & C_2 = 2.446 \\ R_3 = 0.1164 & R_4 = 0.3541 & C_4 = 0.5648 \\ R_5 = 0.7278 & C_5 = 0.1527 & R_6 = 1.000 \end{array}$$

The scatter of element values is greater, but the attenuation is smaller than before.

Inverting the place of the input and output in the network in Fig. 4 and exchanging two resistances (see Fig. 5), the transfer function of the new network has poles and zeros at the same points as the original one, in correspondence with the reciprocity theorem. If $H_1 = 0.5$ the multiplication factor of the new network is $K = 0.188$. The same network would result from synthesis with the initial assumption that the network N' (see Fig. 2) is a high-

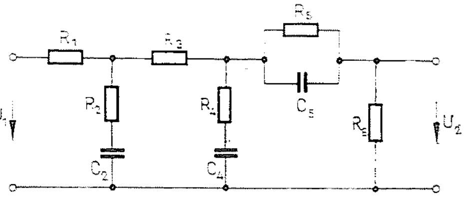

Fig. 4

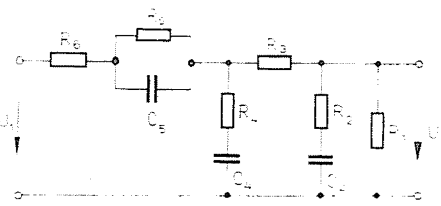

Fig. 5

pass one and N'' is a low-pass one. Accordingly, using the previous expressions for $E'(s)$ and $E''(s)$ the following choice will be made:

$$y'_{22} = \frac{s+9}{s+19.75} \quad \text{and} \quad z''_{11} = \frac{(s+3)(s+5)}{(s+1.368)(s+3.882)}$$

Now, N' and N'' will have the zero at $s = -9$ and both zeros at $s = -5$, respectively.

Let us examine what happens if we choose for H_1 the greatest possible value, i.e. $H_1 = 0.8889$. After a short calculation the following expressions for the polynomials $E'(s)$ and $E''(s)$ result:

$$E'(s) = s(s + 3.78) \qquad E''(s) = s + 49.22$$

and

$$\frac{1 - H_1}{H_1} = 0{,}125\,.$$

Now it is not possible to choose the network N' low-pass and N'' high-pass, as R_1 (see Fig. 4) would be equal to zero because $y'_{22}(0) = 0$. On the contrary:

$$y'_{22} = 8\,\frac{s+9}{s+49.22} \qquad z''_{11} = \frac{(s+3)(s+5)}{s(s+3.78)}\,.$$

The synthesis leads to the network in Fig. 6. The multiplication factor:

$$K = 0.533\,.$$

This is the highest realizable value because the network does not attenuate at zero frequancy. Obviously, the networks in Figs 4 to 6 are more advantageous from several aspects than that in Fig. 3.

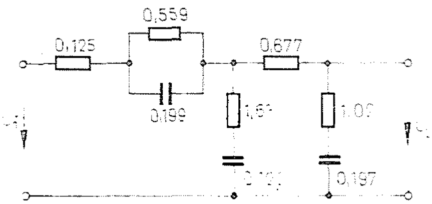

Fig. 6

Example 2

On the basis of the described procedure we have constructed a program or the computer ODRA 1013 made in Poland. This program has been applied or synthesis in case of a complicated transfer function. The zeros of the transfer unction are:

$$z_1 = z_2 = z_3 = z_4 = 0 \quad z_5 = z_6 = z_7 = z_8 = -0.5 \quad z_9 = z_{10} = -1\,.$$

The poles of the transfer function are:

$$\begin{array}{lllll} p_1 = -0.05 & p_2 = -0.1 & p_3 = -0.18 & p_4 = -0.3 & p_5 = -0.4 \\ p_6 = -0.55 & p_7 = -0.62 & p_8 = -0.7 & p_9 = -0.8 & p_{10} = -0.92\,. \end{array}$$

The synthesis resulted in the nerwork shown in Fig. 7.

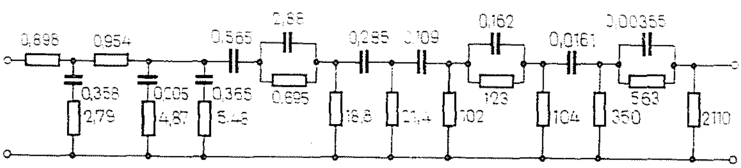

Fig. 7

Summary

Every transfer function with negative real poles and nonpositive real zeros was shown to be realizable by an RC ladder network with as many capacitances as the degree of the denominator of the transfer function, and a method of realization was presented. The essential step of the synthesis was to factorize the transfer function to be realized to the product of two factors, having a low-pass and a high-pass magnitude characteristic, respectively. Accordingly the network was divided into two parts connected in cascade. The low-pass one contained capacitances only in the shunt arms and the high-pass one only in the series arms. The parameters of the two parts could be determined from the transfer function. The two parts could be realized by the method of the removal of poles using resistances to the zero shifting only. An additional possibility with the described method is to optimize the value of the gain to a certain extent.

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