EXACT AND APPROXIMATE SOLUTIONS OF THE STORAGE TIME OF DIODES WITH NON-UNIFORM BASE DOPING

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(Received November 10, 1969)

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Introduction

It is rather complicated to calculate the storage time of diodes with non-uniform base, and even impossible without computers in case of a general doping profile. A widely used method is to determine the charge, created by accumulating minority carriers in the base under statical conditions, and from the time-dependence of this charge to calculate the total recovering time. No accurate result is obtained else than if the reverse current is first constant, then drops to zero. The reverse current of real diodes has never such a shape, the constant-current period is always followed by a phase with decreasing current. The better the real recovery waveform approaches the supposed one, the exacter the solution given by the method mentioned above.

The fall time of the step-recovery diodes is very short, thus MOLL, KRAKAUER and SHEN [1] applied the method mentioned before.

In what follows, an exact solution of the storage time will be presented and its result compared to those of some approximate solutions. For sake of simplicity, in the base a constant field strength is supposed and the effect of high-level injection is neglected. In fact, the field depends more or less on the distance in the base and for higher currents the effect of high-level injection cannot be neglected. Considering all these problems would make the calculation unduly complicated. In the construction of diodes other parameters must be taken into consideration too: for example, the capacitance of the depletion layer, the break-down voltage etc. These problems are not discussed here. A $p^{+}n$ structure is supposed, thus the current consist almost entirely of holes, penetrating the n region. In real diodes the electrons entering the p type region cannot sometimes be ignored. In this case the calculation process is similar, but involving both types of carriers.

For sake of simplicity a one-dimensional model is discussed.

Exact solution

The behaviour of holes in the base is characterized by the continuity equation:

$$\frac{\partial p(x,t)}{\partial t} = D_p \frac{\partial^2 p(x,t)}{\partial x^2} - \frac{qE}{kT} D_p \frac{\partial p(x,t)}{\partial x} - \frac{p(x,t) - p_n}{\tau_p} . \tag{1}$$

where $p(x, t)$ is the density of holes, depending on the time and distance ($x = 0$ being the edge of the depletion region); D_p is the diffusion constant of the holes; E is the field strength in the base ($E =$ const.). τ_p is the life time of holes, k is the Boltzmann-constant and T is the absolute temperature in Kelvin degrees.

Denote the current density in the forward direction by I_f, in the reverse direction by I_r. The initial condition is determined by I_f, for at the beginning of the switching-off, the hole distribution in the base is equal to that created by I_f in the forward direction.

During the switching-off the boundary condition at $x = 0$ is determined by the reverse current, I_r. Supposing a wide-base diode, the other boundary condition is:

$$p(\infty, t) = p_n \tag{2}$$

p_n is the hole concentration in the n side at equilibrium.

If at $x = 0$ the hole concentration decreases to zero, the constant-current period is off, because the boundary conditions change. Thus the storage time t_s can be determined from the condition $p(0, t_s) = 0$.

To ease calculations, Eq. (1) is to be transformed according to EREMIN, MOKEIEV and NOSOV [2]. Introducing the new variables $T = t/\tau_p$, $X = x/L_p$ and $\Delta p = p - p_n$ ($L_p = \sqrt{D_p \tau_p}$ the diffusion length) and applying the transformation:

$$U(X, T) = \Delta p(X, T) \exp\left\{(1 + E_n^2)T - E_n X\right\} \tag{3}$$

$\left(E_n = \frac{q}{kT} \frac{EL_p}{2}\right.$ is the normalized field strength$\Big)$ the continuity equation can be written as:

$$\frac{\partial U(X,T)}{\partial T} = \frac{\partial^2 U(X,T)}{\partial X^2} . \tag{4}$$

The linear combination of the solutions of this linear, partial differential equation also gives a solution, hence the principle of superposition can be applied. After splitting the switching-off waveform (Fig. 1) into two parts, solving Eq. (4) separately for both of them, finally subtracting one from the other, we get the transformed hole distribution:

$$U(X, T) = U'(X, T) - U''(X, T).$$

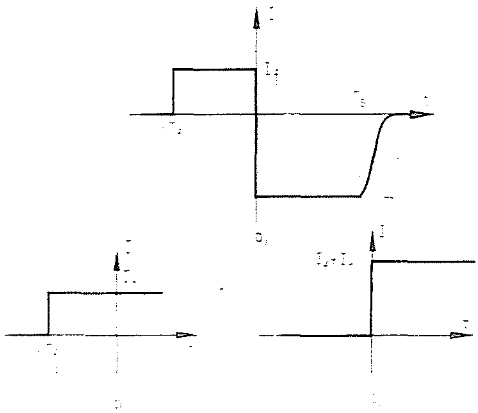

Fig. 1. Disintegration of the switch-in and switch-out waveforms of the diode

The current in Fig. 1*a* is the difference of the currents in Fig. 1*b* and Fig. 1*c*. The advantage of this method is to yield simpler results for currents I_r and $I_f + I_r$ because of the changed initial conditions:

$$U'(X, -T_f) = 0 \quad \text{and} \quad U''(X, 0) = 0. \tag{5}$$

Calculate first the hole distribution caused by the current in Fig. 1c. The total current at $X = 0$ is the sum of the diffusion and drift currents. Thus the boundary condition is:

$$\begin{aligned} I_f + I_r &= q\left(-D_p \frac{\partial \Delta p''}{\partial x} + \mu \Delta p'' E\right)\Big|_{x=0} = \frac{qD_p}{L_p}\left(-\frac{\partial \Delta p''}{\partial X} + 2E_n \Delta p''\right)\Big|_{X=0} = \\ &= \frac{qD_p}{L_p}\left(-\frac{\partial U''}{\partial X} + E_n U''\right)\Big|_{X=0} \exp\{-(1+E_n^2)T\}, \end{aligned} \tag{6}$$

where p is replaced by U making use of Eq. (3).

The solution of Eq. (4) by the initial condition (5) and boundary condition (6) is calculated in the Appendix applying Laplace-transformation. Here only the solution at $X = 0$ is of importance, as the storage time can be calculated from the relevant hole concentration:

$$\begin{aligned} \Delta p''(0, T) = \frac{(I_f + I_r)L_p}{qD_p}\Big\{&\sqrt{1+E_n^2}\,\mathrm{erf}\sqrt{(1+E_n^2)\,T} - \\ &- E_n(1-e^{-T}) - E_n\, e^{-T}\,\mathrm{erf}\,(E_n\sqrt{T})\Big\}. \end{aligned} \tag{7}$$

The hole distribution created by I_f can be calculated similarly replacing $I_f + I_r$ by I_f and T by $T + T_f$ in Eq. (7)

$$\Delta p'(0, T) = \frac{I_f L_p}{q D_p} \Big\{ \sqrt{1+E_n^2}\, \operatorname{erf} \sqrt{(1+E_n^2)(T+T_f)} - \\ - E_n \left(1 - e^{-(T+T_f)}\right) - E_n\, e^{-(T+T_f)} \operatorname{erf}\left(E_n \sqrt{T+T_f}\right) \Big\}. \tag{8}$$

At the end of the storage time

$$\Delta p'(0, T_s) = \Delta p''(0, T_s) \tag{9}$$

(neglecting p_n). From Eq. (7), (8) and (9) the storage time can be determined

$$\frac{I_f}{I_f+I_r} = \frac{\sqrt{1+E_n^2}\, \operatorname{erf} \sqrt{(1+E_n^2)\, T_s} - E_n(1-e^{-T_s}) - E_n e^{-T_s} \operatorname{erf}(E_n \sqrt{T_s})}{\sqrt{1+E_n^2}\, \operatorname{erf} \sqrt{(1+E_n^2)(T_s+T_f)} - E_n(1-e^{-(T_f+T_s)}) - E_n e^{-(T_f+T_s)} \cdot \operatorname{erf}(E_n \sqrt{T_f+T_s})}. \tag{10}$$

If the switching-in impulse is long that is $T_f \gg 1$, $T_f \to \infty$ taking into consideration, that $\operatorname{erf}(\infty) = 1$ and $e^{-\infty} = 0$, Eq. (10) becomes simpler:

$$\frac{I_f}{I_f+I_r} = \frac{\sqrt{1+E_n^2}\, \operatorname{erf} \sqrt{(1+E_n^2) T_s} - E_n(1-e^{-T_s}) - E_n e^{-T_s} \operatorname{erf}(E_n \sqrt{T_s})}{\sqrt{1+E_n^2} - E_n}. \tag{11}$$

If the base has a uniform doping, $E_n = 0$ and Eq. (11) leads to the well-known equation for the storage time of homogeneous, wide-base diodes:

$$\frac{I_f}{I_f+I_r} = \operatorname{erf} \sqrt{T_s}\,. \tag{12}$$

Approximations

Let us examine the case where $E_n < 0$ and $|E_n| \gg 1$. This is known to be the field strength of the ideal step-recovery diode. By this diode (supposing $E_n \ll 0$) the fall time is zero, thus the total recovery time is equal to the storage time. Taking into account that $|E_n| = -E_n$ and $\operatorname{erf}(-x) = -\operatorname{erf}(x)$, the time-dependent factor of Eq. (11) takes the form:

$$f(0, T_s) = |E_n| + |E_n|\,(1 - e^{-T_s}) - |E_n|\, e^{-T_s}. \tag{13}$$

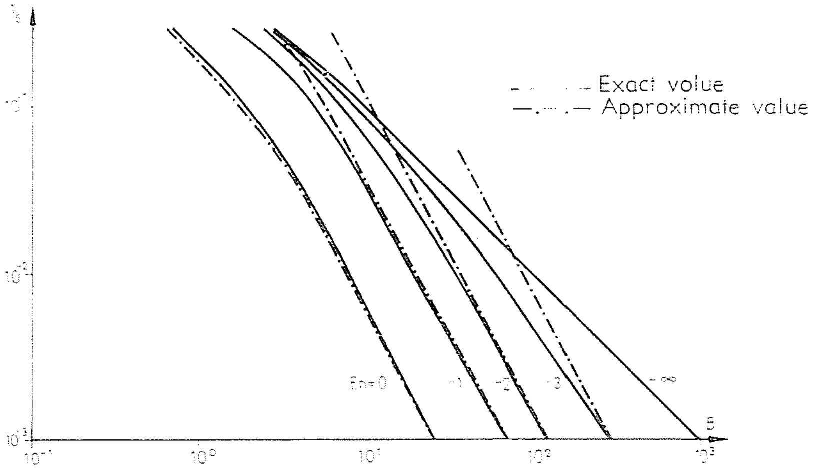

Fig. 2. The exact and approximate values of the storage times

The denominator of Eq. (11) becomes $2\,|E_n|$ to simplify into:

$$\frac{I_f}{I_f+I_r} = 1 - e^{-T_s} . \tag{14}$$

Expressing T_s

$$T_s = \ln\left(1 + \frac{I_f}{I_r}\right) . \tag{15}$$

This equals the result of the charge-controlled calculation, as it was expected.

Considering a finite T_f, Eq. (10) yields in a similar way:

$$\frac{I_f}{I_f+I_r} = \frac{1-e^{-T_s}}{1-e^{-(T_s-T_f)}} . \tag{16}$$

Expressing T_s

$$T_s = \ln\left[1 + \frac{I_f}{I_r}\left(1-e^{-T_f}\right)\right] . \tag{17}$$

The storage times calculated from Eq. (11) are plotted in Fig. (2). Increasing the value of the reverse current, the storage time decreases and becomes more divergent from the approximate values obtained from Eq. (15). On the other hand, by increasing E_n, the two values become more convergent.

From the shapes of the curves in Fig. (2) it appears that there must be other approximation too. By increasing I_r/I_f, each line becomes straight, which indicates a relationship between T_s and I_r/I_f in the form of a power function. For $E_n = -\infty$ Eq. (15), expanded into series:

$$T_s = \frac{I_f}{I_r}\,. \tag{18}$$

For $E_n \neq -\infty$ the function erf (x) is to be expanded into series. By applying erf$(x) \approx \frac{2}{\sqrt{\pi}}$ and $e^{-x} \approx 1 - x$, Eq. (11) takes the form:

$$\frac{I_f}{I_f + I_r} = \frac{\frac{2}{\sqrt{\pi}}\sqrt{T_s}\,(1+E_n^2\,T_s) - E_n\,T_s}{\sqrt{1+E_n^2} - E_n}\,. \tag{19}$$

In the numerator of Eq. (19) $E_n^2 T_s \leqslant 1$ because the whole approximation applies only for $(1 + E_n^2)T_s \leqslant 1$. Then (19) becomes more simple:

$$\frac{I_f}{I_f + I_r} = \frac{\frac{2}{\sqrt{\pi}}\sqrt{T_s} - E_n\,T_s}{\sqrt{1+E_n^2} - E_n}\,. \tag{20}$$

T_s can be expressed from Eq. (20). By solving the second order equation for T_s, only the smaller root satisfies the fundamental condition of the approximate calculation. Applying the new variable, $B = I_r/I_f$ (21)

$$\sqrt{T_s} = \frac{\frac{2}{\sqrt{\pi}} - \sqrt{\frac{4}{\pi} - \frac{4E_n}{1+B}\left(\sqrt{1+E_n^2} - E_n\right)}}{2E_n}\,. \tag{22}$$

In the examined domain $B \gg 1$ (Fig. 2), thus the expression under the square root can be expanded into series and considering only the first member we get for T_s:

$$\sqrt{T_s} = \frac{\sqrt{\pi}}{1+B}\,\frac{\sqrt{1+E_n^2} - E_n}{2}\,. \tag{23}$$

Because of its simplicity, this approximate expression is convenient to use. Before comparing with the exact solution, it should be pointed out that for $E_n = 0$ we get for T_s:

$$\sqrt{T_s} = \frac{1}{1+B}\,\frac{\sqrt{\pi}}{2}\,. \tag{24}$$

The same would result from Eq. (12) by expanding into series the right side.

In Fig. 2 the values calculated from Eq. (23) are plotted with dashed lines, leading to the following conclusions:

(a) If the T_s value calculated from Eq. (23) is much smaller than that calculated from Eq. (17) (valid for $E_n = -\infty$) then Eq. (23) should be used to determine the value of T_s.

(b) In opposite case Eq. (17) should be used.

(c) If the two T_s values are of about the same order of magnitude, for more exact calculations Eq. (11) should be used.

Appendix

Denote the Laplace-transformed, normalized time by s. The transformed form of Eq. (4)

$$sU''(X, s) - U''(X, 0) = \frac{\partial^2 U''(X, s)}{\partial X^2} . \tag{A. 1}$$

The initial condition is: $U''(X, 0) = 0$, thus

$$\frac{\partial^2 U''(X, s)}{\partial X^2} = sU''(X, s) \tag{A. 2}$$

For wide-base diodes $U''(\infty, s) = 0$, therefore the solution of Eq. (A.2) is

$$U''(X, s) = Ce^{-\sqrt{s}X} . \tag{A. 3}$$

The value of C can be calculated from the boundary condition (6). Applying Laplace-transformation to both sides

$$\frac{L_p(I_f+I_r)}{qD_p} \frac{1}{s-(1+E_n^2)} = \left\{ -\frac{\partial U''}{\partial X} + E_n U'' \right\}\Bigg|_{X=0} . \tag{A. 4}$$

Substituting $U''(X, s)$ from Eq. (A.3) into Eq. (A.4) then expressing C finally substituting into Eq. (A.3) one obtains:

$$U''(X, s) = \frac{(I_f+I_r)L_p}{qD_p} \frac{1}{s-(1+E_n^2)} \frac{1}{\sqrt{s}+E_n} e^{-\sqrt{s}X} . \tag{A. 5}$$

For determining the storage time, it is necessary to know the hole density for $X = 0$

$$U''(0, s) = \frac{(I_f+I_r)L_p}{qD_p} \frac{1}{s-(1+E_n^2)} \frac{1}{\sqrt{s}+E_n} . \tag{A. 6}$$

The equation above can be transformed back by the convolution-formula. Second and third member of Eq. (A. 6) take the form:

$$L^{-1}\left\{\frac{1}{s-(1+E_n^2)}\right\} = \exp\left[(1+E_n^2)T\right]$$

$$L^{-1}\left\{\frac{1}{\sqrt{s}+E_n}\right\} = \frac{1}{\sqrt{\pi T}} - E_n \exp\left(E_n^2 T\right)\left[1 - \mathrm{erf}\left(E_n\sqrt{T}\right)\right].$$

Thus transforming back the s-dependent member of Eq. (A.6)

$$L^{-1}\left\{\frac{1}{s-(1+E_n^2)}\,\frac{1}{\sqrt{s}+E_n}\right\} =$$

$$\int_0^T e^{(1+E_n^2)(T-\tau)}\left\{\frac{1}{\sqrt{\pi\tau}} - E_n\, e^{E_n^2\tau}\left[1-\mathrm{erf}\left(E_n\sqrt{\tau}\right)\right]\right\}\mathrm{d}\tau .$$

From the integral above, $e^{(1+E_n^2)T}$ can be taken out. Considering that from Eq. (3)

$$\Delta p''(0,T) = U''(0,T)\exp\left\{-(1+E_n^2)T\right\}$$

the hole density becomes:

$$\Delta p''(0,T) = \frac{(I_f+I_r)\,L_p}{qD_p}\left\{\int_0^T e^{-(1+E_n^2)\tau}\frac{\mathrm{d}\tau}{\sqrt{\pi\cdot\tau}} - E_n\int_0^T e^{-\tau}\left[1-\mathrm{erf}\left(E_n\sqrt{\tau}\right)\right]\mathrm{d}\tau\right\}. \quad (\mathrm{A}.7)$$

By introducing a new variable, the first integral in brackets takes the form erf (x). The first member of the second integral is easy to integrate, the second member can be evaluated by partial integration and then by introducing a new variable. After that, from Eq. (A.7) one obtains Eq. (7).

Summary

The storage time of diodes with nonuniform base doping is examined by supposing constant field strength in the base. Beside the exact solution, some approximations are given facilitating to determine the switching times. The limits of applicability of approximations are presented.

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AN UPPER BOUND FOR THE RELATIVE GAP OF THRESHOLD FUNCTIONS

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(Received February 21, 1970)

1. Introduction

A Boolean function is said to be a threshold function if and only if there exists a weight vector, the scalar product of which with every difference vector is not smaller than zero.

The definition of difference vectors is given in a previous paper [1].

From the viewpoint of practical application, the scalar products of the weight vector and the difference vectors must be greater than zero, because no threshold device can realize other than a threshold domain, a so-called gap, instead of an exact threshold value. It is important to know about a threshold function, whether there is a possibility to realize it by a single given threshold device or not.

2. Terminology

Let $F(x)$ denote an arbitrary Boolean function, where x means the input vector with n bivalued components:

$$x = (x_1, x_2, \ldots, x_n).$$

Let x_i^1 denote an input vector, for which $F(x) = 1$.
Let x_j^0 denote an input vector, for which $F(x) = 0$.
Thus

$$F(x_i^1) = 1 \quad \text{and} \quad F(x_j^0) = 0 \text{ hold}.$$

The number of vectors and x^1 and x^0 depends on the truth table of $F(x)$.

Let y^k denote the difference vector between one of the vectors x^1 and one of the vectors x^0.

It has been shown [1] that Boolean function $F(x)$ is a threshold function or 1-realizable if and only if the inequality

$$wy^k > 0 \tag{1}$$

holds for each y^k derivable from the truth table of $F(x)$.

In the above inequality w denotes the so-called weight vector, with real numbers as components;

wy^k denotes the scalar product of the vectors.

3. An upper bound for the relative gap

Practically, the maintenance of inequality (1) is not sufficient for the realization of a threshold function. If the threshold device has a threshold domain, or gap G then for the sake of the realization the scalar products of the weight vector with every difference vector must not be smaller than G.

$$wy^k \geq G. \tag{2}$$

Introducing the notation of the weight unit-vector (w_u)

$$w_u = \frac{w}{|w|}$$

inequality (2) can be rewritten as follows

$$w_u y^k \geq \frac{G}{|w|}. \tag{3}$$

The absolute value of the weight vector $|w|$ is also limited for every threshold device. Thus there exists a quotient for every threshold device,

$$g_d = \frac{G}{|w|_{\max}}$$

termed the relative gap of the device. References may involve some other meanings for the relative gap [2], [3], but all of them are characteristic of the realizability.

Theorem 1: A threshold function is not realizable with a given weight unit-vector w_u by a single threshold device having a relative gap g_d if there exists at least one difference vector y^k, for which the inequality

$$w_u y^k < g_d \tag{4}$$

holds.

The proof of this theorem is unnecessary, because it is obvious that if inequality (4) holds, then inequality (3) does not hold.

Let g_f denote the minimum value among the scalar products $w_u y^k$.

$$(w_u y^k)_{\min} = g_f$$

thus, for all difference vectors the inequalities

$$\begin{aligned} w_u y^1 &\geq g_f \\ w_u y^2 &\geq g_f \\ &\vdots \\ w_u y^k &\geq g_f \\ &\vdots \\ w_u y^m &\geq g_f \quad \text{hold.} \end{aligned} \tag{5}$$

where m is the number of the difference vectors. For the sake of simplicity m will be considered as the number of the difference vectors differing from each other.

Let g_f be called the minimum relative gap of the threshold function.

Theorem 2: There exists an upper bound for the minimum relative gap of a threshold function:

$$\frac{\left|\sum_{k=1}^{m} y^k\right|}{m},$$

that is

$$g_f \leq \frac{\left|\sum_{k=1}^{m} y^k\right|}{m}, \quad \text{where} \quad \left|\sum_{k=1}^{m} y^k\right| \quad \text{is}$$

is the absolute value of the sum-vector of the difference vectors.

Proof: Summarizing inequalities (5), it follows:

$$w_u \sum_{k=1}^{m} y^k \geq m g_f .$$

The absolute value of w_u being equal to 1, the inequality

$$\left|\sum_{k=1}^{m} y^k\right| \cos \varphi \geq m g_f \quad \text{holds} .$$

where φ is the angle between the sum-vector and the weight unit-vector.

The maximum value of $\cos\varphi$ being equal to 1, the inequality

$$\left|\sum_{k=1}^{m} y^k\right| \geq m g_f$$

is satisfied, and the proof is completed.

The above upper bound can be computed from the truth table of the threshold function previous to the realization procedure and it can be decided whether the realization by a device with a given g_d is possible or not. For example if

$$\frac{\left|\sum_{k=1}^{m} y^k\right|}{m} \leq g_d ,$$

then according to Theorem 1, the realization by the given device is impossible.

Besides if the value of g_f computed with a given weight vector is near to the above upper bound, it will be useless to modify this weight vector in order to get a much greater value of g_f.

4. Examples

A computerizable testing and synthesis algorithm has been written making use of the properties of the difference vectors. A part of this procedure is to calculate the value of g_f and its upper bound. These values and the realizations for some threshold functions are shown in Table 1.

Table 1

Boolean function	Weight vector components	Threshold domain	$\frac{\left\lvert\sum_{i=1}^{m} y^k\right\rvert}{m}$	g_f
$F = x_1(x_2 + x_3 + x_4) + x_2x_3x_4$	2. 1. 1,1	3, 2	0.839	0.378
$F = x_1x_2 + x_1x_3x_4 + x_2x_3x_4$	2. 2. 1. 1	4. 3	0.781	0.316
$F = \sum^4(1. 2. 3. 4. 5. 6. 7. 15)$	−2. 1. 1. 1	1, 0	0.839	0.378
$F = \sum^3(3. 5. 6. 7)$	1. 1. 1	2. 1	0.866	0.577
$F = \sum^4(1. 9. 10. 11. 12. 13. 14. 15) + d(0. 8)$	3. −1. −1, 1	1. 0	0.857	0.289
$F = \sum^4(2. 4. 8) + d(1, 6. 7, 10. 11\ 12. 13. 14. 15)$	1. 1. 1. −2	1. −1	1	0.756

5. Summary

In this paper an upper bound for the relative gap is given, which can be calculated from the truth table of a threshold function. Every threshold device may be characterized by the minimum value of the relative gap possible by that device. Comparing the latter minimum value to the upper bound, the impossibility of the realization by the given device can be predicted.

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