ON THEORETICAL QUESTIONS OF THE ELECTROMAGNETIC FIELD OF TRANSMISSION LINES

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Introduction

Phenomena along the transmission line are a part of the domain of electromagnetic waves. The usual method at writing the relating equations is to express the relationship between voltage and current by the help of electrostatic capacity, of the coefficient of stationary external inductance and conductance, further of the quasistationary resistance and internal inductance [1]. Field theoretical examinations in the literature are primarily concerned with TEM mode waves [2]. In the case of transmission lines with losses, in turn, the arising fundamental mode is of the TM type. For the case of some transmission lines of special arrangement the propagation coefficients were successfully determined exclusively on the basis of considerations of field theory [3, 4]. In the present paper the general field theoretical examination of the field of transmission lines is presented, in the course of which it will be clarified, in the case of wave phenomena, under which conditions is the calculation of line parameters by electrostatic, stationary, or quasi-stationary ways justified. For the general case the value of the propagation coefficient is determined on field theoretical basis, and relationships between the field and network theories of transmission lines are searched for.

The examination of the TM mode field

In the followings an arrangement consisting of two parallel leads will be named a transmission line in the case when the direction of the tangential component of current density on the surface of the leads is longitudinal, i.e. identical with the propagation direction of the waves. (In a different case the arrangement is named a wave guide.) In the followings only the electromagnetic field of transmission lines defined in this way will be examined. In the case of leads with losses electric field has a component also in the direction of the current. Among the possibly arising modes such a TM mode without critical wave-length will be examined which passes over into the TEM mode in the extreme case. In the following the equations of such waves are written.

The Maxwell equations for the space filled with material homogeneous and isotropic in each space section will be started of

$$\mathrm{rot}\,\mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} = \sigma \mathbf{E} + \varepsilon \frac{\partial \mathbf{E}}{\partial t}, \tag{1}$$

$$\mathrm{rot}\,\mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}, \tag{2}$$

$$\mathrm{div}\,\mathbf{H} = 0, \tag{3}$$

$$\mathrm{div}\,\mathbf{E} = \frac{\varrho}{\varepsilon}, \tag{4}$$

where $\mathbf{H}$ and $\mathbf{E}$ are the magnetic and electric field, respectively, $\mathbf{J}$ the current density, ϱ the charge density, μ the permeability, ε the permittivity, σ the specific conductivity, t the time. One of the usual ways of solving the above system of equations is to write $\mathbf{H}$ on the basis of Eq. (3) as the rotation of vector potential $\mathbf{A}$.

$$\mathbf{H} = \mathrm{rot}\,\mathbf{A}. \tag{5}$$

In the case of a TM mode vector potential $\mathbf{A}$ has only a longitudinal component in the direction of the lead axis. Let us choose the direction of the z-axis identical with the longitudinal direction. Thus

$$\mathbf{A} = \mathbf{A}_z = \mathbf{k}A(z, \vartheta_1, \vartheta_2), \tag{6}$$

where $\mathbf{k}$ is the unit vector in the longitudinal direction, ϑ_1 and ϑ_2 denote transversal co-ordinates normal to each other (e.g. in the case of cylindrical co-ordinates, r and φ). Then, considering that $\mathrm{rot}\,\mathbf{k} = 0$, Eq. (5) can be written in the following way:

$$\mathbf{H} = \mathrm{rot}\,\mathbf{A} = \mathrm{rot}\,\mathbf{k}A = \mathrm{grad}\,A \times \mathbf{k}. \tag{7}$$

$\mathrm{grad}\,A$ can be expressed as the sum of the gradient of A with respect to the transversal co-ordinates and of $\mathbf{k}\dfrac{\partial A}{\partial z}$.

$$\mathrm{grad}\,A = \mathrm{grad}_\vartheta A + \frac{\partial A}{\partial z}\mathbf{k}. \tag{8}$$

Accordingly, on the basis of (7)

$$\mathbf{H} = -\mathbf{k} \times \mathrm{grad}_\vartheta A. \tag{9}$$

For the following calculations substitute (5) into (2).

$$\operatorname{rot} \mathbf{E} = -\mu \frac{\partial \operatorname{rot} \mathbf{A}}{\partial t} . \tag{10}$$

The order of forming the rotation and of derivation with respect to time can be interchanged, consequently on the basis of this relationship $\mathbf{E} + \mu \frac{\partial \mathbf{A}}{\partial t}$ can be written as the gradient of a scalar potential, namely

$$\mathbf{E} = -\mu \frac{\partial \mathbf{A}}{\partial t} - \operatorname{grad} \varphi . \tag{11}$$

Decompose this expression into a transversal and a longitudinal component, then, respeetively

$$E_t = - \operatorname{grad}_t \varphi \tag{12}$$

$$E_z = -\mu \frac{\partial A}{\partial t} - \frac{\partial \varphi}{\partial z} . \tag{13}$$

It can be stated on the basis of relationship (12) that the transversal component of the TM mode electric field is of the potential type.

Substitute (5) in relationship (1).

$$\operatorname{rot} \operatorname{rot} \mathbf{A} = \sigma \mathbf{E} + \varepsilon \frac{\partial \mathbf{E}}{\partial t} . \tag{14}$$

The left side of the equation can be rewritten on the basis of the well-known relation of vector analysis, while at the right side the expression (11) for $\mathbf{E}$ is written.

$$\begin{aligned}\operatorname{grad} \operatorname{div} \mathbf{A} - \Delta \mathbf{A} = -\sigma\mu \frac{\partial \mathbf{A}}{\partial t} - \mu\varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} - \\ - \operatorname{grad} \left(\sigma\varphi + \varepsilon \frac{\partial \varphi}{\partial t} \right)\end{aligned} \tag{15}$$

(We made use here of the possibility of interchanging the order of forming the gradient and of derivation with respect to time.) Let us choose $\operatorname{div} \mathbf{A}$ in accordance with the Lorentz condition, namely

$$\operatorname{div} \mathbf{A} = -\left(\sigma\varphi + \varepsilon \frac{\partial \varphi}{\partial t} \right) . \tag{16}$$

Thus Eq. (15) becomes

$$\Delta \mathbf{A} - \sigma\mu \frac{\partial \mathbf{A}}{\partial t} - \mu\varepsilon \frac{\delta^2 \mathbf{A}}{\delta t^2} = 0. \tag{17}$$

On the basis of (6) we obtain

$$\Delta A - \sigma\mu \frac{\partial A}{\partial t} - \mu\varepsilon \frac{\partial^2 A}{\partial t^2} = 0. \tag{18}$$

The solution of Eq. (18) will be discussed later.

Relationship (16) can be written, on the basis of (6), also in the following form

$$-\frac{\partial A}{\partial z} = \left(\sigma + \varepsilon \frac{\partial}{\partial t}\right) \varphi. \tag{19}$$

Let us further substitute (11) in Eq. (4).

$$-\operatorname{div}\left[\mu \frac{\partial A}{\partial t} + \operatorname{grad} \varphi\right] = \frac{\varrho}{\varepsilon} \tag{20}$$

that is

$$\mu \frac{\partial}{\partial t} \operatorname{div} A + \operatorname{div} \operatorname{grad} \varphi = -\frac{\varrho}{\varepsilon}. \tag{21}$$

Upon considering the expression (16),

$$\Delta\varphi - \mu\sigma \frac{\partial \varphi}{\partial t} - \mu\varepsilon \frac{\partial^2 \varphi}{\partial t^2} = -\frac{\varrho}{\varepsilon}. \tag{22}$$

In those parts of the space where there is no space charge ($\varrho = 0$):

$$\Delta\varphi - \mu\sigma \frac{\partial \varphi}{\partial t} - \mu\varepsilon \frac{\partial^2 \varphi}{\partial t^2} = 0. \tag{23}$$

In this way we obtained a differential equation identical in form with Eq. (18).

Line parameters

The solution of differential equations (18) and (23) is searched for by the method of product separation. Further examinations will be restricted to signals changing sinusoidally in time. The solution of the two equations is searched for in the forms

$$A(z, \vartheta_1, \vartheta_2) = A_0 Z_a(z)\, \Theta_a(\vartheta_1, \vartheta_2) \tag{24}$$

and

$$\varphi(z, \vartheta_1, \vartheta_2) = \varphi_0 Z_\varphi(z)\, \Theta_\varphi(\vartheta_1, \vartheta_2), \tag{25}$$

respectively.

Let us substitute Eq. (24) and (25) in (19). Thus

$$-A_0 \frac{\partial Z_a}{\partial z} \Theta_a = \varphi_0 Z_\varphi \Theta_\varphi(\sigma + j\omega\varepsilon). \tag{26}$$

This equation is satisfied, independently of co-ordinates ϑ_1 and ϑ_2, if

$$\Theta_a = \Theta_\varphi = \Theta(\vartheta_1, \vartheta_2). \tag{27}$$

This means that the vector and the scalar potential are described in the transversal plane by identical functions. That is to say, the same function figures also in the expression of gradients formed with respect to the transversal co-ordinates.

$$\mathrm{grad}_\vartheta \varphi = \varphi_0 Z_\varphi \,\mathrm{grad}_\vartheta \Theta \tag{28}$$

$$\mathrm{grad}_\vartheta A = A_0 Z_a \,\mathrm{grad}_\vartheta \Theta. \tag{29}$$

E_ϑ is proportional to $\mathrm{grad}_\vartheta \varphi$, according to (12), while H to $\mathrm{grad}_\vartheta A$, according to (9). It follows of this that field strengths E_ϑ and H are described in the transversal plane by functions of identical character. By comparing (9), (12), (28), (29) it can be established that E_ϑ lies in the direction of $-\mathrm{grad}_\vartheta \varphi$, i.e. of $-\mathrm{grad}_\vartheta A$, and is thus perpendicular to $\mathbf{H}$.

Hereafter our calculations will be limited to the electromagnetic field of transmission lines. We suppose further that in one of the two parallel lines current I is flowing in direction $+z$, while in the other in direction $-z$. This assumption is satisfied in the case of lines of asymmetrical arrangement only approximately [3], namely in such cases the current of the two lines is not identical in general, but the current of one of the lines is closed by the current of the other line and of its dielectric displacement current. This latter, however, can be neglected in most cases in comparison with conduction currents. In the examined case the value of potential φ along the perimeter of the line at a fixed place z is constant. Namely, in the contrary case the current density has also a component perpendicular to z. This in turn would be contrary to the condition that the direction of the tangential component of current density at the surface of the line is identical with the direction of propagation of the waves.

Let φ_1 designate the potential at the surface of one of the lines at a fixed place z, while φ_2 that on the other line. At the place z the integral of $\mathbf{H}$ with respect to the circumference of one of the lines, as a closed loop, is equal to the current in the line arising at z in accordance with Eq. (1). Since displace-

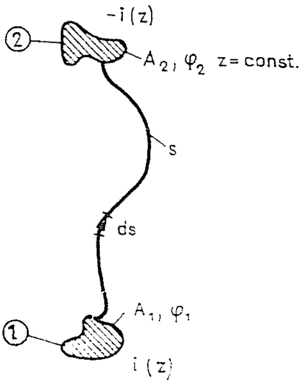

Fig. 1

ment current can be neglected in comparison with conduction current in the conductor, therefore

$$\oint_{l_1} \mathbf{H} d\mathbf{l} = i(z). \tag{30}$$

Let us define capacity C of the transmission line, and external inductance L_k both referred to unit length, by the following relationships.

$$C = \varepsilon \frac{\oint_{l_1} (\mathbf{k} \times \operatorname{grad}_t \varphi) d\mathbf{l}}{\int_{s_1}^{s_1} \operatorname{grad}_t \varphi \, ds} = \varepsilon \frac{-\oint_{l_1} \frac{\partial \varphi}{\partial n} dl}{\varphi_1 - \varphi_2}, \tag{31}$$

$$L_k = \mu \frac{\int_{s_2}^{s_1} \operatorname{grad}_t A ds}{\oint_{l_1} (\mathbf{k} \times \operatorname{grad}_t A) dl} = \mu \frac{A_1 - A_2}{-\oint_{l_1} \frac{\partial A}{\partial n} dl}, \tag{32}$$

where $\frac{\partial}{\partial n}$ indicates a differentiation at z in the direction perpendicular to the surface of the conductor, directed away. In definitions (31) and (32) the curve l_1 indicates the circumference of conductor 1 at the fixed place z, while s is a curve lying in the transversal plane belonging to this same z, which connects one of the points of the circumference l_1 of conductor 1 with the circumference of conductor 2 (Fig. 1). We shall see that capacity, as defined in this way and the external inductance are also functions of the changes of the wave phenomenon in time, beyond geometrical data and material constants. Let A_1 designate the value of the vector potential at the surface of conductor 1 at place z, while A_2 that on the surface of conductor 2.

Substitute in (31) the expression given under (25). Thus we obtain, upon considering also (27), that

$$C = \varepsilon \frac{-\oint\limits_{l_1} \frac{\partial \Theta}{\partial n} dl}{\Theta_1 - \Theta_2}, \tag{33}$$

further of (32), on the basis of (24) and (27),

$$L_k = \mu \frac{\Theta_1 - \Theta_2}{-\oint\limits_{l_1} \frac{\partial \Theta}{\partial n} dl}, \tag{34}$$

where Θ_1 and Θ_2 designate the value of Θ at the fixed place z, along the circumference of conductors 1 and 2, respectively.

On the basis of (33) and (34) we obtain the well known relationship

$$L_k C = \mu\varepsilon. \tag{35}$$

Let us examine now the expressions given under (31) and (32). To this end determine the surface integral of quantity $E_\vartheta = -\operatorname{grad}_\vartheta \varphi$ for a piece of conductor 1 having the length dz. This is proportional, by force of Eq. (4), to the charge on the section of length dz of the conductor, q dz (q is the charge of the conductor of unit length).

$$\varepsilon \int\limits_a E_\vartheta da = -\varepsilon \int\limits_a \operatorname{grad}_\vartheta \varphi da = q dz. \tag{36}$$

We can write that

$$d\mathbf{a} = dz\, d\mathbf{l} \times \mathbf{k} = dz\, \mathbf{n}\, dl \tag{37}$$

and thus

$$\operatorname{grad}_\vartheta \varphi\, d\mathbf{a} = \frac{\partial \varphi}{\partial n} dl\, dz. \tag{38}$$

With this from (36)

$$-\varepsilon\, dz \oint\limits_{l_1} \frac{\partial \varphi}{\partial n} dl = q\, dz. \tag{39}$$

Accordingly

$$q = -\varepsilon \oint_{l_1} \frac{\partial \varphi}{\partial n} dl . \tag{40}$$

Let u designate the difference of φ_1 and φ_2, i.e. u is the voltage between the conductors, in the case of fixed z.

$$u(z) = \varphi_1(z) - \varphi_2(z) . \tag{41}$$

Thus by using (40), relationship (31) can be written in the following form:

$$C = \frac{q}{u} \tag{42}$$

This means that definition (31) is identical with the well-known definition of capacity.

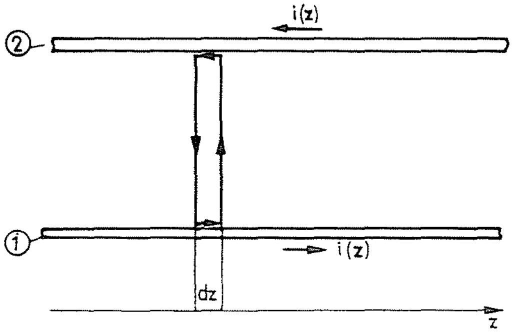

Fig. 2

Consider that

$$d\mathbf{l} \times \mathbf{k} = dl\, \mathbf{n} \tag{43}$$

thus on the basis of (9), for the circumference of conductor 1, we obtain that

$$i = \oint \mathbf{H}\, d\mathbf{l} = \oint (-\mathbf{k} \times \mathrm{grad}_t A)\, d\mathbf{l} = -\oint \mathrm{grad}_t A (d\mathbf{l} \times \mathbf{k}) = -\oint \frac{\partial A}{\partial n} dl \tag{44}$$

Integrate vector potential $\mathbf{A}$ along the loop of width dz shown in Fig. 2. Let $\frac{1}{\mu}\, \Phi\, dz$ designate the value of the integral in accordance with Eq. (5), where $\Phi\, dz$ is the flux passing the surface surrounded by the loop.

$$\frac{\Phi}{\mu} dz = \oint_s \mathbf{A}\, d\mathbf{s} = (A_1 - A_2)\, dz. \tag{45}$$

Accordingly

$$\Phi = \mu(A_1 - A_2). \tag{46}$$

By taking (44) and (46) into consideration, we obtain for L_k from the definition (32) that

$$L_k = \frac{\Phi}{i}, \tag{47}$$

i.e. (32) corresponds to the usual definition of the induction coefficient.

Differential equations for voltage and current

Let us write relationship (13) for the surface of the individual conductors.

$$E_{z1} = -\mu \frac{\partial A_1}{\partial t} - \frac{\partial \varphi_1}{\partial z} \tag{48}$$

$$E_{z2} = -\mu \frac{\partial A_2}{\partial t} - \frac{\partial \varphi_2}{\partial z} \tag{49}$$

Subtract (49) from (48).

$$E_{z1} - E_{z2} = -\mu \frac{\partial (A_1 - A_2)}{\partial t} - \frac{\partial (\varphi_1 - \varphi_2)}{\partial z} \tag{50}$$

E_{z1} and E_{z2} can also be expressed by the internal field of the conductors [1].

$$E_{z1} = R_1 i + L_{b1} \frac{\partial i}{\partial t}, \tag{51}$$

$$E_{z2} = -R_2 i - L_{b2} \frac{\partial i}{\partial t}. \tag{52}$$

where R_1 and R_2, further L_{b1} and L_{b2} denote the resistance and internal inductance of conductors 1 and 2, respectively, obtained by taking the skin effect into consideration. At writing (51) and (52) the direction of the current in the conductors was taken to be $+z$ and $-z$, respectively.

From (50), by considering relationships (41), (51), (52), (46), and (47) we obtain a differential equation for the relationship between u and i.

$$-\frac{\partial u}{\partial z} = (R_1 + R_2)\, i + (L_k + L_{b1} + L_{b2})\frac{\partial i}{\partial t}\,. \tag{53}$$

Write equation (19) for the surface of the two conductors.

$$-\frac{\partial A_1}{\partial z} = \left(\sigma + \varepsilon\frac{\partial}{\partial t}\right)\varphi_1 \tag{54}$$

$$-\frac{\partial A_2}{\partial z} = \left(\sigma + \varepsilon\frac{\partial}{\partial t}\right)\varphi_2\,. \tag{55}$$

Subtract (55) from (54).

$$-\frac{\partial(A_1 - A_2)}{\partial z} = \left(\sigma + \varepsilon\frac{\partial}{\partial t}\right)(\varphi_1 - \varphi_2) \tag{56}$$

From relationships (41), (46), and (47), further from (35), we obtain that

$$-\frac{\partial i}{\partial z} = \frac{\mu}{L_k}\left(\sigma + \varepsilon\frac{\partial}{\partial t}\right)u = \left(\sigma\frac{C}{\varepsilon} + C\frac{\partial}{\partial t}\right)u\,. \tag{57}$$

By introducing the notation

$$G = \sigma\frac{C}{\varepsilon}\,, \tag{58}$$

we can write that

$$-\frac{\partial i}{\partial z} = \left(G + C\frac{\partial}{\partial t}\right)u\,. \tag{59}$$

The denomination of G is the conductance, thus (58) is a relationship expressing the analogy between electrostatic and current fields. (53) and (59) are the well-known Kelvin Telegraph Equations.

The solution of the differential equations

Eqs (18) and (23) are differential equations of the vector potential, and of the scalar potential, respectively. Let us examine the solution of these. To this end decompose the Laplace operator to the sum of the second derivative with respect to z and of the two-dimensional Laplace operator with respect to the transversal co-ordinates.

$$\Delta = \Delta_{\vartheta} + \frac{\partial^2}{\partial z^2} \tag{60}$$

Substitute the expression (24) for A into (18) and divide this equation by A. Thus, the previously mentioned decomposition, in the case of a sinusoidal time change, by considering (27), yields

$$\frac{1}{\Theta} \Delta_{\vartheta} \Theta + \frac{1}{Z_a} \frac{\partial^2 Z_a}{\partial z^2} - j\omega\mu(\sigma + j\omega\varepsilon) = 0 \; . \tag{61}$$

The first member at the left side of this equation is exclusively a function of the transversal co-ordinates, the second member only of the z co-ordinate, while the third member is constant. Thus Eq. (61) can be satisfied only if the individual members are each equal to a constant, that is

$$\Delta_{\vartheta} \Theta - g^2 \Theta = 0 \tag{62}$$

$$\frac{d^2 Z_a}{dz^2} - \gamma^2 Z_a = 0 \; . \tag{63}$$

The separation constants g^2 and γ^2 should satisfy the following equations.

$$g^2 + \gamma^2 = \gamma_0^2 , \tag{64}$$

where

$$\gamma_0^2 = j\omega\mu(\sigma + j\omega\varepsilon) \, . \tag{65}$$

g^2 and γ^2 are constants.

The solution of (63) is known to be

$$Z_a(z) = A_z e^{-\gamma z} + B_z e^{\gamma z} , \tag{66}$$

where constants A_z and B_z can be determined in the knowledge of the excitation and termination at the end of the line, respectively.

For the solution of Eq. (62) a denominate system of co-ordinates is to be chosen [5, 6].

The value of g can be determined of the boundary conditions of the electromagnetic field prescribed for the surface of the conductor, making use of (65).

Inside the conductors, the displacement current can be neglected, thus from (65) we have

$$\gamma_{0v}^2 = j\omega \, \sigma_v \, \mu_v , \tag{67}$$

where the subscript indicates that the symbol refers to the inside of the conductor.

The value of γ is identical both inside and in the space outside the conductor, since the same wave is propagating in the direction z in both space parts. Since a large part of the energy is flowing in the dielectric, γ is of the same order of magnitude than γ_0 which is valid for the dielectric [7]. The value of γ_{ov} which is valid for the conductor is in turn higher by several orders of magnitude than γ_0 or γ, in the case of σ values met in practice. Thus on the basis of (64),

$$g_v^2 \approx \gamma_{ov}^2 = j\omega\,\sigma_v\,\mu_v\,. \tag{68}$$

This means that inside the conductor the functions describing the distribution of the transversal components of the electromagnetic field and the arguments of these are independent of wave phenomena in direction z. The amplitudes of field inside the conductor, however, depend on z, in contrast to the quasi-stationary case, that is to say these values are different in general at different z places. The internal impedance of the conductor, however, is independent of the amplitude of the field and consequently the electromagnetic field formed inside the conductor can be regarded as quasi-stationary from the aspect of the calculation of internal impedance. It follows that the resistance and inductance coefficient values R_1, R_2, L_{b1}, L_{b2} in (53) are identical with the values obtained from the equations of the quasi-stationary field.

The TEM mode solution

In a transmission line the TEM mode arises if the conductors are without losses ($R_1 = R_2 = 0$, $L_{b1} = L_{b2} = 0$) and the dielectric is homogeneous in the individual transversal planes. Then, by force of Eqs (51) and (52), $E_z = 0$ inside the conductors and thus also on their surface. Consequently, on the basis of (48) and (49)

$$-\frac{\partial \varphi_i}{\partial z} = \mu \frac{\partial A_i}{\partial t} \qquad (i = 1, 2)\,. \tag{69}$$

(54) and (55) can be written in the following form:

$$-\frac{\partial A_i}{\partial z} = \left(\sigma + \varepsilon \frac{\partial}{\partial t}\right) \varphi_i \tag{70}$$

From these last two equations we obtain for A_i and φ_i the differential equations

$$\frac{\partial^2 \varphi_i}{\partial z^2} = \left(\sigma + \varepsilon \frac{\partial}{\partial t}\right) \mu \frac{\partial \varphi_i}{\partial t} \tag{71}$$

and

$$\frac{\partial^2 A_i}{\partial z^2} = \left(\sigma + \varepsilon \frac{\partial}{\partial t}\right) \mu \frac{\partial A_i}{\partial t} . \tag{72}$$

By using (25) and (65) we have

$$\frac{\partial^2 Z_\varphi}{\partial z^2} = \gamma_0^2 Z_\varphi \tag{73}$$

and on the basis of (24)

$$\frac{\partial^2 Z_a}{\partial z^2} = \gamma_0^2 Z_a . \tag{74}$$

By comparing these last two relationships with (63) we find that

$$\gamma^2 = \gamma_0^2 \tag{75}$$

and thus, according to (64)

$$g^2 = 0 . \tag{76}$$

From the preceding equations, functions $Z_a(z)$ and $Z_\varphi(z)$ can be determined. These are naturally describing not only the functions φ_1, φ_2, and A_1, A_2, respectively, but also the dependence of φ and A on z, in the case of arbitrary co-ordinates ϑ_1, ϑ_2. That is to say E_z is zero everywhere on the basis of (69) and (13).

According to (76), Eq. (62) will have the following form:

$$\Delta_\vartheta \Theta = 0 , \tag{77}$$

a two-dimensional Laplace equation. Thus, we found that in the case of a fixed z value the same kinds of differential equations refer to φ and potential A as to the electrostatic potential.

Previously it was shown that the boundary conditions for φ were also of similar character (φ is constant at the circumference of the conductor), thus the solution in the case of a given z value was identical with the electrostatic solution. It follows of this that in the case of no losses also the capacity for unit length as defined in (31) is identical with the capacity calculated on the electrostatic way.

A similar consideration is valid for the vector potential. This leads to the relationship

$$\Delta_t A = 0 \tag{78}$$

under the conditions valid for the TEM mode. In the case of fields stationary in the dielectric medium $\Delta A = 0$ and $\frac{\partial}{\partial z} = 0$. Thus (78) is valid also in this case. Accordingly, in the case of no losses, the inductance coefficient defined by the relationship (32) is equal to the value calculated on the basis of the equations for the stationary field.

The TM mode solution

In the case of transmission lines with losses the z-direction component of the electric field arising in the conductor is for the surface of the conductors as given in (51) and (52). In this case not the TEM, but the TM mode arises.

The further calculations will be limited to changes sinusoidal in time. Now (51) and (52) can be written by employing the usual complex way in the following form.

$$E_{z1} = i(R_1 + j\omega L_{b1}) \tag{79}$$

$$E_{z2} = - i(R_2 + j\omega L_{b2}) \tag{80}$$

E_{z1} and E_{z2} can be expressed also in terms of the field arising in the dielectric medium. For this we obtain from (13), by using (19), that

$$E_z = - j\omega\mu A + \frac{1}{\sigma + j\omega\varepsilon} \cdot \frac{\partial^2 A}{\partial z^2} . \tag{81}$$

Considering relationships (63), (24), (65), and (64) we find that

$$E_z = - \frac{\gamma_0^2 - \gamma^2}{\sigma + j\omega\varepsilon} A = - \frac{g^2}{\sigma + j\omega\varepsilon} A . \tag{82}$$

We write this for the surface of the individual conductors and by comparing with (79) and (80) we have

$$E_{z1} = - \frac{g^2}{\sigma + j\omega\varepsilon} A_1 = iZ_{b1} \tag{83}$$

$$E_{z2} = - \frac{g^2}{\sigma + j\omega\varepsilon} A_2 = - iZ_{b2} . \tag{84}$$

Any of these last two equations can be used for determining g^2.

If the geometrical data or the material of the two conductors of the transmission line are not identical then we obtain two different g^2 values. This means that two different modes arise. (This phenomenon is known in the case of the Lecher Conductor [3].) For conductors of asymmetrical arrangement, g^2 can be calculated approximately in such a way that not each of Eqs (83) and (84) is satisfied, but only their difference. (In the case of a symmetrical arrangement this method is naturally exact.)

$$E_{z2} + E_{z1} = \frac{g^2}{\sigma + j\omega\varepsilon}(A_2 - A_1) = i(Z_{b1} + Z_{b2}) \,. \tag{85}$$

Replace A_2—A_1 by the inductance coefficient as defined by (32).

$$\frac{L_k}{\mu}\,\frac{g^2}{\sigma + j\omega\varepsilon}\int\limits_{l_1} |\mathrm{grad}_\vartheta A|\, dl = i(Z_{b1} + Z_{b2}) \,. \tag{86}$$

By multiplying and dividing the left side of the equation by $j\omega$, we may write upon considering (44) that

$$-\frac{j\omega L_k}{j\omega\mu(\sigma + j\omega\varepsilon)}\, g^2 = Z_{b1} + Z_{b2} \,, \tag{87}$$

i.e.

$$g^2 = -(Z_{b1} + Z_{b2})\,\frac{j\omega\mu(\sigma + j\omega\varepsilon)}{j\omega L_k} \,. \tag{88}$$

On the basis of (35) and (58)

$$\frac{j\omega\mu(\sigma + j\omega\varepsilon)}{j\omega L_k} = j\omega C + G. \tag{89}$$

Thus we obtain that

$$g^2 = -(Z_{b1} - Z_{b2})\,(j\omega C + G) \,. \tag{90}$$

Substitute this relationship into (64) and take into consideration (65) and (89),

$$\gamma^2 = \gamma_0^2 - g^2 = j\omega\mu(\sigma + j\omega\varepsilon) + (Z_{b1} + Z_{b2})\,(j\omega C + G) =$$
$$= (j\omega L_k + Z_{b1} + Z_{b2})\,(j\omega C + G) \,. \tag{91}$$

The result obtained corresponds to the known expression for the propagation coefficient. The values C, L_k, and G in this expression, however, correspond only approximately to the respective quantities calculated in static or stationary way. The scalar and vector potentials in the transversal plane are namely only approximately of the same distribution as in the static and stationary case. The closeness of approximation can be judged from a comparison between values calculated on the basis of (33) and (34) for the static and stationary cases, respectively, and values determined on the basis of wave theory. The numerator of (33) and the denominator of (34) are proportional to the current in the conductors, by force of (44). Thus, from the examination of Θ_1—Θ_2 conclusions can be drawn on the closeness of approximation. For $\Theta(\vartheta_1, \vartheta_2)$, as we have seen, differential equation (62) is valid. The solution of this, as written in the form of product separation, is found to be

$$\Theta(\vartheta_1, \vartheta_2) = \Theta_{\mathrm{I}}(\vartheta_1)\,\Theta_{\mathrm{II}}(\vartheta_2)\,. \tag{92}$$

In this form one of the factors depends only on ϑ_1 while the other only on ϑ_2. $g_1\,\vartheta_1$ and $g_2\,\vartheta_2$ occur in the argument of Θ_1, and of Θ_2, respectively, where

$$g_1^2 + g_2^2 = g^2\,. \tag{93}$$

Let us suppose, in accordance with practice, that there is a $\vartheta_1 =$ constant co-ordinate line connecting the two conductors. Let d designate the length of the section of this line between the two conductors. Thus the argument of Θ_1—Θ_2 includes the value $g_1\,d$. The order of magnitude of $|g_1|$ corresponds to that of $|g|$ or is smaller. Accordingly, if $|gd| \ll 1$, the function Θ can be approximated by the value arising in the case of $g \to 0$ and thus, in the place of (62), the relationship

$$\Delta_\vartheta \Theta = 0 \tag{94}$$

can be written for the space between the conductors. This corresponds exactly to the static and the stationary case.

On the basis of the foregoing it can be stated that definition (31) and (32) are valid also in the case of wave phenomena. By the help of C and L_k as defined in this way, the propagation coefficient can be calculated on the basis of (91) exactly for symmetrical arrangements, and approximately for asymmetrical ones. In the static and stationary cases, the definitions go over into the known expressions. (31) and (32) correspond to C and L_k calculated in the usual way for $|gd| \ll 1$. If this neglection is not taken into consideration then A_1—A_2 and φ_1—φ_2 depend on the value of g. The value of g in turn is a function, beyond the geometrical dimensions and the material constants, also of the angular frequency. Accordingly, in the general case L_k and C also depend on ω.

Summary

The paper contains the field-theoretical examination of transmission lines. Correlations between the field and network theory are described. For the case of wave phenomena the conditions of the justification of the calculation of network parameters in a static, stationary, or quasi-stationary way are examined. The value of the propagation coefficient of the transmission line is determined on the basis of the field theory.

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