ON THE CONNECTIONS BETWEEN HELSON SETS AND APPROXIMATION BY CONVOLUTIONS

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§ 1. In the well known book of RUDIN [1]* the following nice characterization of Helson sets can be found: (Thm. 5.6.3)

(H). For any compact set P, let $I(P)$ be the set of all functions in $L^1(\Gamma)$ such that $\hat{f}(x) = 0$ for all $x \in P$ and the dual space of the quotient space $L^1(\Gamma)/I(P)$ be $\Phi(P)$. Then the following three properties of a compact set P in a locally compact Abelian group G are equivalent:

a) P is a Helson set.

b) $\|\mu\|$ and $\|\hat{\mu}\|_\infty$ are equivalent norms on $M(P)$.

c) Each $\Phi \in \Phi(P)$ is (equal almost everywhere to) the Fourier-Stieltjes transform of a $\mu \in M(P)$.

The purpose of this paper is to show that theorem (H) is the particular case for $p = 2$ of a theorem concerning translation-invariant operators in $L^p(G)$ $1 < p < \infty$ and meaningful for non-commutative G too. The investigations leading to the present paper were deeply motivated by EDWARDS [4]. His theorem 1.4 in [4] is essentially identical with the first part of our Theorem 1 on the equivalence of I and II. Assertion III and the cases of $p \neq q$ investigated in Theorem 2 serve to make more clear the connections between theorem (H), the approximation by convolutions and translation invariant operators in $L^p(G)$ $(1 < p < \infty)$. In § 3 we shall show that to a certain extent our results are valid for non-commutative G too. In § 4, by generalizing Theorem 1.4 in [4] to general Banach spaces, the exact correspondence will be shown between Theorem 1 and the main results of EDWARDS [4]

§ 2. A continuous linear operator T from $L^p(G)$ into $L^q(G)$ $(1 \leq p \leq q < < \infty)$ is called *translation invariant* if $TU_t = U_tT$ for $U_t : U_t f(\tau) = f(t + \tau)$. It is easy to verify that the operator T_μ defined by $T_\mu f = \mu * f$, where $\mu * f = = \int_G f(x - y) d\mu(x)$, is a translation invariant operator in $L^p(G)$ $(1 \leq p < \infty)$ for every $\mu \in M(G)$ and we have for the norm of T_μ

$$\|T_\mu\|_p \leq \|\mu\| .$$

* The notations and terminology of [1] will be throughout followed.

In the case of a commutative G, the Banach space of translation invariant operators in L^p $(1 < p < \infty)$ can be characterized as follows: [5]

Let A_p be the set of all functions on G which are of the form $\sum_{k=1}^{\infty} f_k * g_k$ with $f_k \in L^p$, $g_k \in L_{p'}$* and $\sum \|f_k\|_p \|g_k\|_{p'} < \infty$. For $h \in A_p$ define

$$\|h\|_{A_p} = \inf\left\{\sum_{k=1}^{\infty} \|f_k\|_p \|g_k\|_{p'} : h = \sum_{k=1}^{\infty} f_k * g_k\right\},$$

then A_p endowed with the norm defined above is a Banach space.

The Figa-Talamanca's Theorem. The Banach space of translation invariant operators in L^p $(1 < p < \infty)$ is isometric and isomorphic to the dual space A_p^* of A_p. If T is a translation invariant operator in L^p, then T corresponds to the functional φ_T defined by

$$\varphi_T(h) = \sum_{k=1}^{\infty} Tf_k * g_k(0), \tag{1}$$

for $h = \sum_{k=1}^{\infty} f_k * g_k \in A_p$.

Remark. If G is the finite dimensional Euclidian space and $1 \leq p \leq q < < \infty$, then each translation invariant operator T from L^p into L^q can be represented by a tempered distribution T as a convolution operator in the following sense [7]:
For every testing function ψ, $T\psi = T * \psi$.

supp T is the support of the functional φ_T corresponding to T by (1).

Now we can state the generalization of (H) as follows:

Theorem 1. The following three properties of a closed set P in a locally compact Abelian group G are equivalent:

I. For every $F \in C_0(P)$, there exists $R > 0$, $f_k \in L^p(G)$, $g_k \in L^{p'}(G)$, λ_k complex numbers $k = 1, 2, \ldots$ so that $\|f_k\|_p \leq R$, $\|g_k\|_{p'} \leq R$, $\sum_{k=1}^{\infty} |\lambda_k| \leq 1$ and

$$F(x) = \left(\sum_{k=1}^{\infty} \lambda_k f_k * g_k\right)(x) \qquad \text{uniformly for } x \in P.$$

II. There is a number $K > 0$ depending only on p and P such that for every $\mu \in M(P)$

$$K\|\mu\| \leq \|T_\mu\|_p \leq \|\mu\|;$$

i.e. $\|\ldots\|$ and $\|\ldots\|_p$ are equivalent norms on $M(P)$.

* Here and in the following, superscript marked by strokes will mean the conjugate Lebesgue space.

III. For every translation invariant operator T of $L^p(G)$ for which $\operatorname{supp} T \subset P$ there exists $\mu \in M(P)$ so that $T = T_\mu$.

First we shall show that Theorem 1 for $p = 2$ is equivalent to (H).

$A(G)$ consists precisely of the convolutions $f_1 * f_2$ with f_1 and f_2 in $L^2(G)$ ([1], Theorem 1.6.3), hence $A \subset A_2$. On the other hand from

$$\|h\|_A = \|f_1 * f_2\|_A = \|\hat{f}_1 \hat{f}_2\|_1 \leq \|\hat{f}_1\|_2 \|\hat{f}_2\|_2 = \|f_1\|_2 \|f_2\|_2 \tag{2}$$

follows that if $\sum_{k=1}^{\infty} \|f_k\|_2 \|g_k\|_2 < \infty$ then $\sum_{k=1}^{\infty} f_k * g_k$ is a Cauchy sequence in A and hence $\sum_{k=1}^{\infty} f_k * g_k \in A$. Consequently, A and A_2 consist of the same functions.

Similarly it follows from (2) that

$$\left\| \sum_{k=1}^{\infty} f_k * g_k \right\|_A \leq \left\| \sum_{k=1}^{\infty} f_k * g_k \right\|_{A_2}$$

hence from the well-known theorem of Banach we conclude that the norms $\| \ldots \|_A$ and $\| \ldots \|_{A_2}$ are equivalent.

From the Figa-Talamanca Theorem it follows that the Banach space of the translation invariant operators in L^2 is isometric and isomorphic to A_2^* and on the other hand from the definition of $A(G)$ it follows that $A(G)^*$ is isometric and isomorphic to $L^\infty(\Gamma)$. Hence $A_2^* = L^\infty(\Gamma)$ apart from a homeomorphism.

Now it is also clear that the set

$$\{T : \varphi_T \in A_2^*, \ \operatorname{supp} T \subset P\}$$

is identical with $\Phi(P)$ in (H).

Hereby the equivalence of Theorem 1 for $p = 2$ and (H) is proved.

The proof of Theorem 1 is the same as that of (H). It is also a corollary of the following theorem: ([1] Appendix C.11)

(A) Suppose X, Y are Banach spaces, B is a continuous injection (i.e. 1—1) from X into Y and BX is dense in Y. Then each of the following three properties implies the other two:

(a) $BX = Y$

(b) There exists $\delta > 0$ so that $\|B^* y^*\| \geq \delta \|y^*\|$ for every $y^* \in Y^*$.

(c) $B^* Y^* = X^*$.

Indeed, if

$$I(P) = \left\{ h \in A_p, \ h(x) = 0 \ \text{for} \ x \in P \right\},$$

and X is the quotient space $A_p(G)/I(P)$, $Y = C_0(P)$ in (A) then we get Theorem 1.

As an other corollary of the theorem (A) we have an assertion for translation invariant operators from L^p into L^q $(1 \leq p < q < \infty)$ similar to Theorem 1.

It is easy to verify that the operator T_h defined by $T_h f = h * f$ is a translation invariant operator from L^p into L^q if $h \in L^r$ and $\frac{1}{r} + \frac{1}{p} - \frac{1}{q} = 1$. Moreover, we have for the norm of T_h

$$\|T_h\|_p^q \leq \|h\|_r .$$

For commutative G, the Banach space of translation invariant operators from L^p into L^q can be characterized as follows [6]:

Let $A_p^q \left(\frac{1}{p} + \frac{1}{q} > 1\right)$ be the set of all functions on G which are of the form $\sum_{k=1}^{\infty} f_k * g_k$ with $f_k, g_k \in C_c$ and $\sum_{k=1}^{\infty} \|f_k\|_p \|g_k\|_{q'} < \infty$. For $h \in A_p^q$ define

$$\|h\| = \inf \left\{ \sum_{k=1}^{\infty} \|f_k\|_p \|g_k\|_{q'} : h = \sum_{k=1}^{\infty} f_k * g_k \right\}$$

then A_p^q endowed with the norm defined above is a Banach space.

The Theorem of Figa-Talamanca and Gaudry. The Banach space of translation invariant operators from L^p into L^q $(1 \leq p < q < \infty)$ is isomorphic and isometric to the dual space A_p^{q*} of A_p^q. If T is a translation invariant operator then T corresponds to the functional φ_T defined by

$$\varphi_T(h) = \sum_{k=1}^{\infty} Tf_k * g_k(0) \tag{3}$$

for $h = \sum_{k=1}^{\infty} f_k * g_k \in A_q^p$.

Again, we define supp T as the support of the functional φ_T corresponding to T by (3).

If $I(P) = \{h \in A_p^q,\ h(x) = 0 \text{ for } x \in P\}$, X is the quotient space $A_p^q(G)/I(P)$ and $Y = L^{r'}(P)$, $\left(\frac{1}{p} + \frac{1}{q'} - \frac{1}{r'} = 1\right)$ in the Theorem (A) then we obtain:

Theorem 2. The following three properties of a closed set P in a locally compact Abelian group G are equivalent:

I. For every $F \in L^{r'}(P)$ there exist $R > 0$: f_k, $g_k \in C_c(G)$, λ_k complex numbers $k = 1, 2, \ldots$ so that $\|f_k\|_p \leq R$, $\|g_k\|_{q'} < R$, $\sum_{k=1}^{\infty} |\lambda_k| \leq 1$ and

$$F(x) = \left(\sum_{k=1}^{\infty} \lambda_k f_k * g_k\right)(x) \text{ in } L^{r'}\text{-convergence for } x \in P.$$

II. There is a number $K > 0$ depending only on (p, q) and P such that for every $h \in L^r(P)$

$$K\|h\|_r \leq \|T_h\|_p^q \leq \|h\|_r .$$

III. For every translation invariant operator T from L^p into L^q for which $\operatorname{supp} T \subset P$ there exists $h \in L^r(P)$ so that $T = T_h$.

§ 3. The first part of Thm 1 is valid for non commutative G too with the same proof. Considering the II $\Leftrightarrow$ III part, it depends on the validity of the Figa-Talamanca Theorem for non-commutative G.

It follows from the Bipolar Theorem ([2] pp. 35—36.) that A_p^* is the weak operator closure of the operators T_h defined by $T_h f = h * f$ for *any* locally compact G hence, the non-trivial part of the Figa-Talamanca Theorem is that every translation-invariant operator is approximated in such a way.

If $p = 2$ then from the Second Commutant Theorem ([8] Chap. 1. § 5 follows that this is true for any locally compact G.

A similar argument holds for A_p^q $p \neq q$ and Theorem 2.

§ 4. Let X and $\mathbf{Y}$ be Banach spaces and B a continuous homomorphism from X into Y. It is well known that the range of B is dense in $\mathbf{Y}$ if and only if the dual operator B^* is injective (i.e. 1—1). The present paragraph deals with the following stronger approximation property:

(P) For every $y \in \mathbf{Y}$ there exists $K > 0$ and a sequence $\{x_n\}$, $x_n \in X$ so that $\|x_n\| \leq K$ and $y = \lim_{n\to\infty} Bx_n$.

It turns out that there is no proper subset of Y satisfying (P) which means that if (P) is satisfied then B is surjective (i.e. onto).

Theorem 3. The property (b) in the theorem (A) is equivalent with (P).

Proof. $(P) \Rightarrow (b)$: It follows from (P) that the range of B is dense in $\mathbf{Y}$ hence B^* is injective. So, we have only to show that the inverse mapping B^{*-1} from $B^*\mathbf{Y}^*$ onto $\mathbf{Y}^*$ is continuous. I.e. if the sequence $\{B^*y_n^*\}$ is convergent then $\{y_n^*\}$ is also convergent.

It follows from (P) that

$$|< y_n^* - y_m^*, y >| \leq \sup_k |< y_n^* - y_m^*, Bx_k >| =$$
$$= \sup_k |< B^*(y_n^* - y_m^*), x_k >| \leq K\|B^*y_n^* - B^*y_m^*\|$$
$$y_n^*, y_m^* \in Y^* ,$$

or every $y \in \mathbf{Y}$ and from the Banach—Steinhaus
$y_n^*\}$ converges to a certain $y^* \in \mathbf{Y}^*$.

$$= \{x: \|x\| \leq K \quad x \in X\}$$

then $\{S_K^0, K > 0\}$ is a base for X^* ([2] pp. 34—36)
$> 0\}$ is a base for the relative topology of $B^*\mathbf{Y}^*$
3^* is injective and the inverse mapping B^{*-1} from
ιous. Consequently, for every open neighbourhood
:here is a number $K > 0$ such that

$$B^{*-1}(S_K^0 \cap B^* Y^*) \subset \mathcal{U}. \tag{4}$$

that $B^*(BS_K)^0 \subset S_K^0$, hence compared it with (4)

$$(BS_K)^0 \subset \mathcal{U} \tag{5}$$

$\mathcal{U} = \{y^* : |\langle y, y^* \rangle| < 1\}$, then $\mathcal{U}$ is an open
ınce (5) holds. It follows moreover that $y \in (BS_K)^{00}$
: Theorem ([2] pp. 34—36) we have a sequence
$y = \lim_{k \to \infty} y_K$ Q.E.D.

no proper subset of $\mathbf{Y}$ satisfying (P).
ompact group, $L^p \times L^{p'}$ the topological (greatest)
d $L^{p'}(G)$ (see e.g. [2] pp. 130—138) and

$$_k \times g_k: \left(\sum_{k=1}^{\infty} f_k * g_k\right)(x) = 0 \text{ for } x \in P\Big\}.$$

m 3 by the quotient space $L^p \times L^{p'}/N$ and $C_0(P)$
in [4]. Thus Thm 3 is an abstract background of
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ıear.

tion also for two-norm spaces. Let $\langle X, \|\ldots\|,$
>ace, which is not necessarily normal, [3]. If $X =$
e completion of $\langle X, \|\ldots\|^* \rangle$ then we obtain from

-completion and the completion of $\langle X, \|\ldots\|^* \rangle$
ι space $\langle X, \|\ldots\|, \|\ldots\|^* \rangle$ then the norms $\|\ldots\|$

and $\|\ldots\|^*$ are equivalent. Consequently, for a non-trivial two-norm space, the γ-completion is always a proper subset of the completion of $\langle X, \|\ldots\|^*\rangle$.

It is interesting that the following seemingly different approximation property is equivalent to (P).

(P') For every $y \in Y$ with $\|y\| = 1$ there is an $x \in X$ with $\|x\| \leq K$ so that

$$\|y - Bx\| < 1 - \varepsilon$$

where K and ε depend only on B.

Theorem 4. (P') implies property (b) in (A).

Proof. For every $y^* \in Y^*$ we have

$$|\langle y^*, y\rangle| \leq |< y^*, y - Bx >| + |< y^*, Bx >| \leq (1 - \varepsilon) \| y^* \| + $$
$$+ K \| B^*y^* \| .$$

Hence

$$\|y^*\| \leq (1 - \varepsilon) \| y^* \| + K \| B^*y^* \|$$

and

$$\| B^*y^* \| \geq \varepsilon/K \| y^* \| \quad \text{Q.E.D.}$$

Summary

A bounded linear operator from $L^p(G)$ into $L^q(G)$ where $1 \leq p, q < \infty$ and G is a locally compact group is called (p, q)-multiplier if it is commuting with translation. A (p, p)-multiplier is called a p-multiplier. An important class of p-multipliers are the bounded measures as convolution operators but the set of p-multipliers are not exhausted by them. On the basis of a theorem about Helson sets of the well-known book of RUDIN [1], conditions are given for the support of a p-multiplier for being bounded measure.

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