THE CALCULATION OF TRANSMISSION NETWORKS WITH THE AID OF GRAPH THEORY

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Introduction

As is well-known, the Kirchhoff equations of an electric network can be formulated in well arranged matrix equations by using the graph theory. For writing these equations we have to know the topological arrangement of the network, i.e. the arrangement in which the two poles forming the branches of the network are connected to each other. On this basis the graph of the network can be determined, the branches of which correspond to the branches of the electric network.

In the present paper the application of the graph theory for calculating networks consisting of transmission lines or of symmetrical quadripoles (twoterminal-pair) is shown in such a way, that a branch in the graph of the network corresponds to a section of the transmission line or to a quadripole. Accordingly the graph of the transmission line system shown in Fig. 1/a can be seen in Fig. 1/b.

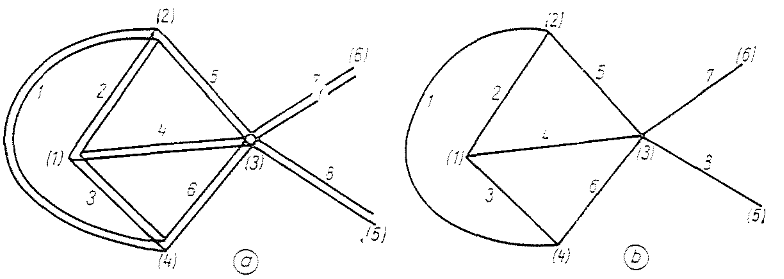

Fig. 1

The connection point between transmission line sections will be denominated as vertex. Branches and vertices will be designated by Arabic figures, and for the sake of discrimination at designating vertices the figures will be placed into brackets. Voltage at the ends of transmission lines connected in a vertex is identical. Beyond the transmission lines, also impedances and generators can be connected to the vertices. In the following that case is examined when a Thevenin generator, i.e. an ideal voltage generator and an impedance connected in series with it, is connected to the vertices (Fig. 2).

In the course of the calculation the source voltage of the generators connected to the vertices and the impedances is assumed to be known. If only a passive impedance is connected to the vertex, the source voltage of the corresponding voltage generator is accordingly zero. If in turn solely the ideal voltage generator is at the vertex, then the impedance is zero. The characteristic impedance, the propagation coefficient, and the length of the individual transmission line sections, and the conductance parameters of the quadripoles in the case of a network consisting of quadripoles are also assumed to be known. In the following the voltages and currents arising at the ends of the transmission line sections, and at the terminal pairs of the quadripoles, respectively, are determined in the knowledge of the above data. From these, voltage and current at any place of whichever transmission line section can be calculated on the basis of the known methods.

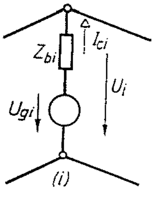

Fig. 2

Characterization of the topology of the network

As mentioned before, the graph of a network is constructed in such a way that a branch corresponds to a transmission line section or to a quadripole, and a vertex of the graph corresponds to the connection point. In the graph constructed in this way there may also be terminal elements, i.e. branches connected to other branches only at one end. For characterizing such a graph, which contains terminal elements as well, the incidence matrix is best suited. In this a vertex corresponds in the order of numbering of the vertices to the individual rows, and a branch corresponds in the order of numbering of the branches to the individual columns. Element a_{ij} of the incidence matrix is equal to 1, if the i-th vertex is in incidence with the j-th branch, and 0, if it is not in incidence. Thus e.g. the incidence matrix of the graph shown in Fig. 1/b is found to be

$$\begin{array}{r} \text{Branch} \longrightarrow \\ \begin{array}{cc} & \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{array}{r} \text{Vertex} \downarrow \; (1) \\ (2) \\ A = \; (3) \\ (4) \\ (5) \\ (6) \end{array} & \left[\begin{array}{cccccccc} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array} \end{array} \qquad (1)$$

In the individual columns two elements are equal to 1, while the others are 0, in accordance with the fact that each branch is in incidence with two vertices.

For writing the equations of the network a direction should be given to each branch (a reference direction should be adopted). This can be chosen arbitrarily, thus e.g. a possible case of the direction of the branches of the graph shown in Fig. 1/b can be seen in Fig. 3. For the directed graph obtained in this way the directed incidence matrix is defined. The elements of the directed

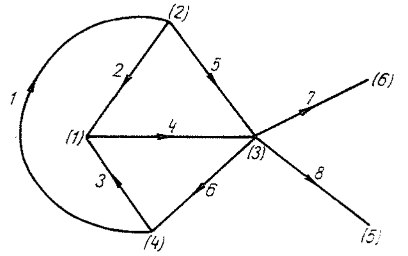

Fig. 3

incidence matrix may be 1, -1, or 0. Namely $a_{ij} = 0$, if the i-th vertex and the j-th branch are not in incidence; $a_{ij} = 1$, if the i-th vertex and the j-th branch are in incidence and the direction of the j-th branch is away from the i-th vertex; $a_{ij} = -1$, if the i-th vertex and the j-th branch are in incidence and the direction of the j-th branch is towards the i-th vertex. Accordingly the directed incidence matrix of the directed graph shown in Fig. 3 is found to be

$$
\boldsymbol{A}_i = \begin{array}{c} \\ (1) \\ (2) \\ (3) \\ (4) \\ (5) \\ (6) \end{array}
\begin{array}{c}
\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\
\begin{bmatrix}
0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & -1 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0
\end{bmatrix}
\end{array}
\tag{2}
$$

In the individual columns one element is equal to 1, one to -1, while the others are 0.

In the followings matrices $\frac{1}{2}(\boldsymbol{A} + \boldsymbol{A}_i)$ and $\frac{1}{2}(\boldsymbol{A} - \boldsymbol{A}_i)$ will also be necessary. In matrix $\frac{1}{2}(\boldsymbol{A} + \boldsymbol{A}_i)$ $a_{ij} = 1$, if the j-th branch is in incidence with the i-th vertex and the direction of the j-th branch is away from the i-th vertex, otherwise $a_{ij} = 0$. In our example:

$$\frac{1}{2}(\boldsymbol{A}+\boldsymbol{A}_i)=\begin{array}{c} \\ (1)\\(2)\\(3)\\(4)\\(5)\\(6)\end{array}\begin{array}{c}\begin{array}{cccccccc}1&2&3&4&5&6&7&8\end{array}\\ \left[\begin{array}{cccccccc}0&0&0&1&0&0&0&0\\0&1&0&0&1&0&0&0\\0&0&0&0&0&1&1&1\\1&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{array}\right]\end{array} \tag{3}$$

In matrix $\frac{1}{2}(\boldsymbol{A}-\boldsymbol{A}_i)$ element $a_{ij}=1$, if the i-th vertex is in incidence with the j-th branch and the j-th branch is oriented towards the i-th vertex, otherwise $a_{ij}=0$. In our example

$$\frac{1}{2}(\boldsymbol{A}-\boldsymbol{A}_i)=\begin{array}{c} \\ (1)\\(2)\\(3)\\(4)\\(5)\\(6)\end{array}\begin{array}{c}\begin{array}{cccccccc}1&2&3&4&5&6&7&8\end{array}\\ \left[\begin{array}{cccccccc}0&1&1&0&0&0&0&0\\1&0&0&0&0&0&0&0\\0&0&0&1&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&1&0\end{array}\right]\end{array} \tag{4}$$

In each column of this last matrix one element is equal to 1, while the others are 0.

Characterization of one branch of the network

In usual networks the branches are formed by one two-pole each. A passive linear branch can be characterized by a single impedance function. This establishes the correlation between branch current and branch voltage. In our problem, however, the branches of the graph are symmetrical quadripoles (Fig. 4). For characterizing a quadripole the correlation between two voltages

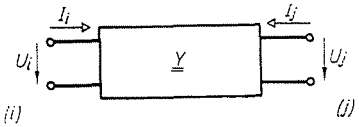

Fig. 4

(U_i, U_j) and two currents (I_i, I_j) should be given. In the case of a linear quadripole this can be given, among others, in terms of the conductance parameters:

$$\begin{bmatrix} I_i \\ I_j \end{bmatrix} = \mathbf{Y} \begin{bmatrix} U_i \\ U_j \end{bmatrix} \tag{5}$$

Among the four elements of the conductance matrix three are independent of each other in the case of a reciprocal network, and only two in the case of a symmetrical reciprocal quadripole. In the following only networks built up of symmetrical reciprocal quadripoles will be examined. In this case the conductance matrix is of the form

$$\mathbf{Y} = \begin{bmatrix} r & p \\ p & r \end{bmatrix} \tag{6}$$

Transmission lines are characterized by the characteristic impedance Z_0, the propagation coefficient γ, and the length l (Fig. 5). It is sufficient to know the characteristic impedance Z_0 and the value $g = e^{-\gamma l}$. In a network consisting of several connected transmission lines neither of the ends of the transmission

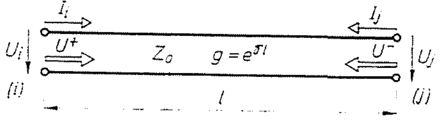

Fig. 5

line are in general preferred. It is practical to write the equations for the transmission line sections accordingly. The reference directions at the two ends of the transmission line section are taken in accordance with Fig. 5. At the ends connected to the same vertex the reference direction of voltages is identical.

As is well known, in general two waves are propagating in the line in opposite direction to each other. Let us designate the wave propagating in the direction coinciding with the reference direction by the sign $+$, and the one propagating in the opposite direction by $-$. Wave equations are written in such a way that the starting point of the waves is taken as the origin. Thus in the case of the transmission line shown in Fig. 5 the zero coordinate of the $+$ wave is the vertex with index i, while that of the $-$ wave the j index vertex. Accordingly

$$U_i = U^+ + e^{-\gamma l} U^- = U^+ + gU^- \tag{7}$$

the value of the voltage at the i-th vertex. The voltage at the j-th vertex can be written similarly:

$$U_j = gU^+ + U^- \tag{8}$$

Equations (7) and (8) can be written also in the form of a matrix equation:

$$\begin{bmatrix} U_i \\ U_j \end{bmatrix} = \begin{bmatrix} 1 & g \\ g & 1 \end{bmatrix} \begin{bmatrix} U^+ \\ U^- \end{bmatrix} = \boldsymbol{T} \begin{bmatrix} U^+ \\ U^- \end{bmatrix} \tag{9}$$

where

$$\boldsymbol{T} = \begin{bmatrix} 1 & g \\ g & 1 \end{bmatrix}. \tag{10}$$

With the denominations used above, the following relationships can be written for the currents:

$$\begin{aligned} I_i &= Y_0[U^+ - gU^-] \\ I_j &= Y_0[-gU^+ + U^-] \end{aligned} \tag{11}$$

where $Y_0 = Z_0^{-1}$ the reciprocal of the characteristic impedance. Taking into consideration that

$$\boldsymbol{T}^{-1} = \frac{1}{1-g^2} \begin{bmatrix} 1 & -g \\ -g & 1 \end{bmatrix} \tag{12}$$

equation (11) in matrix form is

$$\begin{bmatrix} I_i \\ I_j \end{bmatrix} = Y_0 \begin{bmatrix} 1 & -g \\ -g & 1 \end{bmatrix} \begin{bmatrix} U^+ \\ U^- \end{bmatrix} = Y_0(1-g^2)\boldsymbol{T}^{-1} \begin{bmatrix} U^+ \\ U^- \end{bmatrix} \tag{13}$$

The correlation between currents and voltages can be expressed from Equations (9) and (13).

$$\begin{bmatrix} I_i \\ I_j \end{bmatrix} = Y_0(1-g^2)\boldsymbol{T}^{-2} \begin{bmatrix} U_i \\ U_j \end{bmatrix} = \boldsymbol{Y} \begin{bmatrix} U_i \\ U_j \end{bmatrix} \tag{14}$$

what means that the matrix of the conductance parameters of the transmission line section is

$$\boldsymbol{Y} = Y_0(1-g^2)\boldsymbol{T}^{-2} \tag{15}$$

Upon considering (12), using the relationship $g = e^{-\gamma l}$, the elements of matrix $\boldsymbol{Y}$ can be determined.

$$\boldsymbol{Y} = Y_0 \begin{bmatrix} \coth \gamma l & -\dfrac{1}{\sinh \gamma l} \\ -\dfrac{1}{\sinh \gamma l} & \coth \gamma l \end{bmatrix} = \begin{bmatrix} r & p \\ p & r \end{bmatrix} \tag{16}$$

what is naturally identical formally with (6).

Let us form diagonal matrices of the values r_i and p_i, respectively, which characterize the branches.

$$\boldsymbol{R} = \langle r_1 \; r_2 \ldots r_k \rangle$$
$$\boldsymbol{P} = \langle p_1 \; p_2 \ldots p_k \rangle \tag{17}$$

In the following the matrices $\boldsymbol{R}$ and $\boldsymbol{P}$ will be used for characterizing the transmission line sections of the network.

The solvability of the problem

In the following we write the Kirchhoff equations of the network.

A transmission line section or a quadripole are characterized by two voltages, and two current data. Between these, two independent relationships can be written. Upon considering this, each branch means two unknown values. If the network consists of k branches, then the number of unknown values is $2k$.

Let us examine the number of equations that can be written for the network. Voltages at the ends of transmission lines (quadripoles) connected to the same vertex are equal and identical with the voltage at the vertex. Accordingly $h - 1$ independent voltage equations can be written for a vertex to which h branches are connected. Let c designate the number of vertices in the network. Then $\Sigma(h - 1) = 2k - c$ voltage equations can be written in all for all the vertices. Summation should be performed for c vertices, thus $\Sigma h = 2k$.

Two nodes belong to each vertex. For the individual nodes one node equation can be written. Node equations written for the two nodes belonging to the same vertex are identical. Thus the number of independent node equations is identical with the number of vertices c.

Accordingly, a total of $2k - c + c = 2k$, voltage and node equations can be written and the number of unknown values is the same, what means that the problem can be solved unambiguously.

Circuit equations

Circuit equations will be written in the following in such a way that voltage equations should be satisfied automatically and thus only c pieces of independent node equations should be written. In the equations the voltages arising at the vertices are unknown. The number of these is also c, hence the voltages at the vertices can be determined from the node equations.

From vertex voltages we can determine the currents flowing at the ends of the transmission line sections, further the currents in the generators and impedances at the vertices.

Currents flowing out or into one of the nodes of some of the vertices can be written as the sum of three groups. To the first group those currents belong the direction of which is identical with the direction of the corresponding branch. To the second those the direction of which is contrary to the direction of the corresponding branch. Finally the currents flowing through the generator or impedance being between the two nodes of the vertex figure the third group.

Our method is illustrated by the example discussed above. Branches 4. 5, 6, 7, 8 are in incidence with node (3) (Fig. 6). The nodal equation is written

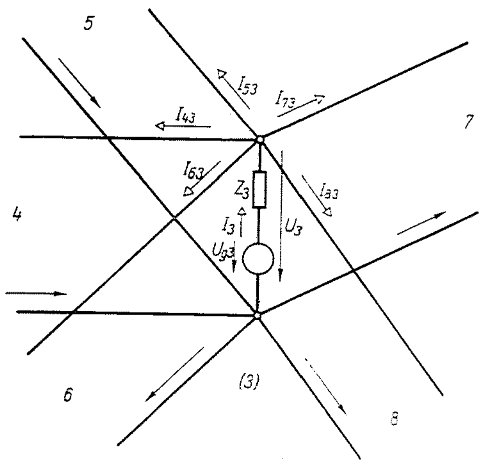

Fig. 6

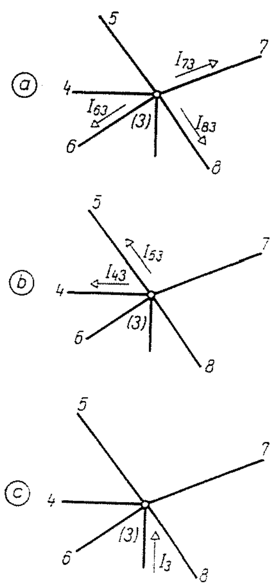

Fig. 7

for that node, from which the voltage of the vertex is directed away. The direction of branches 6, 7, and 8 deviates from the vertex (Fig. 7a). According to the foregoing the sum of these is written in the first group. The direction of branches 4 and 5 points towards vertex (3) (Fig. 7b). The current of these branches is written in the second group. Finally the current of the generator arranged between the nodes of the vertex represents the third group (Fig. 7c).

We write the nodal equation for one of the nodes of each vertex of the network.

If the branch l is in incidence with vertices (i) and (j), and its direction is from (i) towards (j), then the current of the branch in the first group is

$$I_{li} = r_l U_i + p_l U_j \tag{18}$$

Similar equations can be written for each branch. The system of equations obtained can be summarized in the following matrix equation:

$$\mathbf{I}' = \frac{1}{2}\boldsymbol{R}(\boldsymbol{A}^* + \boldsymbol{A}_i^*)\,\mathbf{U} + \frac{1}{2}\boldsymbol{P}(\boldsymbol{A}^* - \boldsymbol{A}_i^*)\mathbf{U} \tag{19}$$

The asterisk designates the transposed value of the matrix.

The column vector formed of the vertex voltages in our example is found to be

$$\mathbf{U} = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{bmatrix} \tag{20}$$

Thus on the basis of (19) the column vector formed of the currents of the first group:

$$\mathbf{I}' = \begin{bmatrix} I_{14} \\ I_{22} \\ I_{34} \\ I_{41} \\ I_{52} \\ I_{63} \\ I_{73} \\ I_{83} \end{bmatrix} = \begin{bmatrix} r_1 U_4 + p_1 U_2 \\ r_2 U_2 + p_2 U_1 \\ r_3 U_4 + p_3 U_1 \\ r_4 U_1 + p_4 U_3 \\ r_5 U_2 + p_5 U_3 \\ r_6 U_3 + p_6 U_4 \\ r_7 U_3 + p_7 U_6 \\ r_8 U_3 + p_8 U_5 \end{bmatrix} \tag{21}$$

The elements of $\mathbf{I}'$ evidently correspond to that what has been written in (18). The subscripts of r and p are serial numbers of the branch, while those of U are the serial numbers of the vertex. The currents forming $\mathbf{I}'$ are shown in Fig. 8a.

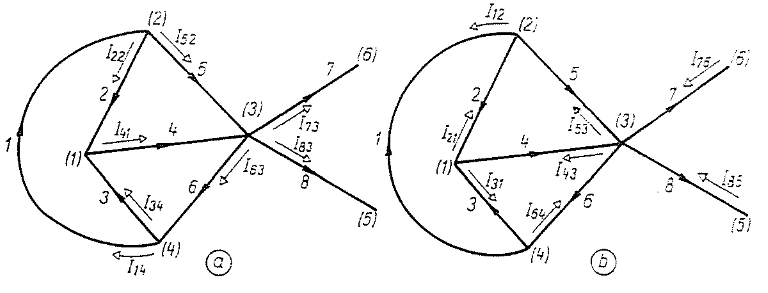

Fig. 8

For the l-th branch, the current belonging to the second group is found to be

$$I_{lj} = p_l U_i + r_l U_j \tag{22}$$

Similar equations can be written for each branch.
This system of equations is the following:

$$\mathbf{I}'' = \frac{1}{2}\boldsymbol{P}(\boldsymbol{A}^* + \boldsymbol{A}_i^*)\mathbf{U} + \frac{1}{2}\boldsymbol{R}(\boldsymbol{A}^* - \boldsymbol{A}_i^*)\mathbf{U} \tag{23}$$

In our example

$$\mathbf{I}'' = \begin{bmatrix} p_1 U_4 + r_1 U_2 \\ p_2 U_2 + r_2 U_1 \\ p_3 U_4 + r_3 U_1 \\ p_4 U_1 + r_4 U_3 \\ p_5 U_2 + r_5 U_3 \\ p_6 U_3 + r_6 U_4 \\ p_7 U_3 + r_7 U_6 \\ p_8 U_3 + r_8 U_5 \end{bmatrix} \tag{24}$$

Currents belonging to the second group are indicated in Fig. 8b.

We have still to determine the currents in the generators and impedances arranged at the vertices. As mentioned before, we are examining the case when between the nodes of the vertex a voltage generator and an impedance are connected in series (Fig. 2). We may write for the i-th vertex that

$$I_{ci} = Y_{bi}(U_{gi} - U_i) \tag{25}$$

where Y_{bi} is the admittance of the branch between the nodes of the vertex, U_{gi} is the source voltage in the branch.

Such equations can be written for each vertex and these can be summarized in the form

$$\mathbf{I}_c = \boldsymbol{Y}_b(\mathbf{U}_g - \mathbf{U}) \tag{26}$$

where $\mathbf{I}_c$ is the column vector formed of the current of the generators and impedances at the vertices, $\mathbf{U}_g$ the column vector of the source voltage of the voltage generators, and $\boldsymbol{Y}_b$ is a diagonal matrix in the principal diagonal of which the values of the admittances at the vertices are figuring.

The currents should satisfy the nodal equations. From one of the nodes the currents written in $\mathbf{I}'$ are flowing away. Let us form from these the sum of current belonging to one node each vertices and designate the column matrix formed of these by $\mathbf{I}_c'$:

$$\mathbf{I}_c' = \frac{1}{2}(\boldsymbol{A} + \boldsymbol{A}_i)\mathbf{I}' \tag{27}$$

In our example we obtain, upon substituting (3) and (21), that

$$\mathbf{I}'_c = \begin{bmatrix} r_4U_1 + p_4U_3 \\ r_2U_2 + p_2U_1 + r_5U_2 + p_5U_3 \\ r_6U_3 + p_6U_4 + r_7U_3 + p_7U_6 + r_8U_3 + p_8U_5 \\ r_1U_4 + p_1U_2 + r_3U_4 + p_3U_1 \\ 0 \\ 0 \end{bmatrix} \tag{28}$$

The branch currents forming $\mathbf{I}''$ are flowing away from one of the nodes of the vertices. Let us form the $\mathbf{I}''_c$ column matrix from currents flowing away from one of the nodes of the vertices.

$$\mathbf{I}''_c = \frac{1}{2}(A - A_i)\mathbf{I}'' \tag{29}$$

Let us write also this for the discussed example, by substituting (4) and (24):

$$\mathbf{I}''_c = \begin{bmatrix} p_2U_2 + r_2U_1 + p_3U_4 + r_3U_1 \\ p_1U_4 + r_1U_2 \\ p_4U_1 + r_4U_3 + p_5U_2 + r_5U_3 \\ p_6U_3 + r_6U_4 \\ p_8U_3 + r_8U_5 \\ p_7U_3 + r_7U_6 \end{bmatrix} \tag{30}$$

The currents represented in $\mathbf{I}_c$ are flowing towards that node of the vertex, from which the corresponding currents of $\mathbf{I}'_c$ and $\mathbf{I}''_c$ are flowing out. Thus the matrix form of the node equation, upon using (26), (28), (29), further (19) and (23), is found to be

$$\begin{aligned} \mathbf{I}'_c + \mathbf{I}''_c - \mathbf{I}_c = \frac{1}{4}(A + A_i)\left[\boldsymbol{R}(A^* + A_i^*) + \boldsymbol{P}(A^* - A_i^*)\right]\mathbf{U} + \\ + \frac{1}{4}(A - A_i)\left[\boldsymbol{P}(A^* + A_i^*) + \boldsymbol{R}(A^* - A_i^*)\right]\mathbf{U} + \\ + \boldsymbol{Y}_b\mathbf{U} - \boldsymbol{Y}_b\mathbf{U}_g = \mathbf{0} \end{aligned} \tag{31}$$

After ordering we obtain that

$$\left[\frac{1}{2}A(\boldsymbol{R} + \boldsymbol{P})A^* + \frac{1}{2}A_i(\boldsymbol{R} - \boldsymbol{P})A_i^* + \boldsymbol{Y}_b\right]\mathbf{U} = \boldsymbol{Y}_b\mathbf{U}_g \tag{32}$$

In equation (32) the multiplication factor of $\mathbf{U}$ can be termed the vertex admittance matrix:

$$Y_c = \frac{1}{2} A(R + P)A^* + \frac{1}{2} A_i(R - P)A_i^* + Y_b \tag{33}$$

Let us write also the first two terms of this for our example.

$$\frac{1}{2} A(R + P)A^* + \frac{1}{2} A_i(R - P)A_i^* =$$

$$= \begin{bmatrix} r_2 + r_3 + r_4 & p_2 & p_4 & p_3 & 0 & 0 \\ p_2 & r_1 + r_2 + r_5 & p_5 & p_1 & 0 & 0 \\ p_4 & p_5 & r_1 + r_5 + r_6 + r_7 + r_8 & p_6 & p_8 & p_7 \\ p_3 & p_1 & p_6 & r_1 + r_3 + r_6 & 0 & 0 \\ 0 & 0 & p_8 & 0 & p_8 & 0 \\ 0 & 0 & p_7 & 0 & 0 & p_7 \end{bmatrix} \tag{34}$$

This matrix is seen to be symmetrical. In the principal diagonal the r values pertaining to branches in incidence with the vertex corresponding to the row (column) are figuring. The other elements are the p values pertaining to the branch connecting the vertices corresponding to the row and column. If the two vertices are not connected then the corresponding matrix element is 0.

If one or several of the generators connected to the vertices are ideal, then the corresponding elements of Y_b are infinitely high. In this case it is practical to rewrite the previous equation in such a way that $Z_b = Y_b^{-1}$ figures in it:

$$\left\{\frac{1}{2} Z_b \left[A(R + P)A^* + A_i(R - P)A_i^*\right] + E\right\} \mathbf{U} = \mathbf{U}_g \tag{35}$$

From these the required matrix $\mathbf{U}$ is found to be

$$\mathbf{U} = \left[\frac{1}{2} A(R + P)A^* + \frac{1}{2} A_i(R - P)A_i^* + Y_b\right]^{-1} Y_b \mathbf{U}_g, \tag{36}$$

and

$$\mathbf{U} = \left\{\frac{1}{2} Z_b\left[A(R + P)A^* + A_i(R - P)A_i^*\right] + E\right\}^{-1} \mathbf{U}_g, \tag{37}$$

respectively.

In the knowledge of $\mathbf{U}$ the branch currents $\mathbf{I}'$ and $\mathbf{I}''$ can be determined on the basis of (19) and (23), thus the problem can be regarded as solved.

Powers, efficiency

On the basis of the results obtained so far the power of the generators and consumers, and in the knowledge of these values also the efficiency of the system can be determined.

The complex power of the generators can be calculated from the source voltage matrix $\mathbf{U}_g$ and from the current matrix $\mathbf{I}_c$.

$$S_g = P_g + jQ_g = \mathbf{U}_g^* \bar{\mathbf{I}}_c = \sum_{m=1}^{c} U_{gm} \bar{I}_{cm} \tag{38}$$

where the dash above designates the conjugate. (The complex power is written as the scalar product of $\mathbf{U}_g$ and $\mathbf{I}_c$.)

$\mathbf{I}_c$ can be determined relatively easily if the network does not contain an ideal voltage generator. Then, by using (26) and (36)

$$S_g = \mathbf{U}_g^* \bar{\boldsymbol{Y}}_b \left\{ \boldsymbol{E} - \left[\frac{1}{2} \boldsymbol{A}(\bar{\boldsymbol{R}} + \bar{\boldsymbol{P}})\boldsymbol{A}^* + \frac{1}{2} \boldsymbol{A}_i(\bar{\boldsymbol{R}} - \bar{\boldsymbol{P}})\boldsymbol{A}_i^* + \bar{\boldsymbol{Y}}_b \right]^{-1} \boldsymbol{Y}_b \right\} \bar{\mathbf{U}}_g \tag{39}$$

If the network does contain also an ideal voltage generator, then $\mathbf{I}_c$ should be calculated on the basis of (31).

$$\mathbf{I}_c = \mathbf{I}_c' + \mathbf{I}_c'' = \frac{1}{2} [\boldsymbol{A}(R + \boldsymbol{P})\boldsymbol{A}^* + \boldsymbol{A}_i(R - \boldsymbol{P})\boldsymbol{A}_i^*]\mathbf{U} \tag{40}$$

Considering expressions (37) and (40), the complex power of the generators can be written in the form of the following expression:

$$\begin{aligned} \mathbf{S}_g = \mathbf{U}_g^* \frac{1}{2} \{\boldsymbol{A}(\bar{\boldsymbol{R}} + \bar{\boldsymbol{P}})\boldsymbol{A}^* + \boldsymbol{A}_i(\bar{\boldsymbol{R}} - \bar{\boldsymbol{P}})\boldsymbol{A}_i^*\} \Big\{ \boldsymbol{E} + \\ - \frac{1}{2} \bar{\boldsymbol{Z}}_b [\boldsymbol{A}(\bar{\boldsymbol{R}} + \bar{\boldsymbol{P}})\boldsymbol{A}^* + \boldsymbol{A}_i(\bar{\boldsymbol{R}} - \bar{\boldsymbol{P}})\boldsymbol{A}_i^*] \Big\}^{-1} \bar{\mathbf{U}}_g \end{aligned} \tag{41}$$

The effective power of the generators is the real part of $\mathbf{S}_g$.

$$P_g = \mathbf{Re}\, S_g \tag{42}$$

In writing the power of the consumers, the elements of $\boldsymbol{Z}_b$ should be separated from the internal impedance Z_{gm} of the generators and the impedance Z_{tm} of the consumers. Let $\boldsymbol{Z}_g$ designate the diagonal matrix formed of the internal impedance $\boldsymbol{Z}_{gm}$ of the generators, while the diagonal matrix $\boldsymbol{Z}_t$ is built of the elements $\boldsymbol{Z}_{tm}$. Then

$$\boldsymbol{Z}_b = \boldsymbol{Z}_g + \boldsymbol{Z}_t \tag{43}$$

Current I_{cm} is flowing in impedance Z_{tm}, thus the expression of the column matrix $\mathbf{U}_t$ consisting of the voltages arising at the consumers is found to be

$$\mathbf{U}_t = \mathbf{Z}_t \mathbf{I}_c \tag{44}$$

The complex power of the consumers is

$$\mathbf{S}_t = \mathbf{U}_t^* \bar{\mathbf{I}}_c = (\mathbf{Z}_t \mathbf{I}_c)^* \bar{\mathbf{I}}_c = \mathbf{I}_c^* \mathbf{Z}_t \bar{\mathbf{I}}_c \tag{45}$$

$\mathbf{I}_c$ can be calculated from (40). If the network does not contain an ideal voltage generator, then in place of (40) the somewhat more simple expression used also in (39) can similarly be employed.

The effective power of the consumers is given by

$$P_t = \mathbf{Re} S_t \tag{46}$$

and the efficiency of the complete network by

$$\eta = \frac{P_t}{P_g} = \mathbf{Re}\,\frac{S_t}{S_g} \tag{47}$$

The complex power and efficiency of the generators and consumers is hereby determined.

Summary

It is known that the Kirchhoff equations of an electric network can be formulated in well arranged matrix equations by using the graph theory. In the paper this method is applied for networks consisting of transmission lines or symmetrical quadripoles. As the final result the correlation between the generator voltages and the voltages arising at the terminal points of the transmission line can be expressed by the help of a single matrix equation. The method is suitable also for determining powers.

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