QUASI-PERIODIC DYNAMIC BEHAVIOUR OF PIECEWISE LINEAR MULTI-PARAMETER SYSTEMS

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1. Introduction

In the solution of power control problems, thyristors and diodes are of rather increasing importance. In conduction they are equivalent to short circuit (condition s), otherwise to open circuit (condition b). A similar situation exists in mercury arc rectifier or relay schemes. In such systems, the steady state condition is usually periodic, with a period of τ. If the system involves, for example, only a single thyristor, then τ may be cut up to $\tau = \tau_s + \tau_b$; in τ_s the thyristor is on and in τ_b it is off. It is assumed that the system may be described by constant coefficient linear differential equations, both for τ_s and τ_b, although the parameters are different in the two states.

Three-phase thyristor circuits are frequently used. The bridge circuit, for example, involves six thyristors and each period consists, generally, of 12 different conduction conditions. In most cases, however, it is quite sufficient to study two conditions since, knowing the data of a single one-sixth period, those of the other one-sixth periods may be obtained through phase and sign changes. In such cases τ indicates one-sixth of the whole period (or sometimes its one-third and in case of single-phase bridge circuit its half).

Condition s is over when the particular thyristor is off; thus τ_s is usually given by the solution of a transcendent equation. At the end of condition b, a thyristor is switched on either in a predetermined instant or depending on the control voltage u_c and the value of the input or output signals of the system. The latter case gives in general a transcendent equation.

First the paper deals with the determination of the periodical condition (Chapter 2); the initial conditions and the τ_s, τ_b must satisfy the periodicity conditions. In such cases one or two transcendent equations must be solved.

Calculating curve-shares it is better to assume values for τ_s and τ_b in advance, consequently two other corresponding variables (e.g. firing angle, control voltage etc.) may be determined more simply. Chapter 3 investigates the small variations compared to the periodic steady state condition; thus, as first approximation, a linear difference equation system — well known from the theory of sampling systems — is obtained. The coefficients of the equations are given from the data of the periodic steady state condition by simple algebraic species whereby the transient and frequency responses, etc., concerning small variations can be readily determined.

Since the system is of multi-parameter character, it is preferable to use matrices. These matrices are indicated by bold type upper case letters (e.g. $\mathbf{A}$), but the special matrices consisting of a single column, the column vectors, are represented by bold type lower case letters (e.g. $\mathbf{x}$); transponation is marked by an asterisk, thus e.g. $\mathbf{x}^*$ is a row vector.

The functions of quadratic matrices, f($\mathbf{A}$) can be calculated by power series, but if several functions of the same matrix $\mathbf{A}$ must be calculated, it is best to convert $\mathbf{A}$ to a normal form. If each eigenvalue is distinguished, it may be written:

$$\mathbf{A} = \mathbf{T}\,\Lambda\,\mathbf{T}^{-1}, \qquad \mathbf{T} = [\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_n], \qquad \Lambda = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}$$

and

$$f(\mathbf{A}) = \mathbf{T} f(\Lambda)\,\mathbf{T}^{-1}, \qquad f(\Lambda) = \begin{bmatrix} f(\lambda_1) & 0 & \ldots & 0 \\ 0 & f(\lambda_2) & \ldots & 0 \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ 0 & 0 & \ldots & f(\lambda_n) \end{bmatrix}$$

where λ_i is the eigenvalue of matrix $\mathbf{A}$, and $\mathbf{s}_i$ is post eigenvector, $\mathbf{A}\mathbf{s}_i = \lambda_i\,\mathbf{s}_i$. In case of complex eigenvalues the column pairs of $\mathbf{T}$, Λ or $f(\Lambda)$ can be modified into an other form, requiring only real algebraic operations.

In Chapter 4 the theoretical results obtained previously are applied for three different practical cases: (1) thyristor inverter with an R, L, C load, (2) D.C. motor fed by a three-phase bridge rectifier, taking into consideration the speed fluctuation, and (3) three-phase induction motor, with back-to-back thyristor pairs connected in series with the stator windings.

2. Periodical condition

The differential equations of the system are written in the following form, with indices s and b referring to the on and off condition of the particular thyristor:

$$\left.\begin{aligned} \boldsymbol{L}_s \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} + \boldsymbol{R}_s \boldsymbol{x} = \boldsymbol{D}_s \boldsymbol{u}\,, \qquad & t_0 \leq t \leq t_0 + \tau_s\,, \\ \boldsymbol{L}_b \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} + \boldsymbol{R}_b \boldsymbol{x} = \boldsymbol{D}_b \boldsymbol{u} \qquad & t_0 + \tau_s \leq t \leq t_0 + \tau \end{aligned}\right\}. \tag{1}$$

As an example, in case of the inverter studied in Paragraph 4.1 (Fig. 1), the column vector elements of dependent variable $\boldsymbol{x}(t)$ have been selected as the characteristic data of the energy storages, where x_1, x_2 and x_5 represent the currents of the respective coils, while x_3 and x_4 are the voltages of the capa-

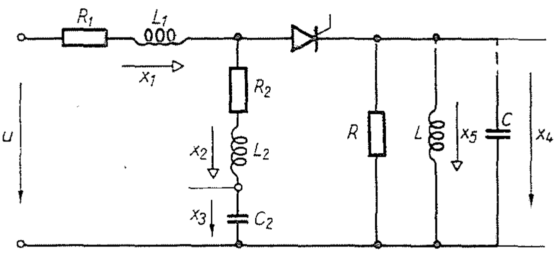

Fig. 1

citors. It is best to select such variables as, in this case, the matrix dimension will not be unnecessarily large and, on the other hand, these variables cannot vary suddenly, that is, they are continuous quantities even during the transition from one condition to the other. In condition s the fundamental equations will be

$$\begin{bmatrix} L_1 & 0 & 0 & 0 & 0 \\ 0 & L_2 & 0 & 0 & 0 \\ 0 & 0 & C_2 & 0 & 0 \\ 0 & 0 & 0 & C & 0 \\ 0 & 0 & 0 & 0 & L \end{bmatrix} \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} + \begin{bmatrix} R_1 & 0 & 0 & 1 & 0 \\ 0 & R_2 & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & \frac{1}{R} & 1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u\,,$$

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} i_1 \\ i_2 \\ u_3 \\ u_4 \\ i_5 \end{bmatrix} \tag{2/a}$$

in condition b

$$\begin{bmatrix} L_1+L_2 & 0 & 0 & 0 & 0 \\ 0 & L_1+L_2 & 0 & 0 & 0 \\ 0 & 0 & C_2 & 0 & 0 \\ 0 & 0 & 0 & C & 0 \\ 0 & 0 & 0 & 0 & L \end{bmatrix} \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} + \begin{bmatrix} R_1 & R_2 & 1 & 0 & 0 \\ R_1 & R_2 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & \frac{1}{R} & 1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u. \tag{2/b}$$

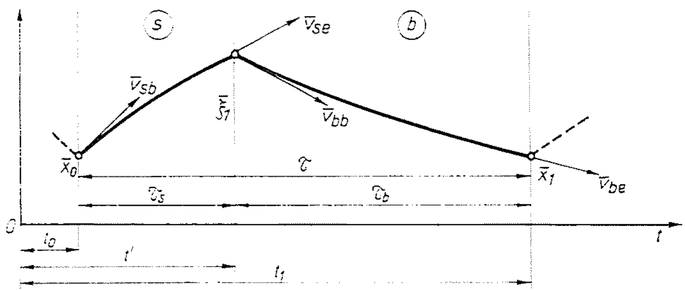

Fig. 2

It will be noted that, by an auxiliary voltage source replacing the thyristor, the two conditions could be described with a single differential equation system, but in practice it is much simpler to apply two equation systems.

In the given example matrices $\boldsymbol{D}_s$ and $\boldsymbol{D}_b$ became column vectors, u is scalar, but there might be more input signals. Multiplying Equ. (1) from the left with the inverse matrix of $\boldsymbol{L}_s$ and/or $\boldsymbol{L}_b$,

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} + \boldsymbol{A}_s\boldsymbol{x} = \boldsymbol{G}_s u\,, \qquad \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} + \boldsymbol{A}_b\boldsymbol{x} = \boldsymbol{G}_b u \tag{3}$$

will be obtained, where

$$\boldsymbol{A}_s = \boldsymbol{L}_s^{-1}\boldsymbol{R}_s\,, \quad \boldsymbol{G}_s = \boldsymbol{L}_s^{-1}\boldsymbol{D}_s\,, \quad \boldsymbol{A}_b = \boldsymbol{L}_b^{-1}\boldsymbol{R}_b\,, \quad \boldsymbol{G}_b = \boldsymbol{L}_b^{-1}\boldsymbol{D}_b\,.$$

The symbols applied for times and initial values are presented in Fig. 2. The solutions of Equ. (3) are

$$\left.\begin{aligned} \boldsymbol{x}_s(t) &= \mathrm{e}^{-\boldsymbol{A}_s(t-t_0)}[\boldsymbol{x}_0 - \boldsymbol{z}_s(t_0)] + \boldsymbol{z}_s(t) = \mathrm{e}^{-\boldsymbol{A}_s(t-t_0)}\boldsymbol{x}_0 + \boldsymbol{y}_s(t)\,, \\ \boldsymbol{x}_b(t) &= \mathrm{e}^{-\boldsymbol{A}_b(t-t')}[\boldsymbol{\xi}_1 - \boldsymbol{z}_b(t')] + \boldsymbol{z}_b(t) = \mathrm{e}^{-\boldsymbol{A}_b(t-t')}\boldsymbol{\xi}_1 + \boldsymbol{y}_b(t)\,, \end{aligned}\right\} \tag{4}$$

where $\boldsymbol{z}(t)$ represents any (e.g. simplest) particular solution of the inhomogeneous equation, $\boldsymbol{y}(t)$ is that of a zero initial condition at the beginning of the respective section.

In simple cases the condition of the periodicity is
$\boldsymbol{x}_1 = \boldsymbol{x}_0(t_1) = \boldsymbol{x}_0$, and in general

$$\boldsymbol{x}_1 = \boldsymbol{P}\boldsymbol{x}_0 \tag{5}$$

where $\boldsymbol{P}$ is, in simple cases, a unit matrix and, for example in three-phase connections, a matrix expressing phase and sign changes. Taking into consideration that $\boldsymbol{x}_s(t') = \boldsymbol{x}_b(t') = \boldsymbol{\xi}_1$, the initial value of the periodical condition is obtained from (4) and (5):

$$\left.\begin{aligned} &\boldsymbol{x}_0 = (\boldsymbol{P} - \boldsymbol{X}_b\boldsymbol{X}_s)^{-1}(\boldsymbol{X}_b\boldsymbol{y}_s + \boldsymbol{y}_b)\,, \\ &\boldsymbol{\xi}_1 = \boldsymbol{X}_s\boldsymbol{x}_0 + \boldsymbol{y}_s\,, \qquad\qquad \boldsymbol{x}_1 = \boldsymbol{P}\boldsymbol{x}_0\,, \end{aligned}\right\} \tag{6}$$

where

$$\boldsymbol{X}_s = \mathrm{e}^{-\boldsymbol{A}_s\tau_s}, \quad \boldsymbol{X}_b = \mathrm{e}^{-\boldsymbol{A}_b\tau_b}, \quad \boldsymbol{y}_s = \boldsymbol{y}_s(t')\,, \quad \boldsymbol{y}_b = \boldsymbol{y}_b(t_1)$$

To calculate Equ. (6), the approximative value of τ_s and τ_b must be assumed in advance, then the condition of thyristor extinction and firing must be checked upon:

$$\boldsymbol{k}^*\boldsymbol{\xi}_1 = 0\,, \quad \boldsymbol{l}_x^*\boldsymbol{P}\boldsymbol{x}_0 + \boldsymbol{l}_u^*\boldsymbol{u}(t_1) = u_c(t_1) \tag{7}$$

In Fig. 1, for example, the thyristor current is $i_t = x_1 - x_2$, thus $\boldsymbol{k}^* = (1, -0, 0, 0)$. Thyristor firing is controlled by voltage u_c which is compared with a certain linear combination of the variables and input signals. If, for example in Fig. 1, u_c is compared with voltage x_4 of the load, then

$$\boldsymbol{l}_x^* = \boldsymbol{e}_4^* = (0, 0, 0, 1, 0)\,, \quad \boldsymbol{l}_u = \boldsymbol{0}\,.$$

The accurate values may be determined from Equs (6) and (7) by iteration, then Equ. (4) determines the values of $\boldsymbol{x}(t)$ in both states.

In practice usually the mean values and the harmonics of the variables are of greatest interest. Thus, for example, the following relations apply to the ν-th harmonic:

$$\left.\begin{aligned} &\boldsymbol{x}_\nu = \boldsymbol{a}_\nu \cos \nu\omega_1 t + \boldsymbol{b}_\nu \sin \nu\omega_1 t = \mathrm{Re}\{(\boldsymbol{a}_\nu - j\boldsymbol{b}_\nu)\,\mathrm{e}^{j\nu\omega_1 t}\}\,, \\ &\boldsymbol{a}_\nu - j\boldsymbol{b}_\nu = \frac{2}{\tau}\int_{t_0}^{t_0+\tau} \boldsymbol{x}(t)\mathrm{e}^{-j\nu\omega_1 t}\mathrm{d}t = \frac{2}{\tau}(\boldsymbol{I}_s + \boldsymbol{I}_b)\,, \end{aligned}\right\} \tag{8}$$

where $\boldsymbol{I}_s$ and $\boldsymbol{I}_b$ designate the integral value for periods τ_s and τ_b, respectively. Substituting the time functions (4) yields:

$$\boldsymbol{a}_\nu - j\boldsymbol{b}_\nu = \frac{2}{\tau}\{\boldsymbol{H}_{s\nu}[\boldsymbol{x}_0 - \boldsymbol{z}_s(t_0)] + \boldsymbol{H}_{b\nu}[\boldsymbol{x}_1 - \boldsymbol{z}_b(t_1)]\} + \boldsymbol{c}_\nu - j\boldsymbol{d}_\nu, \tag{9}$$

where $\boldsymbol{c}_\nu - j\mathbf{d}_\nu$ is the amplitude of the particular solution $\mathbf{z}(t)$:

$$e^{j\nu\omega_1 t_0}\boldsymbol{H}_{s\nu} = \int_0^{\tau_s} e^{-(\boldsymbol{A}_s + j\nu\omega_1\boldsymbol{E})t}\,dt\,; \qquad e^{j\nu\omega_1 t_1}\boldsymbol{H}_{b\nu} = \int_{-\tau_b}^{0} e^{-(\boldsymbol{A}_b + j\nu\omega_1\boldsymbol{E})t}\,dt. \tag{10}$$

In Equ. (9), when $\nu = 0$, $2/\tau$ is to be replaced by $1/\tau$. Integrations can be performed by using the normal forms referred to in the Introduction, but if the matrices in the exponents are not singulars, the following formula, much more suitable for calculation purposes, can be derived:

$$\left.\begin{aligned} e^{j\nu\omega_1 t_0}\boldsymbol{H}_{s\nu}[\boldsymbol{x}_0 - \boldsymbol{z}_s(t_0)] &= [\boldsymbol{A}_s + j\nu\omega_1\boldsymbol{E}]^{-1}\left[(\boldsymbol{z}_s(t') - \boldsymbol{\xi}_1)\,e^{-j\nu\omega_1\tau_s} - \boldsymbol{z}_s(t_0) + \boldsymbol{x}_0\right], \\ e^{j\nu\omega_1 t_1}\boldsymbol{H}_{b\nu}[\boldsymbol{x}_1 - \boldsymbol{z}_b(t_1)] &= [\boldsymbol{A}_b + j\nu\omega_1\boldsymbol{E}]^{-1}\left[(\boldsymbol{\xi}_1 - \boldsymbol{z}_b(t'))\,e^{j\nu\omega_1\tau_b} + \boldsymbol{z}_b(t_1) - \boldsymbol{x}_1\right]. \end{aligned}\right\} \tag{11}$$

For the periodical condition, the derivatives of the variables in the cut-off points may also be calculated which facilitates curve plotting and, on the other hand, are necessary for transitions to small variations. Thus, for example, on the basis of (3),

$$\left(\frac{d\boldsymbol{x}}{dt}\right)_{t_0-\varepsilon} = \boldsymbol{v}_{sb} = \boldsymbol{G}_s\boldsymbol{u}(t_0) - \boldsymbol{A}_s\boldsymbol{x}_0. \tag{12}$$

3. Small variations from the periodic steady state condition

3.1. *Basic equations*

Interpretation of the symbols is shown in Fig. 3. Variations may be

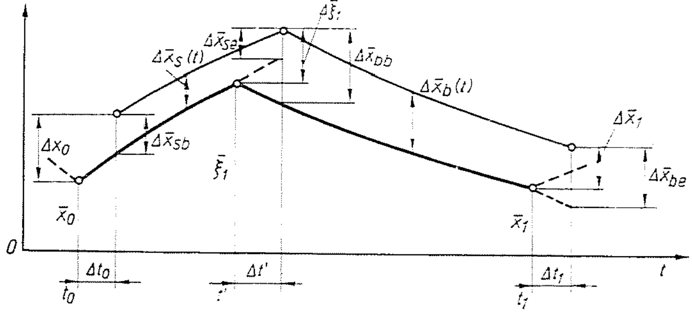

Fig. 3

caused by input signals $\Delta\mathbf{u}(t)$, variations of the Δu_c control voltage, or the deviation $\Delta\boldsymbol{x}_0$, $\varDelta t_0$ from the periodic steady state values, that is from the initial conditions. Variations $\Delta\boldsymbol{x}(t)$ are generally not periodic. If $\varDelta t_0$, $\varDelta t'$, $\varDelta t_1$ are positive, then

$$\left.\begin{aligned} \Delta\boldsymbol{x}(t) &= \Delta\boldsymbol{x}_s(t)\,, & \frac{\mathrm{d}\Delta\boldsymbol{x}_s}{\mathrm{d}t} + \boldsymbol{A}_s\Delta\boldsymbol{x}_s &= \boldsymbol{G}_s\Delta\boldsymbol{u}\,; & t_0 + \varDelta t_0 &\le t \le t'\,, \\ \Delta\boldsymbol{x}(t) &= \Delta\boldsymbol{x}_b(t)\,, & \frac{\mathrm{d}\Delta\boldsymbol{x}_b}{\mathrm{d}t} + \boldsymbol{A}_b\Delta\boldsymbol{x}_b &= \boldsymbol{G}_b\Delta\boldsymbol{u}\,; & t' + \varDelta t' &\le t \le t_1\,. \end{aligned}\right\} \tag{13}$$

In intervals $\varDelta t_0$, $\varDelta t'$, $\varDelta t_1$ the above equations are not satisfied by $\Delta\boldsymbol{x}(t)$, instead of them on the basis of Fig. 3, the following relations may be written (the second b and e index, respectively, refers to the beginning and end of the section):

$$\left.\begin{aligned} \Delta\boldsymbol{x}_0 &= \boldsymbol{v}_{sb}\varDelta t_0 + \Delta\boldsymbol{x}_{sb}\,, & \Delta\xi_1 &= \boldsymbol{v}_{se}\varDelta t' + \Delta\boldsymbol{x}_{se}\,, \\ \Delta\xi_1 &= \boldsymbol{v}_{bb}\varDelta t' + \Delta\boldsymbol{x}_{bb}\,, & \Delta\boldsymbol{x}_1 &= \boldsymbol{v}_{be}\varDelta t_1 + \Delta\boldsymbol{x}_{be}. \end{aligned}\right\} \tag{14}$$

The relation between $\Delta\boldsymbol{x}_{sb}$ and $\Delta\boldsymbol{x}_{se}$ is given by differential equation (13). Since these are small quantities of first order, the small variation of first order of the time will cause only a small error of second order and, therefore, the solution of (13) might be applied to range $t_0 \le t \le t'$ instead of $t_0 + \varDelta t_0 \le$ $\le t \le t' + \varDelta t'$. Making use of the quantities calculated for the periodic solution, it may be written that

$$\Delta\boldsymbol{x}_{se} = \boldsymbol{X}_s\Delta\boldsymbol{x}_{sb} + \Delta\boldsymbol{y}_s(t)\,, \qquad \Delta\boldsymbol{x}_{be} = \boldsymbol{X}_b\Delta\boldsymbol{x}_{bb} + \Delta\boldsymbol{y}_b(t_1) \tag{15}$$

where $\Delta\boldsymbol{y}_s$ and $\Delta\boldsymbol{y}_b$ represent the particular solutions for voltage variation Δu, with initial conditions $\Delta\,\boldsymbol{y}_s(t_0) = \boldsymbol{0}$ and $\Delta\,\boldsymbol{y}_b(t') = \boldsymbol{0}$, respectively.

Intervals $\varDelta t'$ and $\varDelta t_1$ are determined by the extinction and firing conditions of the thyristors:

$$\left.\begin{aligned} 0 &= \boldsymbol{k}^*\Delta\xi_1 = \boldsymbol{k}^*(\boldsymbol{v}_{bb}\varDelta t' + \Delta\boldsymbol{x}_{bb})\,, \\ \varDelta u_c(t_1) + \left(\frac{\mathrm{d}u_c}{\mathrm{d}t}\right)_{t_1}\varDelta t_1 &= \boldsymbol{l}_x^*\Delta\boldsymbol{x}_1 + \boldsymbol{l}_u^*\left[\Delta\boldsymbol{u}(t_1) + \left(\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}\right)_{t_1}\varDelta t_1\right]. \end{aligned}\right\} \tag{16}$$

Equs (14)—(16) yield the following final result:

$$\begin{bmatrix} \Delta\boldsymbol{x}_1 \\ \varDelta t_1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{Z}_{11}\boldsymbol{z}_{12} \\ \boldsymbol{z}_{21}^*z_{22} \end{bmatrix} \begin{bmatrix} \Delta\boldsymbol{x}_0 \\ \varDelta t_0 \end{bmatrix} + \begin{bmatrix} \boldsymbol{f}_1(\Delta\boldsymbol{u}) \\ f_2(\Delta\boldsymbol{u}) \end{bmatrix} + \begin{bmatrix} \boldsymbol{v}_1 \\ v_2 \end{bmatrix} \varDelta u_c(t_1)\,. \tag{17}$$

The coefficients can be calculated from the periodic condition data:

$$\left.\begin{aligned}
&\boldsymbol{C}_1 = \boldsymbol{E} + \frac{(\boldsymbol{v}_{bb} - \boldsymbol{v}_{se})\boldsymbol{k}^*}{\boldsymbol{k}^*\boldsymbol{v}_{se}}, \quad \boldsymbol{C}_2 = \boldsymbol{X}_b\boldsymbol{C}_1, \quad \boldsymbol{C}_3 = \boldsymbol{C}_2\boldsymbol{X}_s, \quad \boldsymbol{Z}_{11} = \boldsymbol{C}_4\boldsymbol{C}_3, \\
&\frac{1}{c} = \boldsymbol{l}_u^* \frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} - \boldsymbol{l}_x^*\boldsymbol{v}_{be} - \left(\frac{\mathrm{d}u_c}{\mathrm{d}t}\right)\Bigg|_{t=t_1}, \quad \boldsymbol{c}_2 = -\boldsymbol{C}_2\boldsymbol{v}_{sb}, \\
&\boldsymbol{z}_{12} = \boldsymbol{C}_4\boldsymbol{c}_2, \quad \boldsymbol{z}_{21}^* = -c\boldsymbol{l}_x^*\boldsymbol{C}_3, \quad z_{22} = -c\boldsymbol{l}_x^*\boldsymbol{C}_2, \\
&\boldsymbol{f}_3(\Delta\boldsymbol{u}) = \boldsymbol{C}_2\Delta\boldsymbol{y}_s + \Delta\boldsymbol{y}_b, \quad \boldsymbol{f}_1(\Delta\boldsymbol{u}) = \boldsymbol{C}_4\boldsymbol{f}_3(\Delta\boldsymbol{u}) + c\boldsymbol{v}_{be}\boldsymbol{l}_u^*\Delta\boldsymbol{u}(t_1), \\
&f_2(\Delta\boldsymbol{u}) = -c[\boldsymbol{l}_x^*\boldsymbol{f}_3(\Delta\boldsymbol{u}) + \boldsymbol{l}_u^*\Delta\boldsymbol{u}(t_1)], \\
&\boldsymbol{v}_1 = -c\boldsymbol{v}_{be}, \quad v_2 = c, \quad \boldsymbol{C}_4 = \boldsymbol{E} + c\boldsymbol{v}_{be}\boldsymbol{l}_x^*.
\end{aligned}\right\} \tag{17a}$$

These calculations can be readily carried out on computers, by using matrix sub-routines. In a number of practical cases, the formulae are much simpler. If, for example, $\Delta t_0 = \Delta t_1 = 0$, then $\boldsymbol{Z}_{11} = \boldsymbol{C}_3$ and $\boldsymbol{z}_{12}, \boldsymbol{z}_{21}^*, z_{22}$ are not needed; $\boldsymbol{f}_1(\Delta\boldsymbol{u}) = \boldsymbol{f}_3(\Delta\boldsymbol{u})$.

In Equ. (17), for $\Delta\boldsymbol{x}$ and Δt, indices 1 and 0 may be replaced by k and $k-1$, thus a difference equation is obtained for the further periods. If the $\boldsymbol{P}$ periodicity matrix is not a unit matrix, i.e. $\boldsymbol{P} \neq \boldsymbol{E}$, then it is best to apply the reduced differences $\Delta\boldsymbol{x}_1' = \boldsymbol{P}^{-1}\Delta\boldsymbol{x}_1$ instead of $\Delta\boldsymbol{x}_1$.

3.2 *Transient response, time constants*

If, for example, Δu_c is varied by a step change, or the input signals are changed to develop a new periodic steady state condition, then it is preferable to consider the new steady state condition as an operating point, and to calculate with the differences therefrom. In this case, difference equation (17) will be homogeneous, since $\Delta\boldsymbol{u}$ and Δu_c are equal to zero.

$$\begin{bmatrix} \boldsymbol{P}^{-k}\Delta\boldsymbol{x}_k \\ \Delta t_k \end{bmatrix} = \begin{bmatrix} \boldsymbol{P}^{-1}\boldsymbol{Z}_{11} & \boldsymbol{P}^{-1}\boldsymbol{z}_{12} \\ \boldsymbol{z}_{21}^* & z_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{P}^{-(k-1)}\Delta\boldsymbol{x}_{k-1} \\ \Delta t_{k-1} \end{bmatrix} \tag{18}$$

or, with a shorter expression,

$$\Delta\boldsymbol{w}_k = \boldsymbol{Z}\Delta\boldsymbol{w}_{k-1} \tag{18a}$$

The initial value is obtained by the difference between the previous and new steady state condition, respectively. If $\Delta t_0 = 0$, then

$$\Delta\boldsymbol{w}_0 = \begin{bmatrix} \Delta\boldsymbol{x}_0 \\ \Delta t_0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}_{0,pr} - \boldsymbol{x}_{0,new} \\ 0 \end{bmatrix} \tag{19}$$

and the solution of (18a):

$$\Delta\boldsymbol{w}_k = \boldsymbol{Z}^k\Delta\boldsymbol{w}_0 .$$

This way, the variation of quantities in the instance of thyristor firing is obtained. In the periodic condition, the instance of firing is $t_k = t_0 + k\,\tau$. If in (19) k is expressed by t_k, and the formula thus obtained is considered as valid not only for the discrete t_k values, some formal time constants may be introduced. Designating the eigenvalues of $\boldsymbol{Z}$ by λ_i, the equivalent time constants and oscillation frequencies can be calculated from the following formula:

$$\frac{\ln \lambda_i}{\tau} = -\frac{1}{T_i} + j\omega_i\,. \tag{20}$$

Obviously, the eigenvalue (or eigenvalues) $\lambda = 0$ must be neglected here.

Another approximative continuous substituting scheme is obtained, if Equ. (17) is written as follows:

$$\frac{\Delta \boldsymbol{w}_k - \Delta \boldsymbol{w}_{k-1}}{\tau} = \frac{1}{\tau}(\boldsymbol{Z} - \mathbf{E})\Delta \boldsymbol{w}_k + \boldsymbol{f}(\Delta \boldsymbol{u}) + \boldsymbol{v}\Delta u_c \tag{21}$$

and the left-hand side quantity is considered as an approximate derivative. hus the eige nvalues of matrix $\frac{1}{\tau}(\boldsymbol{Z} - \boldsymbol{E})$ give the time constants. From Equ. (20), Equ. (21) makes possible to plot a block diagram for the difference equation, by using a digital integrator (summator) or derivative element.

3.3 *Harmonic analysis*

In practice the mean values, fundamental harmonics, etc. are more nteresting than the values at switch-over points. Although small variations do not represent periodical signals, harmonics can be interpreted approximately for any individual period, although their amplitude and phase will vary from one period to the other. Since in the integrals taken for intervals τ_s and τ_b the parts taken for Δt_0, $\Delta t'$, and Δt_1 are small values of second order, the ntegration range can be set in a simple way:

$$\Delta \boldsymbol{a}_\nu - j\Delta \boldsymbol{b}_\nu = \frac{2}{\tau}\int_{t_0}^{\tau_0+\tau_s} \Delta \boldsymbol{x}_s\, e^{-j\nu\omega_1 t}\,\mathrm{d}t + \frac{2}{\tau}\int_{\tau_1-\tau_b}^{t_1} \Delta \boldsymbol{x}_b\, e^{-j\nu\omega_1 t}\,\mathrm{d}t\,. \tag{21a}$$

If $\Delta \boldsymbol{u} = \boldsymbol{0}$, then $\Delta \boldsymbol{x}_s$ and $\Delta \boldsymbol{x}_b$ contain only the solution of the homogeneous equation, and a simple formula is obtained:

$$\Delta \boldsymbol{a}_\nu - j\Delta \boldsymbol{b}_\nu = \frac{2}{\tau}\left[\boldsymbol{H}_s \Delta \boldsymbol{x}_{sb} + \boldsymbol{H}_b \Delta \boldsymbol{x}_{be}\right]. \tag{22}$$

Substituting the values of Δx_{sb} and Δx_{be} from (14), it can be seen that the harmonics depend only on Δx_0, and the result can be readily generalized to the k-th period.

3.4 *Frequency response*

With the Δu_c control signal (or any input signal) sinusoidally oscillated, and the steady state solution of (17) found, the frequency response will be obtained.

In case of

$$\Delta u_c = \delta \cos \omega t = \mathrm{Re}\left\{\delta\, e^{j\omega t}\right\} \tag{22a}$$

the steady state solution of equation

$$\Delta \boldsymbol{w}_k = \boldsymbol{Z}\Delta \boldsymbol{w}_{k-1} + \boldsymbol{v}\Delta u_c(t) \tag{22b}$$

will be

$$\Delta \boldsymbol{w}_k = \mathrm{Re}\{[\boldsymbol{E} - \boldsymbol{Z}e^{-j\omega\tau}]^{-1}\, \boldsymbol{v}\delta\, e^{j\omega t_k}\} = \tag{22c}$$

$$= \mathrm{Re}\{\boldsymbol{E} - 2\boldsymbol{Z}\cos\omega\tau + \boldsymbol{Z}^2]^{-1}\,[\boldsymbol{E} - \boldsymbol{Z}\cos\omega\tau - j\boldsymbol{Z}\sin\omega\tau]\,\boldsymbol{v}\delta\, \mathrm{e}^{j\omega t_k}\}\,, \tag{22d}$$

thus the frequency response is

$$\boldsymbol{y}(j\omega) = [\boldsymbol{E} - 2\boldsymbol{Z}\cos\omega\tau + \boldsymbol{Z}^2]^{-1}\,[\boldsymbol{E} - \boldsymbol{Z}\cos\omega\tau - j\boldsymbol{Z}\sin\omega t]\,\boldsymbol{v}. \tag{23}$$

The first factor in (22c) to be inverted has the determinant

$$\det[\mathbf{E}-\mathbf{Z}]e^{-j\omega\tau}] = \prod_{i=1}^{n+1}(1-\lambda^{ie}e^{-j\omega\tau}). \tag{23a}$$

This can be similarly applied as the denominator of the transfer function in the calculation of Bode diagrams and from this factorized form the cutoff frequencies can be easily determined. Here the values λ_i represent the eigenvalues of the $\boldsymbol{Z}$ matrix.

Above $\Delta \boldsymbol{x}$ represents the output signals. On ground of the Fourier analysis, the harmonics are the functions of $\Delta \boldsymbol{x}$ and thus the corresponding frequency responses can also be determined.

4. Applications

Some applications are shown in this chapter. Only the basic equations are presented. On that basis the computer analyses can be readily carried out.

4.1 *Inverter*

The differential equations obtained from the circuit diagram (Fig. 1) have been presented at the beginning of Chapter 2 (Equs 2a, b). It will be noted here that in condition b $x_1 = x_2$, thus the second equation could have been omitted but, because of the uniform interpretation of vector $\boldsymbol{x}$ in both ranges, this would not have been reasonable. Owing to the general form (3), $\boldsymbol{L}_b$ must be invertable. For this reason $(L_1 + L_2)\, dx_2/dt$ had to be used.

The vector of the condition of extinction is: $\boldsymbol{k}^* = (1, -1, 0, 0, 0)$, that of the firing condition: $\boldsymbol{l}_x^* = (0, 0, 0, 1, 0)$, $\boldsymbol{l}_u^* = 0^*$, while the control voltage u_c is constant in steady state. The periodicity matrix is: $\boldsymbol{P} = \boldsymbol{E}$. Steady state condition can be determined for two u_c values, with the first one analyzed in detail.

In this case the a_0 mean values of the variables $x(t)$, the $a_1 - jb_1$ amplitude of their $\nu = 1$ fundamental harmonics, and the $x_{1\text{eff}}$ effective values of the latter are essential.

4.2 *D. C. motor fed by thyristor rectifier*

The example assumes a symmetrically controlled three-phase SCR bridge rectifier (Fig. 4), where u_a, u_b, and u_c are the open circuit phase voltages of the

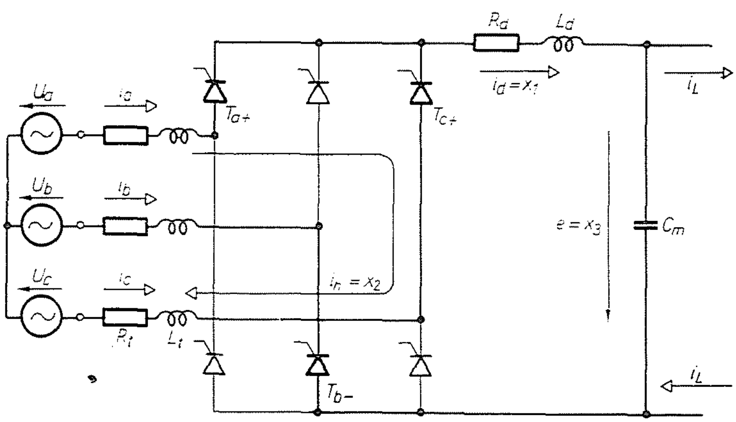

Fig. 4

three-phase transformer, R_t and L_t are the short circuit data of the transformer (as reduced to the secondary side), R_d and L_d are the direct current circuit data including those of the motor armature, e is the e.m.f. induced by rotation of the motor, $e = k\,\Phi\, w$, where Φ is the flux of the motor (constant), and w is the angular velocity. The equation of motion is:

$$J \frac{\mathrm{d}w}{\mathrm{d}t} = m - m_l \tag{24}$$

which can be written, by substituting $e = k\,\Phi\,w$, $m = k\,\Phi\,i_d$, and $m_l = k\,\Phi\,i_l$ as follows:

$$i_d - i_l = \frac{J}{k^2\Phi^2}\frac{\mathrm{d}e}{\mathrm{d}t} = C_m \frac{\mathrm{d}e}{\mathrm{d}t}. \tag{25}$$

Here J is the rotor inertia moment, i_d is the direct current of the armature, and i_l is the m_l loading torque in current scale. On the basis of (25) the mechanical relations can be taken into account with a capacity C_m in the equivalent circuit.

Before t_0 the thyristors T_{c+} and T_{b-} are on and at t_0 the thyristor T_{a+} is turned on, thus the condition s represents the commutation from T_{c+} to T_{a+}.

The direct current $x_1 = i_d$, loop current $x_2 = i_h$ and the e.m.f. $x_3 = e$ are used as variables. The phase currents are $i_a = i_h + i_d/2$, $i_b = -i_d$, and $i_c = -i_h + i_d/2$. The differential equation for state s is

$$\begin{bmatrix} \left(L_d + \frac{3}{2}L_l\right) & 0 & 0 \\ 0 & L_l & 0 \\ 0 & 0 & C_m \end{bmatrix} \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} + \begin{bmatrix} \left(R_d + \frac{3}{2}R_l\right) & 0 & 1 \\ 0 & R_l & 0 \\ -1 & 0 & 0 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} -\frac{3}{2}u_b \\ \frac{u_a - u_c}{2} \\ -i_l \end{bmatrix}$$

At the end of the τ_s, T_{c+} is extinguished: the precondition of extinction is $i_c = 0 = -x_2 + x_1/2$, that is $\boldsymbol{k}^* = (1, -2, 0)$. The differential equation of condition b is:

$$\begin{bmatrix} (L_d + 2L_l) & 0 & 0 \\ 0 & L_l & 0 \\ 0 & 0 & C_m \end{bmatrix} \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} + \begin{bmatrix} (R_d + 2R_l) & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} u_a - u_b \\ 0 \\ -i_l \end{bmatrix}$$

At t_{1c} ($\tau = 1/6$ period), the thyristor T_{c-} is fired; the matrix of periodicity is:

$$\boldsymbol{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Calculating two steady-state conditions, the changes during the transition from one state to another can be determined from Equ. (18).

4.3 *Induction motor fed through back-to-back thyristor pairs*

Connection is shown by Fig. 5. Before t_0 the thyristors T_{a-} and T_{c+} are on, at $t = t_0$ the T_{b-} thyristor is turned on, thus in section τ_s all three primary

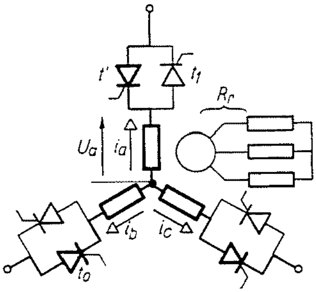

Fig. 5

terminals of the motor are connected to the network. At the end of condition s, when i_a becomes zero, T_{a-} will be off, thus in condition b only two terminals are connected to the system. In t_1 the T_{a+} thyristor is turned on and τ amounts to 1/6 of the period of the supply. The condition of periodicity, as expressed with phase currents, is:

$$\begin{bmatrix} i_{a1} \\ i_{b1} \\ i_{c1} \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} i_{a0} \\ i_{b0} \\ i_{c0} \end{bmatrix}$$

Since $i_a + i_b + i_c = 0$, two free variables will be left. Applying the $x, y, 0$ transformation used in the theory of alternating current machines (see, for example Reference No 1), $i_a = i_x$, $i_b = -i_x/2 + i_y\sqrt{3}/2$, $i_c = -i_x/2 - i_y\sqrt{3}/2$, and the condition of periodicity is:

$$\begin{bmatrix} i_{x1} \\ i_{y1} \end{bmatrix} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix} \begin{bmatrix} i_{x0} \\ i_{y0} \end{bmatrix}$$

With the rotor currents expressed also in the x, y coordinate system fixed to the stator then by using $\boldsymbol{x}^* = (i_x, i_y, i_{rx}, i_{ry})$, the periodicity condition will assume the form

$$\boldsymbol{x}_1 = \boldsymbol{P}\boldsymbol{x}_0, \qquad \boldsymbol{P} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 & 0 & 0 \\ \sqrt{3}/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & -\sqrt{3}/2 \\ 0 & 0 & \sqrt{3}/2 & 1/2 \end{bmatrix}.$$

It will be noted here that, when thyristor-diode pairs are applied or three thyristors in delta are in the disconnected neutral point of the motor, the value

τ equals to one-third of the period, and the matrix of periodicity is $\boldsymbol{P}^2$. The condition of extinction is: $i_a = 0$, that is, $\boldsymbol{k}^* = (1, 0, 0, 0)$.

The motor equations in the s condition of three-phase conduction are:

$$\boldsymbol{L}\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} + \boldsymbol{R}\boldsymbol{x} + \boldsymbol{W}\boldsymbol{L}\boldsymbol{x} = \boldsymbol{u}\,, \tag{26}$$

where

$$\boldsymbol{L} = \begin{bmatrix} L & 0 & L_m & 0 \\ 0 & L & 0 & L_m \\ L_m & 0 & L_r & 0 \\ 0 & L_m & 0 & L_r \end{bmatrix}, \quad \boldsymbol{R} = \begin{bmatrix} R & 0 & 0 & 0 \\ 0 & R & 0 & 0 \\ 0 & 0 & R_r & 0 \\ 0 & 0 & 0 & R_r \end{bmatrix},$$

$$\boldsymbol{W} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & W \\ 0 & 0 & -W & 0 \end{bmatrix}, \quad \boldsymbol{u} = \begin{bmatrix} u_x \\ u_y \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} i_x \\ i_y \\ i_{rx} \\ i_{ry} \end{bmatrix}$$

In condition b (when $i_x = 0$), the first row and column of matrices **R** and $\mathbf{W}L$ must be made equal to zero and, in the matrix $\boldsymbol{L}$ of the $\mathrm{d}x/\mathrm{d}t$ coefficient, the Lm in the first row and column must be replaced also by zero, furthermore $u_x = 0$. The symbols are those of Fig. 6 and W is the angular velocity of the rotor in electrical angle.

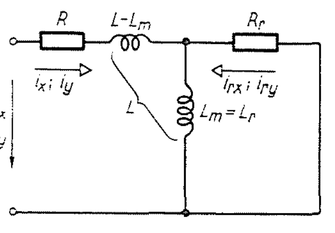

Fig. 6

It must be noted here that, since the torque equations of A. C. motors are non-linear, Equ. (26) can be simply analyzed in case of constant W. Since, however, pulsating torque is produced in periodic condition as well, W will pulsate. In case of small variations, the inertia moment can be taken into account with good approximation by assuming an additional pulsating torque opposite to that of the periodic condition. Then the equations for small variations will be linear.

Summary

In this paper a general theory, developed for the analysis of deviations from periodic steady-state condition of quasi-periodic, multi-parameter systems, linearized for small variations by means of matrix calculation, is discussed. The theoretical results are well applicable for the digital computer analysis of, for instance, thyristor and diode circuits.

First the paper considers the determination of the periodical condition (Chapter 2.) then investigates the small variation from the periodic steady-state condition, that the transient and frequency responses, etc., can be readily determined from Chapter 3.

This general theory was used to carry out digital computer analyses on the following non-linear, practical systems: SCR-inverter involving five energy-storage element, D. C. motor drive, supplied by three-phase bridge rectifier, and finally a three-phase voltage-controlled induction motor drive (Chapter 4).

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