ON SOME GENERALIZATIONS OF THE ISOMETRICAL MAPPINGS AND THEIR APPLICATIONS

THE METHOD OF ITERATION*

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Introduction

In this paper the different generalizations of the isometrical mappings (metric-true in classical meaning) are dealt with. The first part gives the simplest generalizations; the second part is a stronger generalization as a generalization of the Lipschitz-condition and examination of their properties; the third part deals with their applications and illustrations; namely with the fixed point theorems for the contractive mappings (on completely regular topological spaces), a local property (more properly local contractibility) and some illustrations of Banach's fixed point theorem as the unique existence of solution of the distinct type of integral equations. In the end a numerical example of the iteration method (an integral equation) and an interesting condition about the roots of algebraic equations in the complex plane will be presented.

§ 1.

Let us consider the sets X and X' with pseudostructures Σ and Σ' ([3], [6], [7]). If Σ is a pseudostructure then the following axioms must be true:

A.1. σ is a mapping from $X * X$ into R^+ where R^+ is the set of the non-negative numbers.

A.2. σ is a symmetrical mapping, i.e.

$$\sigma(x; y) = \sigma(y; x)$$

A.3. $\sigma(x; x) = 0$ for every x from X.

A.4. for every x, y, z from X

$$\sigma(x; y) \leqq \sigma(x; z) + \sigma(z; y)$$

In the followings, suppose for every $(X,, \Sigma)$ and (X', Σ') space considered that there is a one-to-one mapping between Σ and Σ' (denoted by k: $\Sigma \rightarrow \Sigma'$).

* Presented in abbreviated form in Rostock, February 19, 1968. See [9].

1.1. *Definiton:* Let f be a mapping from the space (X, Σ) into the space (X', Σ'), then the mapping f is called k-isometrical mapping* if there is a k one-to-one mapping between Σ and Σ' and for all $x, y \in X$; $\sigma \in \Sigma$ such that

(1.1.1.) $\sigma(x; y) = k\sigma\,[f(x); f(y)]$ is satisfied.

1.2. *Theorem* [1], [2], [7]: Let f be an isometrical mapping of the space (X, Σ) into (X', Σ'), then

(1.2.1.) f is continuous (according to $\tau_{\Sigma} \to \tau'_{\Sigma'}$)

(1.2.2.) f is proximity continuous (according to $\delta_{\Sigma} \to \delta'_{\Sigma'}$)

(1.2.3.) f is uniform continuous (according to $\mathfrak{U}_{\Sigma} \to \mathfrak{U}'_{\Sigma'}$)

(1.2.4.) If $f(X) = X'$ holds, then f is an open and closed mapping.

(1.2.5.) If f is a one-to-one mapping then space X and X' are homeomorph (in the topological sense), isomorph (in the uniform sense) and equimorph (in the proximity sense).

Proof: (1.2.1.) and (1.2.2.) follow from (1.2.3.) thus it is enough to show (1.2.3.): it follows from the fact that

(1.2.6.) $U_{\varepsilon;\,\sigma_1 \dots \sigma_r} \subset (f * f)^{-1}\,[U_{\varepsilon;\,k\sigma_1 \dots k\sigma}]$ is true.

(1.2.4.) is evident from

(1.2.7.) $U_{\varepsilon;\,\sigma_1 \dots \sigma_r}(x) = f^{-1}\,[U_{\varepsilon;\,k\sigma_1 \dots k\sigma_r} < f(x) >]$

(1.2.5.) follows from (1.2.1.); (1.2.2.); (1.2.3.) and since f^{-1} is an isometrical mapping. Q. E. D.

Separable questions (See [1], [7])

1.3. *Lemma.* Let (X, Σ) be a T_2-space, $f(X) = X'$ and f an isometrical mapping, then (X', Σ') is also a T_2-space.

Proof: Let x' and y' be two different points of X', because of $f(X) = X'$ there are two points x, y of X such that $x \in f^{-1}(x')$ and $y \in f^{-1}(y')$. If(**) $\sigma'[f(x); f(y)] = \sigma(x'; y') = 0$ could be satisfied, then $\sigma(x; y) = 0$ should follow. This fact cannot be satisfied for all $\sigma' \in \Sigma'$, because X is a T_2-space, and thus $x = y$ follows from $\{\sigma(x; y) = 0$: for every $\sigma \in \Sigma\}$. Consequently there is σ' from Σ' such that $\sigma'(x'; y') > 0$; i.e. X' is a T_2-space. Q. E. D.

1.4. *Theorem:* Let the conditions of lemma (1.3.) be met then f is a one-to-one mapping too.

Proof: Because of lemma (1.3.) X' is a T_2-space. Let $x \neq y$; then there exists a σ from Σ such that $\sigma(x, y) > 0$ and thus $k\sigma[f(x); f(y)] = \sigma(x; y) > 0$. In consequence $f(x) \neq f(y)$. If $x' \neq y'$; then there is a $\sigma' \in \Sigma'$ such that $\sigma(x'; y') > 0$. Thus

$$\sigma'(x'; y') = k^{-1}\sigma'[f^{-1}(x'); f^{-1}(y')] = \sigma(x; y) > 0$$

where $f(x) = x', f(y) = y'$, therefore $x \neq y$. Q. E. D.

Quotient space. (See [1])

* k-isometrical mapping will be called below isometrical mapping.
** $k^{-1}\sigma' = \sigma$.

Let (X, Σ) be a pseudostructure space. Let us define equivalent relation $\sim$ as: "$x \sim y$ if and only if $\sigma(x; y) = 0$ for every σ of Σ" (i.e. x and y are in separable points of X).

Evidently it is an equivalent relation following from axioms A.1., A.2., A.3., A.4.

Let us denote the set of the equivalent classes $\tilde{x}$ by $\tilde{X} \cdot \tilde{\Sigma}$ contains each pseudometric $\tilde{\sigma}$ where $(\tilde{x}; \tilde{y}) = \sigma(x; y)$ whenever $x \in \tilde{x}$ and $y \in \tilde{y}$. We have to show that $\tilde{\sigma}(\cdot)$ is a pseudometric.

Let us choose $x_1; x_0 \in \tilde{x}$ and $y_1; y_0 \in \tilde{y}$ using axiom A.4. and definition of $\tilde{x}$ leading to the following inequality:

$$\sigma(x_1; y_1) \leqq \sigma(x_1; x_0) + \sigma(x_0; y_1) \leqq \sigma(x_1; x_0) + \sigma(x_0; y_0) + \sigma(y_1; y_0) = \\ = \sigma(x_0; y_0)$$

Similarly we obtain $\sigma(x_0; y_0) \leqq \sigma(x_1; y_1)$. Thus it follows $\sigma(x_0; y_0) = \sigma(x_1; y_1) = = \tilde{\sigma}(\tilde{x}; y)$.

The described method gives separate classification of the space (X, Σ) into the space $(\tilde{X}, \tilde{\Sigma})$. The space $(\tilde{X}, \tilde{\Sigma},)$ is the same as quotient space $(X, \Sigma)/\sim$.

1.5. *Definition:* Let us call the spaces (X, Σ) and (X', Σ') isometrical spaces, if there is a one-to-one isometrical mapping f between them.

Let $f: (X, \Sigma) \to (X', \Sigma')$, then the mapping f can be defined as: "$\tilde{f}(\tilde{x}) = \tilde{f}(\tilde{x})$ where x is an arbitrary representation of $\tilde{x}$. Let us remark that this definition is usually incorrect, but it can be easily proved on the isometrical mappings that $\tilde{f}$ is an unambiguously defined mapping.

1.6. *Lemma:* Let f be an isometrical mapping from (X, Σ) into (X', Σ'), then f is an isometrical mapping of $(\tilde{X}, \tilde{\Sigma})$ into $(\tilde{X}', \tilde{\Sigma}')$.

Proof: It is trivial from the definition of $\sim$.

1.7. *Theorem:* If f is an isometrical mapping of the space (X, Σ) onto the space (X', Σ') then spaces $(X', \Sigma')/\sim$ and $(X, \Sigma)/\sim$ are isometrical spaces.

Proof: The quotient spaces are T_2-spaces thus it follows from lemma (1.6.) and theorem (1.4.) that $\tilde{f}$ is a one-to-one isometrical mapping. Q. E. D.

§ 2.

In this part some further generalizations of the isometrical mappings will be given.

2.1. *Definition:* Let (X, Σ) and (X', Σ') be pseudostructure spaces. Let us call the mapping f from the space (X, Σ) into (X', Σ') weak isometrical mapping, if there is a substructure Σ'_f of Σ' such that f is an isometrical mapping from (X, Σ) into (X', Σ'_f).

2.2. *Note:* If $\tau'_{\Sigma'_f} > \tau'_{\Sigma'}$, then f is a real weak isometrical mapping; if $\tau'_{\Sigma'_f} = \tau'_{\Sigma'}$, then only "profusion" of the structure Σ' can be spoken of. This is the case of extending a pseudostructure with some pseudometrics which did not make a change in the primary topology nevertheless there is not a one-to-one mapping between Σ'_f and Σ'.

2.2.1. *Example:* Let (E, ϱ) and (E', ϱ') be metrical spaces, f an isometrical (metric-true) mapping from E into E', in the classical sense. Extend the metric ϱ' to the pseudostructure:

$$\Sigma' = \{\varrho'_i: \varrho'_i(x'; y') = 2^{-i}\varrho'(x'; y');\ i = 1,2, \dots\}$$

then f is a weak isometrical mapping, not an isometrical mapping, nevertheless f is not a real weak isometrical mapping.

2.2.2. *Example:* Let (X, Σ) be a no metrizable space, and let Σ_0 be a countable subsystem of Σ. Then there exists no one-to-one mapping between Σ and Σ_0, because $\overline{\overline{\Sigma}} > \aleph_0$. Moreover τ_Σ is more refined than τ_{Σ_0}, because τ_{Σ_0} is a metrizable topology.

Let us consider the identical mapping of the space (X, Σ_0) onto the space (X, Σ). This mapping is a real weak isometrical one.

In the following an extensive generalization of the isometrical mappings will be given, more properly, the Lipschitz condition will be generalized for the class of completely regular topological spaces.

2.3. *Definition:* Let f be a mapping from the space (X, Σ) into the space (X', Σ'), it is called *L-isometrical* mapping, if there are some positive real numbers a and b such that

(2.3.1.) $a\sigma(x; y) \leqq k\sigma[f(x); f(y)] \leqq b\sigma(x; y)$

for all $\sigma \in \Sigma$ $(k\sigma \in \Sigma')$ and $x, y \in X$.

2.4. *Definition:* The mapping f as defined in 2.3. is a weak L-isometrical mapping, if a and b are positive real functions of σ; i.e.:

(2.4.1.) $a(\sigma)\sigma(x; y) \leqq k\sigma[f(x); f(y)] \leqq b(\sigma)\sigma(x; y)$

is met.

2.5. *Definition:* The mapping f, as defined in 2.3, is called weak contractive mapping, whenever

(2.5.1.) $k\sigma[f(x); f(y)] \leqq b(\sigma)\sigma(x; y)$

is satisfied.

The mapping f is called weak expansive mapping, whenever

(2.5.2.) $a(\sigma)\sigma(x; y) \leqq k\sigma[f(x); f(y)]$

is true for every $\sigma \in \Sigma$ and $x, y \in X$.

Separation (See [1], [7])

2.6. *Lemma:* If f is a weak expansive mapping from a T_2-pseudostructure space (X, Σ) into a pseudostructure space (X', Σ') then $f(X) \subset X'$ provided the customary subspace topology is a T_2-space.

Proof: Let x_1, x_2 be two different points of X. So there is a $\sigma \in \Sigma$ that $\sigma(x_1; x_2) > 0$. Thus for $\sigma' = k\sigma \in \Sigma'$

$$k\sigma[f(x_1); f(x_2)] = \sigma'(y'_1; y'_2) \geqq b(\sigma)\sigma(x_1; x_2) > 0$$

is satisfied.

In consequence there are disjoint neighbourhoods of $f(x_1)$ and $f(x_2)$

$f(x_1) \neq f(x_2)$. That is for each y_1', y_2' where $y_1' \neq y_2'$ and $y_1'; y_2' \in f(X)$, there exist $x_1; x_2 \in X$, such that $x_1 \neq x_2$. From this fact the statement of 2.6. follows, because $y_1' = f(x_1) \neq f(x_2) = y_2'$. Q. E. D.

2.7. *Lemma:* If f is a weak contractive one-to-one mapping of X onto X' and X' is T_2-space, then X is also a T_2-space.

Proof: Choose different x and y from X, then $f(x) \neq f(y)$, because of the condition of lemma. Consequently there exists a $\sigma' \in \Sigma'$ such that the following inequality is satisfied:

$$\sigma'[f(x); f(y)] > 0.$$

Thus hence $\sigma = k^{-1}\sigma'$ and because of the properties of contractive mapping:

$$(*) \qquad 0 < \sigma'[f(x); f(y)] \leqq a(\sigma)\sigma(x; y)$$

It follows from $a(\sigma) > 0$, $\sigma(x; y) > 0$ and (*) that x and y have some disjoint neighbourhoods. Q. E. D.

2.8. *Remark:* The separational results mentioned in the first part are easy to deduce from this part's result. Evidently, every isometrical mapping has the properties from 2.1. to 2.5.

Continuity (See [1], [2], [3], [6], [7])

2.9. *Theorem:* Let $(X; \Sigma)$ and $(X'; \Sigma')$ be given pseudostructure spaces, moreover a mapping f from X into X'.

a) If f is a weak expansive mapping and $f(X) = X'$ then f is an open mapping.

b) If f is a weak contractive mapping then f is a continuous mapping.

Proof:

a) G is an open set from X, so for all x from G there exist pseudometrics $\sigma_1 \ldots \sigma_r$ of Σ and a positive number ε such that the following relation is satisfied:

$$U_{\sigma_1 \ldots \sigma_r; \varepsilon}(x) \subset G$$

Let us denote $k\sigma_i$ by σ_i'. For each point x_0 from $U_{\sigma_1 \ldots \sigma_r; \varepsilon}(x)$

$$\sigma_i(x_0; x) < \varepsilon$$

and

$$\sigma_i'[f(x_0); f(x)] \geq a(\sigma_i)\, \sigma(x_0; x)$$

are satisfied. Thus:

$$\frac{1}{a(\sigma_i)}\, \sigma_i'[f(x_0); f(x)] \geq \sigma_i(x_0; x)$$

Let us choose $f(x_0)$ to fulfil

$$\varepsilon > \frac{1}{a(\sigma_i)} \sigma_i'[f(x) ; f(x_0)]$$

$(i = 1, \ldots r)$.

Furthermore

$$\varepsilon a(\sigma_i) > \sigma_i'[f(x_0) ; f(x)]$$

is true.

If $\varepsilon' = \min_{i=1\ldots r} \{\varepsilon \cdot a(\sigma_i)\}$ is chosen, then from the preceding

$$\varepsilon' > \sigma_i'[f(x_0) ; f(x)]$$

is satisfied for each $f(x_0) \in U'_{\sigma_1' \ldots \sigma_i' ; \varepsilon'}[f(x)]$ and thus

$$\varepsilon > \frac{\varepsilon'}{a(\sigma_i)} > \frac{\sigma_i'[x_0' ; f(x)]}{a(\sigma_i)} > \sigma_i(x_0 ; x)$$

where $x_0' = f(x_0)$. In consequence:

$$U'_{\sigma_1' \ldots \sigma'_i ; \varepsilon'}[f(x)] \subset f[U_{\sigma_1 \ldots \sigma_r \varepsilon}(x)] \subset f(G),$$

i.e.: if G is an open set then $f(G)$ is also an open one.

b) If f is a weak contractive mapping and G' is an open set which belongs to X' then for every x from $G = f^{-1}(G')$ there are some pseudometrics $\sigma_1 \ldots \sigma_r$ from Σ and a positive number ε' such that $U'_{\sigma_1' \ldots \sigma_r' ; \varepsilon'}(x') \subset G$ where $x' = f(x)$.

Moreover for each $x_0' \in U'$

$$\sigma_i'(x_0' ; x') < \varepsilon'$$

is satisfied, and introducing $x_0' = f(x_0)$, because of the contractive property of f it is:

$$\sigma_i'(x_0' ; x') < a(\sigma_i)\, \sigma_i(x_0 ; x)$$

where $\sigma_i = k^{-1}\sigma_i'$. Thus, if $a(\sigma_i)\sigma_i(x_0 ; x) < \varepsilon'$, i.e. the following relation is realized:

$$\frac{\varepsilon'}{a(\sigma_i)} > \sigma_i(x_0 ; x) \text{ is satisfied, then with } \varepsilon = \min \left\{ \frac{\varepsilon'}{a(\sigma_i)} : 1 \leqq i \leqq r \right\}$$

$$U_{\sigma_1 \ldots \sigma_r ; \varepsilon (x)} \subset f^{-1}[U'_{\sigma_1' \ldots \sigma_r' ; \varepsilon'(x')}] \subset G$$

It means that the set G is an open set. Q. E. D.

2.10. Let f be a weak L isometrical mapping from X into X', i.e. the inequality (2.3.1.) is satisfied:

$$a(\sigma)\sigma(x;y) \leqq k\sigma[f(x);f(y)] \leqq b(\sigma)\sigma(x;y)$$

Evidently, if $a(\sigma)$ holds (2.3.1.), then every positive real function $a'(\sigma)$ which satisfies the inequality $0 < a'(\sigma) \leqq a(\sigma)$ also satisfies (2.3.1.). Since $a(\sigma) < b(\sigma)$, that follows from (2.3.1.) for all $\sigma \in \Sigma$, there always exist following functions:

(2.10.1.) $\bar{a}(\sigma) = \sup a'(\sigma)$

and

(2.10.2.) $\underline{b}(\sigma) = \inf b'(\sigma)$

where $b'(\sigma)$ can be chosen similarly.

2.11. *Theorem:* A weak L-isometrical mapping is an L-isometrical mapping if and only if (for fixed Σ and Σ')

$$\lim\inf \{\bar{a}(\sigma) : \sigma \in \Sigma\} = A > 0$$

and

$$\lim\sup \{(\sigma):\underline{b} : \sigma \in \Sigma\} = B < +\infty$$

are true for the functions $\bar{a}(\sigma)$, $\underline{b}(\sigma)$ defined in (2.10.1.), (2.10.2.).

(2.11.1.) *Remark:* If either A or B exists, then it follows just the expansive or contractive property of f.

Proof: Let f be a weak L-isometrical mapping and let the conditions of the theorem be satisfied then the inequality

$$A\sigma(x;y) \leqq a(\sigma)\sigma(x;y) \leqq \sigma'[f(x),f(y)] \leqq b(\sigma)\sigma(x;y) \leqq B\sigma(x;y)$$

is evident and thus (2.3.1.) is going to be true; that is f is a weak L-isometrical mapping.

Conversely: let f be an L-isometrical mapping. If the conditions of theorem 2.11. are not satisfied, then

$$\lim\inf \bar{a}(\sigma) = A = 0 \quad \text{or} \quad \lim\sup \underline{b}(\sigma) = B = \infty$$

must be true.

For example let $\lim\inf \bar{a}(\sigma) = 0$. Because of (2.3.1.), there are positive numbers A_0, B_0 such that: $A_0\,\sigma(x;y) \leq \sigma'(x';y') \leq B_0\sigma(x;y)$, thus there exists σ_0 from Σ that $\bar{a}(\sigma_0) < A_0$, because of $\lim\inf \bar{a}(\sigma) = 0$. This contradicts to the definition of $\bar{a}(\sigma_0)$.

The existence of lim sup $\underline{b}(\sigma)$ can be proved similarly. Q. E. D.

The following theorem shows that the term "isometrical" is correctly applied for the weak L-isometrical mapping.

2.12. *Theorem:* Let f be a one-to-one weak L-isometrical mapping from (X, Σ) into (X', Σ'). Then there is a pseudostructure Σ_L' which is equivalent to Σ' on $f(X)$, such that f is an isometrical mapping from (X, Σ) onto $\{f(X), \Sigma_L'\}$.

Proof: Because of (2.3.1.),

$$a(\sigma)\sigma(x; y) \leqq \sigma'(x'; y') \leqq b(\sigma)\sigma(x; y)$$

holds for all $x, y \in X$ $(\sigma \in \Sigma)$. The structure Σ_L' on $f(X) \subset X'$ can be constructed in the following way:

$\sigma_L' \in \Sigma_L'$ if and only if $\sigma_L'(x'; y') = \sigma(x; y)$ for each $x', y' \in f(X)$, where $x' = f(x)$ and $y' = f(y)$. This definition is evidently correct since f is a one-to-one mapping from X onto $f(X)$.

Moreover it is evident that σ_L' is always a pseudometric and there is a one-to-one mapping between Σ_L' and Σ.

Let U' be from $\mathfrak{U}'_{\Sigma'|_{f(x)}}$, then:

$$U' = [U'_{\sigma_1' \ldots \sigma_r'; \varepsilon}] \cap [f(X) * f(X)],$$

moreover

$$U'_{\sigma_1' \ldots \sigma_r'; \varepsilon} = \bigcap_{i=1}^{r} U'_{\sigma_i'; \varepsilon}$$

is true.

It is sufficient to show that there exist $\sigma_{L_1}'; \sigma_{L_2}'$ from Σ_L' and positive $\varepsilon_1, \varepsilon_2$ for some σ' of Σ and a positive number ε such that the relation

$$U'_{\sigma_{L_1}'; \varepsilon_1} \subset U'_{\sigma'; \varepsilon} \subset U'_{\sigma_{L_2}'; \varepsilon_2}$$

is satisfied. Since (2.3.1.) holds, we obtain

$$\frac{\varepsilon}{b(\sigma)} > \sigma_L' (x' ; y') \geqq \frac{1}{b(\sigma)} \sigma' (x' ; y')$$

whenever

$$\frac{\varepsilon}{b(\sigma)} > \sigma (x ; y) \geqq \frac{1}{b(\sigma)} \sigma'(x' ; y')$$

is satisfied. Thus the relation $U'_{\sigma_L};\frac{\varepsilon}{b(\sigma)} \supset U''_{\sigma',\varepsilon}$ follows from (*). Similarly if

$$\frac{\varepsilon}{a(\sigma)} > \frac{1}{a(\sigma)}\,\sigma'(x';y') \geqq \sigma'_L(x;y)$$

is true we get from (2.3.1.) that

$$\frac{\varepsilon}{a(\sigma)} > \frac{1}{a(\sigma)}\,\sigma'(x';y') \geqq \sigma'_L(x';y')$$

Then from (**) $U'_{\sigma'_L};\frac{\varepsilon}{a(\sigma)} \supset U'_{\sigma';\varepsilon}$ follows and from the relation

$$\mathfrak{U}_{\Sigma'_L} <\subset \mathfrak{U}_{\Sigma'} <\subset \mathfrak{U}_{\Sigma'_L}$$
$$\mathfrak{U}_{\Sigma'_L} \langle\sim\rangle \mathfrak{U}_{\Sigma'}$$

is given. Q. E. D.

2.13. *Remark:* From the proof it reads off easily that if f is not an L-isometrical mapping, but it is only an expansive or contractive mapping, then the resulted uniform structure is coarser or finer than the primary one.

§ 3.

Applications

In the theory of differential equations the differential operators with less than one norm are very important. The Thyhonov—Cacapulli contractive principle or fixed point theorem of Banach are valid to them. Now, the Banach fixed point theorem can be generalized to the class of completely regular topological spaces. It is well-known fact that there is a pseudostructure for every completely regular space such that the topology defined by it is equivalent to the primary topology. Pseudostructures have to satisfy the axioms A.1.; A.2.; A.3.; A.4. The topology defined by the pseudostructure is the following: for every x of X the sets'system

$$U_{\sigma_1 \ldots \sigma_r;\,\varepsilon}(x) = \{y : \sigma_i(x;y) < \varepsilon\,,\ 1 \leqq i \leqq r\}$$

— where $\varepsilon > 0$ and r's are arbitrary positive integers — there is a neighbourhood system of the point x.

Let us note that every topological vector space has a pseudostructure which defines an equivalent topology with the primary one.

3.1. ***Theorem*** **(Banach fixed point theorem):**

Let (X, Σ) be a sequentially complete T_2-space concerning the uniform structure $\mathfrak{U}_\Sigma$; moreover let f be a weak contractive mapping from X into itself and $b(\sigma) < 1$ for each σ from Σ. Suppose that $k\sigma = \sigma$, then there exists a unique fixed point x^* from X to mapping f; i.e. $f(x^*) = x^*$.

Proof: Let us show first that not more than one fixed point exists. If the existence of two different fixed points (x_1 and x_2) is supposed then there is at least one σ_0 from Σ for that (3.1.1.)

$$\sigma_0(x_1; x_2) > O$$

is true, because X is a T_2-space. Hence f is a contractive mapping so the following inequality has to be satisfied:

$$(3.1.2.)\ \sigma_0[f(x_1); f(x_2)] \leqq b(\sigma_0)\sigma_0(x_1; x_2) < \sigma_0(x_1; x_2) = \sigma_0[f(x_1); f(x_2)];$$

and from this $\sigma_0[f(x_1); f(x_2)] = \sigma_0(x_1; x_2) = O$ follows. Thus (3.1.1.) is impossible, since $\sigma(x_1; x_2) = O$ for all $\sigma \in \Sigma$; i.e. $x_1 = x_2$.

Let us take a fixed point. Let x be an arbitrary point from X, and $x_0 = x_1$; $x_2 = f(x_1) \ldots x_n = f(x_{n-1}) \ldots$

Then

$$(3.1.3.)\ \sigma(x_{n+1}; x_n) = \sigma[f(x_n); f(x_{n-1})] \leqq b(\sigma)\,\sigma[f(x_{n-1}); f(x_{n-2})] \leqq \ldots \leqq b[(\sigma)\ ^{n-1}\sigma[(x_1); x_0]$$

is satisfied for each integer n, and σ from Σ.

Let us consider $\sigma[f(x_n); f(x_m)]$ (let $n > m$), and using axiom A.4. and the relation (3.1.3.):

$$\sigma(x_{n+1}; x_{m+1}) = \sigma[fx_n); f(x_m)] \leqq \sigma[f(x_n); f(x_{n-1})] + \sigma[f(x_{n-1}); f(x_m)] \leqq \ldots \leqq \sigma[f(x_n); f(x_{n-1})] + \ldots$$

$$\ldots + \sigma[f(x_{m+1}); fx_m)] \leqq \sigma[f(x_1); (x_0)] \cdot \{[b(\sigma)]^{n-1} + \ldots + [b(\sigma)]^m\} = \sigma[f(x_1); x_0][b(\sigma)]^m \{[b(\sigma)]^{n-m-1} + \ldots + 1\} =$$

$$= \sigma[f(x_1); x_0][b(\sigma)]^m \frac{1 - b(\sigma)}{1 - [b(\sigma)]^{n-m}} < \sigma[f(x_1); x_0] \frac{[b(\sigma)]^m}{1 - b(\sigma)} = C(\sigma)[b(\sigma)]^m$$

where

$$(3.1.4.)\qquad C(\sigma) = \frac{\sigma[f(x_1); x_0]}{1 - b(\sigma)} \geqq 0$$

for all $\sigma \in \Sigma'$. $C(\sigma) = 0$ for all $\sigma \in \Sigma$ if and only if

$$\sigma[f(x_1), x_0] = \sigma[f(x_0), x_0] = 0,$$

then $f(x_0) = x_0$.

In order to prove $\{ x_n \}$ is a Cauchy-sequence choose an arbitrary neighbourhood $U_{\sigma_1 \ldots \sigma_r;\varepsilon}$ of $\mathfrak{U}_\Sigma$ and an integer m_0 such that for every $m > m_0$

$$\varepsilon > C(\sigma_i) \left[b(\sigma_i)\right]^m$$

is satisfied ($i = 1, \ldots r$).

This is always correct because of $\lim_{n \to \infty} [b(\sigma)]^n = 0$.

Consequently $\sigma_i(x_n; x_m)$ for $n, m > m_0$ ($i = 1, \ldots r$) and thus $(x_n ; x_m) \in U_{\sigma_1 \ldots \sigma_r;\varepsilon}$.

It could be shown that $\{ x_n \}$ is a Cauchy-sequence, so there has to exist a limit point x^* for this sequence. Because of continuity of f, (see 2.9. b.) $f(x_n) \to f(x^*)$, moreover $x_n = f(x_{n-1})$ and thus $x_n \to f(x^*)$.

X is a T_2-space, hence $\{x_n\}$ might have no more than one limit point, in consequence $f(x^*) = x^*$ holds. Q. E. D.

3.2. *Remark:* The condition $b(\sigma) < 1$ of the theorem 3.1. cannot be weakened. There may be a mapping f with the property $\sigma(x_1; x_2) > \sigma[f(x_1); f(x_2)]$, but there exists no fixed point.

For example; Let f be a function from $[0, \infty)$ into itself, of the form $f(x) = x - g(x) + C$ where function $g(x)$ satisfies the following conditions:

(3.2.1.) $0 < g(x) < C$ for all x from $(0, \infty)$

(3.2.2.) $g(x)$ is differentiable in $(0, \infty)$ and
$0 < g'(x) < 1$ for all x from $(0, \infty)$,

(3.2.3.) $\lim_{x \to \infty} g(x) = C$.

(The condition 3.2.2. might be replaced by the weaker condition 3.2.4.)

(3.2.4.) $g(x) \in L_1(1)$ where $L_1(1)$ is the class of functions which satisfy the 1 — power Lipschitz condition and their Lipschitz constant is less than one.)

So $|f(x_2) - f(x_1)| < |x_2 - x_1|$ for every $x_2 \neq x_1$ and the equation $f(x) = x$ has no solution because of the condition 3.2.1.

There exists such a function, for example $f(x) = x - arctg\ x + \dfrac{\pi}{2}$;

or another one $f(x) = x\dfrac{x}{x+1} + 1 = \dfrac{x^2+x+1}{x+1}$.

3.3. *A question:* Let f be a contractive mapping from a completely bounded set* of $\mathfrak{U}$ complete space (X, Σ) into itself such that f $\sigma[f(x); f(y)] \leqq \sigma(x; y)$ holds. 2 Does f have a fixed point?

* i.e. this set is compact

(3.3.1.) *Negative exameple:* Let $C(0, 1)$ be the class of continuous functions in interval $[0, 1]$ with the following norm:

$$\sigma(f, g) = \|f - g\| = \sup_{t \in [0, 1]} |f(t) - g(t)|$$

Let us denote by $1(C)$ the set of the functions from $C(0, 1)$ such that $0 \leqq g(t) \leqq 1$. Moreover let $g(t) = \frac{t}{\pi}\left(\frac{\pi}{2} + \operatorname{arctg}\frac{1}{t}\right)$ if $t \neq 0$ and $g(0) = 0$, then $g(t) \in C(0, 1)$.

The operator $P[x(t)] = (1 - t)x(t) + g(t)$ meets the following conditions:

$$P : 1(C) \to 1(C)$$

$$\| P[x(t)] - P[y(t)] \| < \| x(t) - y(t) \|.$$

But, there is no function $x(t)$ from $1(C)$ to be as a solution for equation $P[x(t)] = x(t)$; because this equation has a unique solution:

$$x_0(t) = \frac{1}{\pi}\left(\frac{\pi}{2} + \operatorname{arctg}\frac{1}{t}\right)$$

and this function does not belong to $C(0, 1)$, since $x_0(t)$ cannot be defined so that it is continuous in the point $t = 0$. (See other example and examination in [4].)

The preceding results were always related to the whole spaces or a compact sets of a space. Let us now examine a local property.

3.4. *Definition:* Let f be a mapping from (X, Σ) into (X', Σ'), moreover let k be a one-to-one mapping between Σ and Σ'. f is (weak) contractible in x_0 from X, if there is (a mapping $L_{x_0}(\sigma)$ from Σ into R^*) a constant L_{x_0} and a real function h from X such that in some neighbourhood U of x_0:

$$(3.4.1.) \quad k\sigma[f(x_0); f(y)] \leqq L_{x_0}(\sigma)\sigma(x_0; y) + h(y)\sigma(x_0; y)$$

is satisfied and $h(y) \to 0$ whenever $\sigma(x_0; y) \to 0$ for every σ from Σ.

Evidently from 2.5.1. the following theorem is true:

3.5. *Theorem:* All of the (weak) contractive mappings are (weakly) contractible.

(3.5.1.) *Note:* Every differentiable function is contractible; all of the functions from the function class $L(1)$ (1 — power Lipschitz functions'class) are also contractible.

3.6. *Theorem:* If f is a (weak) contractible mapping in x_0 then f is there continuous.

The proof is evident.

An important question of numerical analysis is the following:

There is a solution x_0 of the equation $f(x) = x$. Which beginning point does the iteration converge to the solution x_0 from?

This question will be generalized for the completely regular spaces in the following theorem.

3.7. *Theorem* (Local-contraction): Let (X, Σ) be a completely regular space, moreover let f be a mapping from X into itself which satisfies the following conditions:

(3.7.1.) There is a fixed point to $f; f(x_0) = x_0$

(3.7.2.) f is weak contractible in x_0 and $k; \Sigma \to \Sigma$ is the identical mapping.

(3.7.3.) $L_{x_0}(\sigma) : \Sigma \to [0, q] \subset R^+$, where $0 < q < 1$.

Then there is a positive number ε such that $f^n(y) \to x$, if $n \to \infty$ (where f^n is the n-th iterated of f) for every y which holds $\sigma(x_0; y) < \varepsilon$ for every $\sigma \in \Sigma'$.*

(3.7.4.) *Note:* Here the minimal system Σ can be assumed. For the following classes of spaces the theorem 3.7. can give a real neighbourhood of x_0 in which the iteration surely converges to the fixed point x_0:

Metrical spaces,

Metrizable space (countable pseudometrics).

A space (X, Σ) is called locally metrizable in its point x_0, if there is a neighbourhood V, which is given the customary subspace topology and then it is a metrizable topological space. Then there exists a countable subsystem $\Sigma/_V$ of Σ such that it gives an equivalent topology to the primary topology. Let this system $\Sigma/_V = \{\sigma_i\}_{i=1}^{\infty}$ then the form $\varrho(x, y) = \sum_{k=1} 2^{-k}\sigma_k(x; y)$ is a metric for the system $\{\sigma_i\}_{i=1}^{\infty}$; moreover f is contractible again with respect to the metric ϱ. For this class of spaces there is a real neighbourhood for the iterations beginning points.

Proof of theorem 3.7.: Let us choose r according to the condition $L_{x_0}(\sigma) \leqq q < r < 1$. Hence $\lim \{(x \to x_0)h(y)\} = 0$ thus there exists some positive number ε such that $h(y) < (1 - r)\sigma(x_0 : y)$ if $\varepsilon > \sigma(x_0 ; y)$. ($\varepsilon$ can be supposed less than one.) We show that ε is the demanded number to the theorem 3.7.

Indeed, because of contractibility:

$$\sigma[x_0 ; f(y)] = \sigma[f(x_0) ; f(y)] \leqq L_{x_0}(\sigma)\sigma(x_0 ; y) + h(y)\sigma(x_0 ; y) <$$
$$< L_{x_0}(\sigma)\sigma(x_0 ; y) + (1 - r)\sigma^2(x_0 ; y) = \sigma(x_0 ; y)[L_{x_0}(\sigma) + (1 - r)\sigma(x_0 ; y)] \leqq$$
$$\leqq \sigma(x_0 ; y)[L_{x_0}(\sigma) + 1 - r] \leqq \sigma(x_0 ; y)(q + 1 - r) = \sigma(x_0 ; y)P;$$

where $P = q + 1 - r$ and evidently $0 < P < 1$.

* See a special case of the theorem 3.7 in [5].

Let us consider $\sigma[f^2(y);f(x)]$; hence

$$(*)\ \sigma(f(x_0);f(y)] = \sigma[f(y);x_0] < P\sigma(x_0;y)$$

thus using the form 3.4.1. i.e.

$$\sigma[f^2(y);f(x_0)] < L_{x_0}(\sigma)\sigma[f(x_0);f(y)] + h[f(y)]\sigma[f(x_0);f(y)] \leqq$$
$$\leqq L_{x_0}(\sigma)\sigma[f(x_0);f(y)] + (1-r)\sigma^2[f(x_0);f(y)] < \sigma[f(y);f(x_0)](q+1-r) =$$
$$= P\sigma[f(y);f(x_0)] < P^2\sigma(x_0;y)$$

— because of the form (*) —

Similarly, with complete induction the following form for arbitrary integer n arises:

$$\sigma[f^n(y);f(x_0)] < P^n\sigma(x_0;y)$$

Because of $0 < P < 1$, thus $P^n \to 0$ $(n \to \infty)$, it follows:

$$\sigma[f^n(y), x_0] \to 0;$$

hence f is a continuous mapping and the fact $f^n(y) \to x_0 (n \to \infty)$ is true. Q.E.D.

3.8. *Application and illustration for some integral equations*

Let Ω be a finite μ-measure set of a locally compact Haussdorf space. (See [8].) Let us consider the class of integrable real functions over Ω, moreover let the bounded real function $K(x, t)$, which is defined over $\Omega * \Omega$, be a μ-multiplicant, if there exists $\int_\Omega K(x, t)g(t)d\mu_t$ for every $x \in \Omega$ and the μ-integrable function $g(t)$.

Let us denote the space of these functions $f(t)$ by $L_{\Omega,\varkappa,\mu}$. The norm in this space is the following:

$$\sigma(f;g) = ||f-g||_{L\Omega,\varkappa,\mu} = \int_\Omega |f-g|\,dt.$$

(3.8.1.) *Theorem:* The operator

$$T^\circ(f,\Omega') = \begin{cases} p(x) + \int_\Omega f(t)\,K(x,t)\,d\mu_t : x\in\Omega' \\ 0 : x\in \Omega - \Omega' \end{cases}$$

has a fixed point (that is the equation $T^0 f = f$ has a unique solution) in suitable subset Ω' of Ω.

Proof: Because of Fubini's theorem $T^0 f$ belongs to $L_{\Omega,\varkappa,\mu}$ again, moreover $L_{\Omega,\varkappa,\mu}$ is a complete space, thus we have to show that T^0 is a contractive mapping with L less than one.

Let us consider $\sigma(T^0 f\,;\,T^0 g)$.

$$\sigma(T^0 f; T^0 g) \leq \int_{\Omega'} \Big(\int_{\Omega'} (K(x,t) \,|\, |f(t) - g(t)|\, dt\Big)\, dx \leq$$

$$\leq \int_{\Omega} (\sup |(K(x,t)| \int_{\Omega'} |f(t) - g(t)|\, dt)\, dx \leq \mu(\Omega')\, K\sigma(f; g)\,.$$

If $\mu(\Omega) K\sigma(f\,;\,g) < L\sigma(f\,;\,g)$, where $L < 1$, then there is a fixed point to $T^0 f$. That holds whenever $\mu(\Omega')K < 1$; that is $\mu(\Omega') < \dfrac{1}{K}$. Q.E.D.

Let us define the classof the functions B_q. The function $f(t)$ belongs to B_p if and only if

$$\int_{\Omega} \Big(\int_{\Omega} |f(t)\, K(x,t)|\, dt\Big)\, dx < q$$

is satisfied.

(3.8.2.) *Theorem:* Let q be less than one. Then for every $p(x)$ from B_q and $0 < \alpha < 1$ there exists an integer n_0 such that the operator

$$T^{n_0} f = p(x)\,(1 - \alpha) + \alpha\Big[\int_{\Omega} f(t)\, K(x,t)\, dt\Big]^{n_0}$$

has a unique fixed point in the set B_p (that is the operator equation $T^{n_0} f = f$ has a unique solution), moreover if $T^{n_0} f = f$ has a solution, then for some $n > n_0$ the operator equation $T^n f = f$ has also a unique solution in the set B_q.

(3.8.2.1.) *Lemma:* $T^n(B_q) \subset B_q$ for suitable n.

Proof: Let us consider

$$\int_{\Omega}\int_{\Omega} |(T^n f)\, K(x,t)|\, dx\, dt:$$

$$\int_{\Omega}\int_{\Omega} |(T^n f)\, K(x,t)|\, dx\, dt \leq \int_{\Omega} \Big[\int_{\Omega} (1 - \alpha)$$

$$p(x)\, K(x,t)\, dx\Big]\, dt + \int_{\Omega} \Big\{ \int_{\Omega} \Big| \Big[\int_{\Omega} f(t)\, K(x,t)\, dt\Big]^n K(x,t)\, dx \Big|\, dt \leq (1 - \alpha)$$

$$q + \alpha q \mu^2(\Omega)\, q^{n-1} \sup_{(x,t) \in \Omega * \Omega} |K(x,t)| = (1 - \alpha)\, q + \alpha\, qr$$

Because q is less than one, there is a suitable n, denoted by n_1, to satisfy:

$r = \mu_2(\Omega) q^{n-1} \sup |K(x,t)| \leq 1$

So $(1 - \alpha) q + \alpha q = q(1 - \alpha + \alpha) = q < 1$.

That is $T^n f \in B_q$ for every $n > n_1$.

(3.8.2.2.) *Lemma:* $\sigma(T^n f ; T^n g) < L\sigma(f ; g)$, where L is less than one for a suitable n.

$$Proof: \sigma(T^n f ; T^n g) = \int_\Omega |T^n (f(t)) - T^n (g(t))| \, dt =$$

$$= \int_\Omega |\varkappa[(\int_\Omega f(x) K(x,t)\,dx)^n - (\int_\Omega g(x) K(x,t)\,dx)^n]| \, dt =$$

(Let us choose $n = 2k + 1$ first)

$$= \varkappa \int_\Omega |(\int_\Omega f(x) K(x,t)\,dx) - \int_\Omega g(x) K(x,t)\,dx) [(\int_\Omega fK dx)^{2k} -$$

$$- (\int_\Omega fK\,dx)^{2k-1} (\int_\Omega gK\,dx) \pm \ldots + (\int_\Omega gK\,dx)^{2k}]| \, dt \leq$$

$$\leq \varkappa \int_\Omega (\int_\Omega |K(x,t)| \, |f(x) - g(x)| \, dx) [(\int |fK| \, dx)^{2k} + \ldots +$$

$$+ (\int |gK| \, dx)^{2k}] \, dt \leq \sup_{(x,t) \in \Omega * \Omega} |K(x,t)| \, \sigma(f ; g) \, n \, \mu(\Omega) \, M^{2k} \text{ — where}$$

$$M = \max_{f,g} \{ \int_\Omega |fK| \, dx ; \int_\Omega |gK| \, dx \} .$$

Hence $f, g \in B_q$, thus $M = q < 1$. Then:

$$\sigma(T^n f ; T^n g) \leq \varkappa \sup |K(x,t)| \, n\mu(\Omega) \, q^{n-1} \, \sigma(f ; g) ;$$

because of $q < 1$; thus there exists an integer n_2 such that $\sigma(T^n f ; T^n g) \leqq \leqq L\sigma(f, g)$ — where $L < 1$ —.

Let us choose $n = 2k$ then:

$$a^{2k} - b^{2k} = (a - b) (a^{2k-1} + a^{2k-2} b + \ldots + b^{2k-1})$$

a result similar to $\sigma(T^n f ; T^n g)$. Let this index be denoted by n_3. If $n_0 = \max \{n_1 ; n_2 ; n_3\}$, then

$$T^n(B_q) \subset B_q$$

and

$$\sigma(T^n f ; T^n g) < L \, \sigma(f ; g)$$

where $L < 1$ for all $f, g \in B_q$ and $n > n_0$. Q.E.D.

(3.8.2.3.) *Lemma*: B_q is a complete space.

Proof: $B_q \in L_{\Omega,\varkappa,\mu}$ because of the completeness of $L_{\Omega,\varkappa,\mu}$ it is enough to show that the set Bq is a closed set of $L_{\Omega,\varkappa,\mu}$. This fact is made evident by the

Fatou theorem [8]

$$\lim_{n\to\infty} \int_\Omega f_n(t)\, K(x, t)\, dt = \int_\Omega (\lim f_n(t))\, K(x, t)\, dt$$

if

$$\lim f_n = f, \text{ then from } \int_\Omega f_n(t)\, K(x, t)\, dt \leq q$$

the inequality $\int_\Omega f(t)\, K(x, t)\, dt \leq q$ follovs. Q. E. D.

Proof of theorem (3.8.2.): From (3.8.2.3.) and (3.8.2.2.) follows the possibility to apply the theorem 3.1. The solution is unambiguous except a zero measure set. Q.E.D.

Example to 3.8.2.

Let Ω be $[0, 1]$, $q = \alpha = 1/2$, $k(x, t) = \frac{1}{2}(x^2 + t^2)$, $p(x) = x/2$ then the equation

$$(*)\ \frac{x}{2} + \frac{1}{2}\left[\frac{1}{2}\int_0^1 f(t)\,(x^2 + t^2)\, dt\right]^n = f(x)$$

has a unique solution for every positive integer n.

Indeed, from $\mu(\Omega) = 1$, $\sup\limits_{(t,x)\in[\Omega*\Omega]} (x^2 + t^2)/2 = 1$

and $\mu^2(\Omega) \sup \frac{1}{2}(x^2 + t^2)\, p^{n-1} \leq 1$

$$\alpha\mu(\Omega) \sup \frac{1}{2}(x^2 + t^2) q^{n-1} n \leq 1$$

it follows for every $n > 0$ (see the proof of theorem) 3.8.2. that equation (*) has a unique solution, to be found by the method of iteration.

Let $f_1(x) = 1 \in B$,be the beginning point of the iteration, then

$$f_2^1(x) = \frac{1}{6} + \frac{x}{2} + \frac{x^2}{2}$$

$$f_3^1(x) = \frac{1}{6^2} + \frac{7x}{12} + \frac{x^2}{6} + \frac{x^3}{4} + \frac{x^4}{4}$$

$$f_4^1(x) = \frac{1}{6^3} + \frac{25x}{36} + \frac{x^2}{24} + \frac{x^3}{3} + \frac{x^4}{6} + \frac{x^5}{8} + \frac{x^6}{8}$$

$$f_5^1(x) = \frac{1}{6^4} + \frac{133x}{216} + \frac{5x^2}{432} + \frac{29x^3}{72} + \frac{x^4}{16} + \frac{5x^5}{24} + \frac{x^6}{6} + \frac{x^7}{18} + \frac{x^8}{16}$$

An interesting application of the theorem 3.7 (or 3.1)

Let $P_n(z)$ be a polynomial function of the complex variable z.
$P_n(z) = a_0 z^n + a_1 z^{n-1} + \ldots + a_{n-1} z + a_n \quad (a_{n-1} \neq 0)$

if

$$\left| \frac{a_0}{a_{n-1}} n z^{n-1} + \frac{a_1}{a_{n-1}} (n-1) z^{n-2} + \ldots + \frac{a_{n-2}}{a_{n-1}} 2z \right| < 1$$

and

$$P_n(z) \in \Omega, \qquad z \in \Omega$$

is satisfied in certain closed subset Ω of complex plane then there is only one root of $P_n(z)$ in Ω.

For example: Every $P_n(z) = a_0 z_n + \ldots + n^2 z + a_n = 0$ with $|a_i| < 1 \; i \neq n-1$ has only one root in the closed unit circle $S(0, 1)$. Let $\frac{P_n(z) - a_{n-1} z}{a_{n-1}} = Q_n(z)$, then the equation $Q_n(z) = z$ has the same roots as the equation $P_n(z) = 0$. Moreover the condition

$$|Q_n(z_1) - Q_n(z_2)| < q\, |z_1 - z_2| \quad \text{and} \quad q < 1$$

is satisfied for every $z_1; z_2 \in S(0, 1)$, if $|Q'_n(z)| < 1$ for every $z \in S(0, 1)$. Thus from the compactness of the closed unit circle (and $Q_n(z) \in S(0, 1)$) it follows by the theorem 3.1 (or 3.7).

Summary

Different generalizations of the isometrical mappings are dealt with. In the first and second part some generalizations are given as the generalizations of the Lipschitz-condition and their properties examined, in the third part their applications are presented together with illustrations, namely the fixed point theorems, local contractible mappings, unique existence of solution of the distinct type of integral equations. In proving the fixed point theorem the condition of this theorem cannot be weakened. The local contractibility shows how a neighbourhood of the solution of the equations $Tx = x$ can be found assuming it has a unique solution and local contractibility, from where beginning, the iteration always converges to the solution. Finally a numerical example of the method of iteration (an integral equation) and an interesting condition about the roots of algebraic equation in the complex plane are presented.

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