THE CALCULATION OF STATIONARY STATE TRANSMISSION LINE SYSTEMS BY TOPOLOGICAL METHODS

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Introduction

Electric energy is distributed by the help of transmission line systems. The applied transmission lines are forming a system in two aspects. A transmission line section namely contains more than two conductors. (Also the earth is to be included in the number of the conductors.) The electromagnetic field of the conductors is creating a coupling between the conductors and thus a transmission line section forms a system consisting of coupled transmission lines. Beyond this the individual transmission line sections are connected to each other at the ends. At the connection places also generators and consumers may be connected to the transmission line sections. Such a network may be regarded also as a system.

In practice the requirement may arise that the equations for the whole energy distributing transmission line system should be handled jointly. A problem of this kind is e.g. to determine the generator voltages to be employed to obtain the necessary consumer voltages in the case of a given arrangement of the network, or to establish in a given network the consumer voltages arising in the case of known generator voltages and consumers. Various approximative methods were published for solving these problems [8]. The majority of these is disregarding the fact that the transmission line is a network with distributed parameters, in some of the methods the transmission line is simply substituted by a two-pole, in others the joint handling of all the equations necessary for the solution of the problem is not possible.

In the present paper a general method is described, which solves the above problems by jointly writing all the necessary equations for the stationary state transmission line system. The calculation described in the followings is utilizing results published earlier on the theory of coupled transmission lines [5, 7] and on transmission networks [9].

The examinations refer to an excitation changing in time sinusoidally on the following conditions. The system is linear. The conductors have a circular cross-section. Earth is regarded as limited by a plane and homogeneous within a transmission line section. In the individual transmission line sections the conductors are parallel to each other and to the earth. The electromagnetic field of different transmission line sections does not create a coupling between the sections. The connection of the network, the material constants essential from the aspect of the electromagnetic field (ε, μ, σ), generator voltages, and the individual impedance values are assumed to be known. Only such cases are

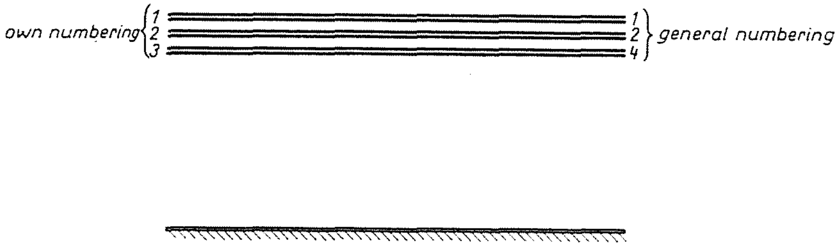

Fig. 1

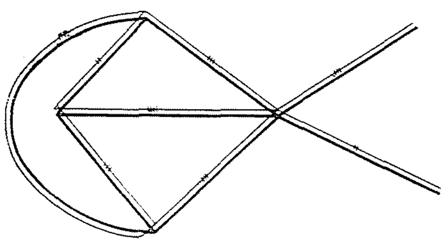

Fig. 2

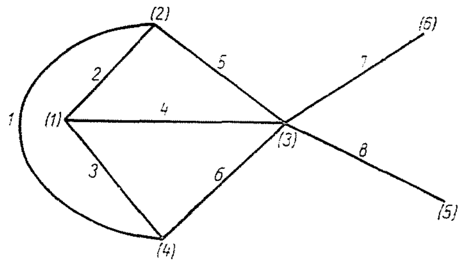

Fig. 3

examined where the displacement current in the earth is negligible with respect to the conduction current.

Our further conditions on the connection of transmission line sections will be given in full details later.

The topology of the network

The examined transmission line system is built up of k transmission line sections. The number of the conductors of the hth section is designated by m_h and these conductors are given an order number $(1, 2, \ldots, m_h)$. The number of conductors in the section containing the highest number of overhead conductors is n $(n \geq m_h,\ h = 1, 2, \ldots, k)$. The conductors of all transmission line sections are given also a general numbering relating to the complete system, where the order number of conductors connected at the vertices is identical.

In the followings only such systems are to be examined in which such a numbering is possible. Let us define the matrix $\mathbf{t}_h$ describing the hth section in such a way that the jth element in the ith row $t_{ij} = 1$ if the general numbering of the jth conductor according to the own numbering of the section is i, and otherwise $t_{ij} = 0$. The rows of matrix $\mathbf{t}_h$ correspond to the general numbering, while the columns to the own one. Let the section consist of e.g. three overhead conductors ($m = 3$) (Fig. 1) and be in the system $n = 4$. A possible own numbering is indicated in Fig. 1 at the left side, while the general numbering at the right side. Matrix $\mathbf{t}_h$ pertaining to the section is then found to be

$$\mathbf{t}_h = \begin{array}{c c} & \overrightarrow{\text{own}} \\ \text{general} \downarrow & \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \end{array} \tag{1}$$

The expression

$$\mathbf{a}_h = \mathbf{t}_h \, \mathbf{t}_h^* \tag{2}$$

will be necessary which is a characteristic of the selected hth section. (The asterisk * denotes the transposed matrix.) In our previous example

$$\mathbf{a}_h = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3}$$

$\mathbf{a}_h$ is a diagonal matrix of n order, both the rows and columns of which correspond to the general numbering of the conductors. The element in the main diagonal is 1 if the section is containing the conductor with an order number corresponding to the row (column), otherwise it is 0. If the section characterized by $\mathbf{a}_h$ contains n conductors, then $\mathbf{a}_h = \mathbf{E}$ ($\mathbf{E}$ is the unit matrix of the nth order).

Let us form from the matrices $\mathbf{t}_h$ pertaining to the individual sections a hypermatrix describing the topology of the complete system.

$$\mathbf{T} = \begin{bmatrix} \mathbf{t}_1 & \mathbf{o} & \cdots & \mathbf{o} \\ \mathbf{o} & \mathbf{t}_2 & \cdots & \mathbf{o} \\ \cdot & \cdot & \cdots & \cdot \\ \mathbf{o} & \mathbf{o} & \cdots & \mathbf{t}_k \end{bmatrix} \tag{4}$$

A graph (Fig. 3) can be made to correspond to the examined transmission network (Fig. 2) [1, 3]. The branches of the graph correspond to the transmission line sections, and the vertices of the graph to the connection places. A branch consisting of m conductors represents from the aspect of network theory a $2 \times (m+1)$ pole (earth is namely counted as a conductor). The branches and vertices of the graph are given an order number. The order number designating the vertices is in brackets for the sake of distinction. The number of vertices is designated by c.

The graph may have such branches too, which are connected to further branches only at one end. Such a graph, containing also terminal elements,

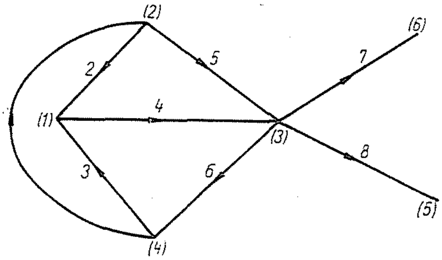

Fig. 4

can best be described by a vertex matrix, indicating the coincident branches and vertices. For our problem let us define a vertex matrix which gives information also on the order number of the conductors belonging to the individual transmission line sections, according to the general numbering. The vertex matrix is given in the form of a hypermatrix. A vertex corresponds to the rows formed of the matrix elements, and a section to the columns. The hth element of the ith row is the matrix $\mathbf{a}_h$ as defined in (2), if the hth section is coincident with the ith vertex, otherwise the matrix $\mathbf{o}$ of the nth order.

Thus e.g. the vertex matrix of the network shown in Fig. 2 is found to be

$$\mathbf{A} = \begin{bmatrix} \mathbf{o} & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{o} & \mathbf{o} & \mathbf{o} & \mathbf{o} \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{o} & \mathbf{o} & \mathbf{a}_5 & \mathbf{o} & \mathbf{o} & \mathbf{o} \\ \mathbf{o} & \mathbf{o} & \mathbf{o} & \mathbf{a}_4 & \mathbf{a}_5 & \mathbf{a}_6 & \mathbf{a}_7 & \mathbf{a}_8 \\ \mathbf{a}_1 & \mathbf{o} & \mathbf{a}_3 & \mathbf{o} & \mathbf{o} & \mathbf{a}_6 & \mathbf{o} & \mathbf{o} \\ \mathbf{o} & \mathbf{o} & \mathbf{o} & \mathbf{o} & \mathbf{o} & \mathbf{o} & \mathbf{o} & \mathbf{a}_8 \\ \mathbf{o} & \mathbf{o} & \mathbf{o} & \mathbf{o} & \mathbf{o} & \mathbf{o} & \mathbf{a}_7 & \mathbf{o} \end{bmatrix} \qquad (5)$$

If all the sections of the transmission network contain n conductors, then the elements of matrix $\mathbf{A}$ are $\mathbf{E}$ and $\mathbf{o}$ matrices of the nth order.

For writing the equations of the network the sections of the network and the branches of the graph, should be given a direction (a reference direction indicated). This direction is accordingly valid for the conductors in the transmission line sections as well. Reference directions may be chosen arbitrarily. (Thus e.g. a possible case of the direction of the graph shown in Fig. 3 is indicated in Fig. 4.) For the directed graph obtained in this way the directed vertex matrix may be defined. The build up of the directed vertex matrix is similar to that of the vertex matrix. The matrix element in the hth column of the ith row is $\mathbf{a}_h$, if the hth section is coincident with the ith vertex and its direction is away from the vertex. If in turn the hth section is coincident with the ith vertex and its direction is towards the vertex, then the hth element

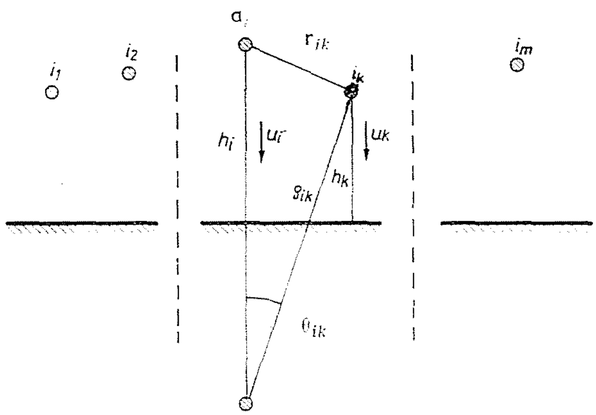

Fig. 5

of the ith row is $-\mathbf{a}_n$. If the hth section does not coincide with the ith vertex, then the hth matrix element of the ith row is the $\mathbf{o}$ matrix of nth order.

The directed vertex matrix of the network described by the directed graph shown in Fig. 4 is found to be

$$\mathbf{A}_i = \begin{bmatrix} \mathbf{o} & -\mathbf{a}_2 & -\mathbf{a}_3 & \mathbf{a}_4 & \mathbf{o} & \mathbf{o} & \mathbf{o} & \mathbf{o} \\ -\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{o} & \mathbf{o} & \mathbf{a}_5 & \mathbf{o} & \mathbf{o} & \mathbf{o} \\ \mathbf{o} & \mathbf{o} & \mathbf{o} & -\mathbf{a}_4 & -\mathbf{a}_5 & \mathbf{a}_6 & \mathbf{a}_7 & \mathbf{a}_8 \\ \mathbf{a}_1 & \mathbf{o} & \mathbf{a}_3 & \mathbf{o} & \mathbf{o} & -\mathbf{a}_6 & \mathbf{o} & \mathbf{o} \\ \mathbf{o} & \mathbf{o} & \mathbf{o} & \mathbf{o} & \mathbf{o} & \mathbf{o} & \mathbf{o} & -\mathbf{a}_8 \\ \mathbf{o} & \mathbf{o} & \mathbf{o} & \mathbf{o} & \mathbf{o} & \mathbf{o} & -\mathbf{a}_7 & \mathbf{o} \end{bmatrix} \tag{6}$$

Overhead transmission line system

In the followings the results of the theory of overhead transmission line systems [4, 7] which will be necessary for our further calculations, are briefly summarized.

The examined arrangement is shown in Fig. 5. Let us form the column vectors $\boldsymbol{u}$ and $\boldsymbol{i}$ of the voltages of the individual conductors with respect to the earth, and of the current of the conductors, respectively. The dependence of these values on the coordinate z in the direction of the transmission line is described by the equations

$$u(z) = e^{-\Gamma z}\, u^{(+)} + e^{\Gamma z}\, u^{(-)}$$
$$i(z) = \mathbf{y}_0(e^{-\Gamma z}\, u^{(+)} - e^{\Gamma z}\, u^{(-)}) \tag{7}$$

where $u^{(+)}$ and $u^{(-)}$ are the column vectors formed at the place $z = 0$ from voltages advancing in directions $+z$ and $-z$, respectively. The propagation coefficient matrix $\mathbf{\Gamma}$ and the wave admittance matrix $\mathbf{y}_0$ can be determined from the parallel admittance matrix $\mathbf{y}_p$ for unit length and from the series impedance matrix $\mathbf{z}_s$ for unit length, respectively:

$$\mathbf{\Gamma}^2 = \mathbf{z}_s\mathbf{y}_p\,; \qquad \mathbf{\Gamma} = \sqrt{\mathbf{z}_s\mathbf{y}_p}\,; \qquad \mathbf{y}_0 = \mathbf{z}_s^{-1}\,\mathbf{\Gamma} \tag{8}$$

The expressions for matrices $\mathbf{y}_p$ and $\mathbf{z}_s$ are

$$\mathbf{y}_p = \mathrm{j}\omega\varepsilon\mathbf{M}^{-1}$$
$$\mathbf{z}_s = \frac{\mathrm{j}\omega\mu_0}{\pi}\,\mathbf{M} + \mathbf{z}_b \tag{9}$$

Matrix $\mathbf{M}$ can be determined from geometrical data. The kth element of the ith row is found to be

$$m_{ik} = \frac{1}{2}\ln\frac{\varrho_{ik}}{R_{ik}} \tag{10}$$

If $i \neq k$, the meaning of ϱ_{ik} and R_{ik} can be read in Fig. 5. If $i = k$, then $\varrho_{ii} = 2h_i$ and $R_{ii} = a_i$, where h_i and a_i can similarly be determined from Fig. 5. $\mathbf{z}_b$ is the sum of two matrices:

$$\mathbf{z}_b = \mathbf{z}_v + \mathbf{z}_f \tag{11}$$

where $\mathbf{z}_v$ is a diagonal matrix, with the internal impedances of the individual conductors related to unit length in the main diagonal, while the earth impedance matrix $\mathbf{z}_f$ is a symmetrical matrix. For writing an element of $\mathbf{z}_f$, let us introduce the designation

$$r_{ik}^2 = \omega\mu_0\,\sigma_f\,\varrho_{ik}^2 \tag{12}$$

With this, an approximative value for the kth element in the ith row of $\mathbf{z}_f$

is found to be

$$z_{jik} = \frac{\omega\mu_0}{\pi}\left[\frac{\pi}{4}\frac{\sqrt{j}}{r_{ik}}\left(e^{-j\Theta_{ik}} H_1(r_{ik}\, e^{j\Theta_{ik}}) + e^{j\Theta_{ik}} H_1(r_{ik}\, e^{-j\Theta_{ik}}) - \right.\right.$$

$$\left.\left. - e^{-j\Theta_{ik}} N_1(r_{ik}\, e^{j\Theta_{ik}}) - e^{j\Theta_{ik}} N_1(r_{ik}\, e^{-j\Theta_{ik}})\right) - \frac{\cos 2\Theta_{ik}}{r_{ik}}\right] \tag{13}$$

The first order Struve function $H_1(z)$ and the first order Neumann function $N_1(z)$ can be approximated by some terms of their series. At a given frequency z_{jik} depends only on the value of the material constants and on geometrical dimensions.

The matrix functions in Equation (7) can be interpreted by the help of the Lagrange polynomials. To write these, the eigenvalues γ_i^2 of matrix $\mathbf{\Gamma}^2$ have to be determined by means of the roots of the algebraic equation of mth order

$$\det|\mathbf{\Gamma}^2 - \gamma^2\,\mathbf{E}| = 0 \tag{14}$$

In the general case (14) has m different roots.

The calculation of the eigenvalues of matrix $\mathbf{\Gamma}^2$ is reduced on the basis of (8) and (9) to those of the matrix

$$\mathbf{G}^2 = j\,\omega\varepsilon\pi\,\mathbf{z}_b\,\mathbf{M}^{-1} \tag{15}$$

We find that

$$\det|\mathbf{\Gamma}^2 - \gamma^2\,\mathbf{E}| = \det|\mathbf{G}^2 - g^2\,\mathbf{E}| = 0 \tag{16}$$

where

$$\mathbf{\Gamma}^2 = \mathbf{G}^2 + k^2\,\mathbf{E} \tag{17}$$

$$\gamma^2 = g^2 + k^2 \tag{18}$$

k denotes the propagation coefficient of the plane wave advancing in the dielectric medium surrounding the conductors,

$$k^2 = j\omega\mu_0 j\omega\varepsilon, \tag{19}$$

g_i^2 denotes the eigenvalues of matrix $\mathbf{G}^2$.

Having determined γ_i^2 $(i = 1,2, \ldots, m)$, the matrix Lagrange polynomials can be written.

$$\mathbf{L}_i\,(\mathbf{\Gamma}^2) = \prod_{\substack{l=1\\ l\neq i}}^{m} \frac{\mathbf{\Gamma}^2 - \gamma_l^2\mathbf{E}}{\gamma_i^2 - \gamma_l^2} \tag{20}$$

By the help of the Lagrange polynomials, some function $\mathbf{f}(\mathbf{\Gamma}^2)$ can be calculated as follows.

$$\mathbf{f}(\mathbf{\Gamma}^2) = \sum_{i=1}^{m} f(\gamma_i^2)\, \mathbf{L}_i\, (\mathbf{\Gamma}^2) \tag{21}$$

On the basis of this the matrix functions in (7) can be written.

$$e^{\mp \mathbf{\Gamma} z} = \sum_{i=1}^{m} e^{\mp \gamma_i z}\, \mathbf{L}_i(\mathbf{\Gamma}^2) \tag{22}$$

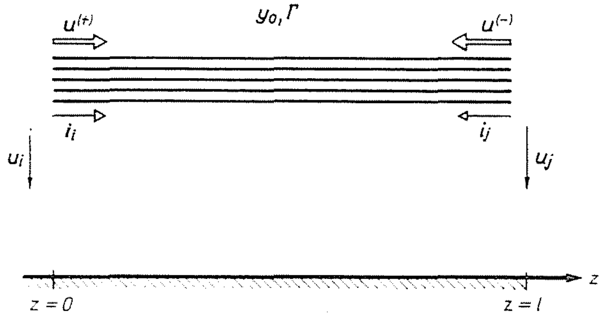

Fig. 6

By considering (21) the propagation coefficient matrix $\mathbf{\Gamma}$ can be determined.

$$\mathbf{\Gamma} = \sum_{i=1}^{m} \gamma_i\, \mathbf{L}_i(\mathbf{\Gamma}^2) \tag{23}$$

and in the knowledge of this matrix $\mathbf{y}_0$ can also be calculated on the basis of (8).

The admittance matrix of the transmission line section

We have seen that in the transmission line section consisting of m conductors both in directions $+z$ and $-z$ a bundle of waves advanced, both of which were superpositions of m waves.

Neither of the ends of the transmission line sections is a preferred one in the transmission network. The starting coordinate of the two bundles of waves advancing in the individual sections in directions opposite to each other is chosen at that end of the section where the wave actually starts. Let us choose that bundle of waves as advancing in positive direction the propagation direction of which is identical with the reference direction of the section.

Accordingly the voltage at the ends of the transmission line section pertaining to vertices (i) and (j), respectively (Fig. 6), is found to be

$$u_i = u^{(+)} + e^{-\Gamma l}\, u^{(-)} \tag{24}$$

$$u_j = e^{-\Gamma l}\, u^{(+)} + u^{(-)}, \tag{25}$$

where l denotes the length of the transmission line section. By introducing the designation

$$\mathbf{b} = e^{-\Gamma l}, \tag{26}$$

formulae (24) and (25) can be summarized in the form

$$\begin{bmatrix} u_i \\ u_j \end{bmatrix} = \begin{bmatrix} \mathbf{E} & \mathbf{b} \\ \mathbf{b} & \mathbf{E} \end{bmatrix} \begin{bmatrix} u^{(+)} \\ u^{(-)} \end{bmatrix} \tag{27}$$

The currents arising at the ends of the transmission line section belonging to vertices (i) and (j) can be written on the basis of (7) as follows.

$$i_i = \mathrm{y}_0(u^{(+)} - \mathbf{b}u^{(-)}) \tag{28}$$

$$i_j = \mathrm{y}_0(-\mathbf{b}u^{(+)} + u^{(-)}) \tag{29}$$

that is

$$\begin{bmatrix} i_i \\ i_j \end{bmatrix} = \mathrm{y}_0 \begin{bmatrix} \mathbf{E} & -\mathbf{b} \\ -\mathbf{b} & \mathbf{E} \end{bmatrix} \begin{bmatrix} u^{(+)} \\ u^{(-)} \end{bmatrix} \tag{30}$$

By using (27) it can be written:

$$\begin{bmatrix} i_i \\ i_j \end{bmatrix} = \mathrm{y}_0 \begin{bmatrix} \mathbf{E} & -\mathbf{b} \\ -\mathbf{b} & \mathbf{E} \end{bmatrix} \begin{bmatrix} \mathbf{E} & \mathbf{b} \\ \mathbf{b} & \mathbf{E} \end{bmatrix}^{-1} \begin{bmatrix} u_i \\ u_j \end{bmatrix} = \mathrm{y} \begin{bmatrix} u_i \\ u_j \end{bmatrix} \tag{31}$$

The admittance matrix of the transmission line section is accordingly found to be

$$\mathrm{y} = \mathrm{y}_0 \begin{bmatrix} \mathbf{E} & -\mathbf{b} \\ -\mathbf{b} & \mathbf{E} \end{bmatrix} \begin{bmatrix} \mathbf{E} & \mathbf{b} \\ \mathbf{b} & \mathbf{E} \end{bmatrix}^{-1} \tag{32}$$

Let us take correlation (26) into consideration and perform the indicated operations. Then we obtain that

$$\mathrm{y} = \mathrm{y}_0 \begin{bmatrix} \cosh \boldsymbol{\Gamma} l & -\mathbf{E} \\ -\mathbf{E} & \cosh \boldsymbol{\Gamma} l \end{bmatrix} \sinh^{-1} \boldsymbol{\Gamma} l = \begin{bmatrix} \mathbf{r}' & \mathbf{p}' \\ \mathbf{p}' & \mathbf{r}' \end{bmatrix}, \tag{33}$$

where

$$\mathrm{r}' = \mathrm{y}_0 \cosh \boldsymbol{\Gamma} l \cdot \sinh^{-1} \boldsymbol{\Gamma} l \tag{34}$$

and

$$\mathrm{p}' = -\,\mathrm{y}_0 \sinh^{-1} \boldsymbol{\Gamma} l\,. \tag{35}$$

$\mathbf{r}'$ and $\mathbf{p}'$ are quadratic matrices of the mth order, where m is the number of the overhead conductors in the transmission line section.

Let us form a hypermatrix from matrices $\mathbf{r}'$ and $\mathbf{p}'$ in the following way.

$$\mathbf{R}' = \begin{bmatrix} \mathbf{r}_1' & \mathbf{o} & \mathbf{o} & \ldots & \mathbf{o} \\ \mathbf{o} & \mathbf{r}_2' & \mathbf{o} & \ldots & \mathbf{d} \\ \cdot & \cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot \\ \mathbf{o} & \mathbf{o} & \mathbf{o} & \ldots & \mathbf{r}_k' \end{bmatrix} \tag{36}$$

$$\mathbf{P}' = \begin{bmatrix} \mathbf{p}_1' & \mathbf{o} & \mathbf{o} & \ldots & \mathbf{o} \\ \mathbf{o} & \mathbf{p}_2' & \mathbf{o} & \ldots & \mathbf{o} \\ \cdot & \cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot \\ \mathbf{o} & \mathbf{o} & \mathbf{o} & \ldots & \mathbf{p}_k' \end{bmatrix} \tag{37}$$

The order of the matrices obtained in this way is identical with the number of conductors in the system. Let us still form the nth order quadratic matrices

$$\mathbf{r}_j = \mathbf{t}_j \mathbf{r}_j' \mathbf{t}_j^* \tag{38}$$

and

$$\mathbf{p}_j = \mathbf{t}_j \mathbf{p}_j' \mathbf{t}_j^* \tag{39}$$

We shall need also the matrices

$$\mathbf{R} = \mathbf{T}\mathbf{R}'\mathbf{T}^* = \begin{bmatrix} \mathbf{r}_1 & \mathbf{o} & \ldots & \mathbf{o} \\ \mathbf{o} & \mathbf{r}_2 & \ldots & \mathbf{o} \\ \cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot \\ \mathbf{o} & \mathbf{o} & \ldots & \mathbf{r}_k \end{bmatrix} \tag{40}$$

and

$$\mathbf{P} = \mathbf{T}\mathbf{P}'\mathbf{T}^* = \begin{bmatrix} \mathbf{p}_1 & \mathbf{o} & \ldots & \mathbf{o} \\ \mathbf{o} & \mathbf{p}_2 & \ldots & \mathbf{o} \\ \cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot \\ \mathbf{o} & \mathbf{o} & \ldots & \mathbf{p}_k \end{bmatrix} \tag{41}$$

The elements of these matrices are of the nth order and the individual admittance parameters are at the places corresponding to the general numbering.

Characterization of the network components at the vertices

There may be consumers or generators at the vertices. These network components are regarded as active linear $(n+1)$ poles. As limiting cases, networks containing only passive consumers, and solely ideal generators, respectively, can be considered.

The terminal of the examined $(n + 1)$ pole connected to the earth (Fig. 7) is designated by the index 0. The numbering of the other terminals corresponds to the general numbering of the conductors to which they are connected.

Let us form the column vector u_{ci} out of the voltages of terminals designated by $1, 2, \ldots, n$ with respect to the earth, and column vector i_{ci} out of the respective currents. For a number m of the terminals of the network components connected to the vertex $(m < n)$, in column vectors u_{ci} and i_c there is a 0 at the places corresponding to the order number of the absent

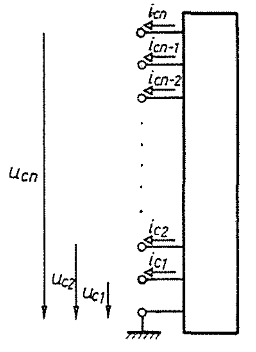

Fig. 7

terminals. From the voltages of the terminals of the network components connected to the vertex, with respect to the earth, the column vector can be written for an open-circuit line, designated by u_{gi}. By employing the principle of superposition we obtain the correlation

$$u_{gi} = u_{ci} + \mathbf{z}_{gi} \, \boldsymbol{i}_i \,, \tag{42}$$

where $\mathbf{z}_{gi}$ is the impedance matrix of the network component. For the deactivated network component $u_{gi} = 0$ and then

$$u_{ci} = - \mathbf{z}_{gi} \, \boldsymbol{i}_{ci} \tag{43}$$

In the knowledge of the connection and of the electric characteristics of the network part this matrix equation and from this $\mathbf{z}_{gi}$ can be written.

If the network part contains exclusively ideal voltage generators, then $\mathbf{z}_{gi} = 0$, and thus

$$u_{ci} = u_{gi} \tag{44}$$

If in turn the network part is passive, then $u_{gi} = 0$, and

$$\mathbf{z}_{gi} = \mathbf{z}_{ti} \tag{45}$$

where $\mathbf{z}_{li}$ is the impedance matrix of the consumers.

Equation (42) can be written for some of the vertices ($i = 1, 2, \ldots, c$) and these can be summarized in the following relationship.

$$U_g = U_c + \mathbf{Z}_g I_c \tag{46}$$

hence:

$$I_c = \mathbf{Y}_g U_g - \mathbf{Y}_g U_c \tag{47}$$

where U_c, U_g, and I_c denote the column vectors formed from the matrices u_{ci}, u_{gi} and i_{ci}, respectively, in accordance with the order numbers of the vertices.

$$U_c = \begin{bmatrix} u_{c1} \\ u_{c2} \\ \cdot \\ \cdot \\ \cdot \\ u_{cc} \end{bmatrix}; \quad U_g = \begin{bmatrix} u_{g1} \\ u_{g2} \\ \cdot \\ \cdot \\ \cdot \\ u_{gc} \end{bmatrix}; \quad I_c = \begin{bmatrix} i_{c1} \\ i_{c2} \\ \cdot \\ \cdot \\ \cdot \\ i_{cc} \end{bmatrix} \tag{48}$$

$\mathbf{Z}_g$ is the hypermatrix

$$\mathbf{Z}_g = \begin{bmatrix} \mathbf{z}_{g1} & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & \mathbf{z}_{g2} & \ldots & \mathbf{0} \\ \cdot & \cdot & \cdots & \cdot \\ \mathbf{0} & \mathbf{0} & \ldots & \mathbf{z}_{gc} \end{bmatrix} \tag{49}$$

formed of the matrices $\mathbf{z}_{gi}$, and $\mathbf{Y}_g$ is its reciprocal:

$$\mathbf{Y}_g = \mathbf{Z}_g^{-1} \tag{50}$$

Circuit equations

In the foregoing the determination of the circuit equations for the network was prepared.

At the ith vertex ends of transmission line sections, voltages u_i are equal and identical with vertex voltage u_{ci}. In the followings the vertex voltages are assumed to be unknown and with this the circuit equations are written in such a way that the Kirchhoff loop equations should be satisfied automatically.

To a vertex n nodes belong at maximum. The node rule should be satisfied for each node. The node equations for the nodes at a vertex are written in a single matrix equation.

Currents flowing to or from a node of some of the vertices can be considered as the sum of three groups. To the first group those currents belong the direction of which is identical with the reference direction of the respective branch. The second group contains the currents the direction of which is contrary to the reference direction of the respective branch. Finally the third group contains the currents of the generators and impedances at the vertex.

If the hth section is connected to vertices (i) and (j) and its reference direction is given from (i) towards (j), then the current of the section in the first group is found to be

$$i_{hi} = \mathrm{r}_h u_i + \mathrm{p}_h u_j \tag{51}$$

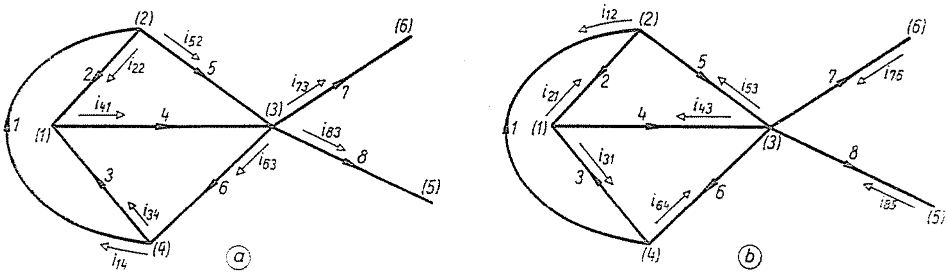

Fig. 8

Similar equations can be written for each section. The system of equations obtained in this way can be summarized as follows.

$$I' = \frac{1}{2} \mathbb{R}(\mathrm{A}^* + \mathrm{A}_i^*) U + \frac{1}{2} \mathbb{P}(\mathrm{A}^* - \mathrm{A}_i^*) U \tag{52}$$

The column vector formed of the currents of the first group is given in our example by

$$I' = \begin{bmatrix} i_{14} \\ i_{22} \\ i_{34} \\ i_{41} \\ i_{52} \\ i_{63} \\ i_{73} \\ i_{83} \end{bmatrix} = \begin{bmatrix} \mathrm{r}_1 u_4 + \mathrm{p}_1 u_2 \\ \mathrm{r}_2 u_2 + \mathrm{p}_2 u_1 \\ \mathrm{r}_3 u_4 + \mathrm{p}_3 u_1 \\ \mathrm{r}_4 u_1 + \mathrm{p}_4 u_3 \\ \mathrm{r}_5 u_2 + \mathrm{p}_5 u_3 \\ \mathrm{r}_6 u_3 + \mathrm{p}_6 u_4 \\ \mathrm{r}_7 u_3 + \mathrm{p}_7 u_6 \\ \mathrm{r}_8 u_3 + \mathrm{p}_8 u_5 \end{bmatrix} \tag{53}$$

The subscript of r and p is the order number of the section, that of u the order number of the vertex. The first subscript of the currents forming I' indicates the branch, while the second one the vertex (Fig. 8/a).

The current of the hth section belonging to the second group is

$$i_{hj} = \mathrm{p}_h u_i + \mathrm{r}_h u_j \tag{54}$$

Similar equations can be written for each section. This system of equations is the following.

$$I'' = \frac{1}{2}\,\mathbf{P}(\mathbf{A}^* + \mathbf{A}_i^*)\,U + \frac{1}{2}\,\mathbf{R}(\mathbf{A}^* - \mathbf{A}_i^*)\,U \tag{55}$$

In our example

$$I'' = \begin{bmatrix} \mathbf{p}_1 u_4 + \mathbf{r}_1 u_2 \\ \mathbf{p}_2 u_2 + \mathbf{r}_2 u_1 \\ \mathbf{p}_3 u_4 + \mathbf{r}_3 u_1 \\ \mathbf{p}_4 u_1 + \mathbf{r}_4 u_3 \\ \mathbf{p}_5 u_2 + \mathbf{r}_5 u_3 \\ \mathbf{p}_6 u_3 + \mathbf{r}_6 u_4 \\ \mathbf{p}_7 u_3 + \mathbf{r}_7 u_6 \\ \mathbf{p}_8 u_3 + \mathbf{r}_8 u_5 \end{bmatrix} \tag{56}$$

These currents are indicated in Fig. 8/b.

The currents of the generators and impedances arranged at the vertices were expressed in the previous chapter [see (47)].

$$I_c = \mathbf{Y}_g (U_g - U_c) \tag{57}$$

The currents should satisfy the node equations. From the individual nodes of the vertices the currents written in I' flow away. Let us form from these the sum of currents belonging to the individual vertices and let I_c' denote the column matrix formed from these.

$$I_c' = \frac{1}{2}\,(\mathbf{A} + \mathbf{A}_i)\,I' \tag{58}$$

By using our previous results, we obtain in our example, that

$$I_c' = \begin{bmatrix} \mathbf{r}_4 u_1 + \mathbf{p}_4 u_3 \\ \mathbf{r}_2 u_2 + \mathbf{p}_2 u_1 + \mathbf{r}_5 u_2 + \mathbf{p}_5 u_3 \\ \mathbf{r}_6 u_3 + \mathbf{p}_6 u_4 + r_7 u_3 + \mathbf{p}_7 u_6 + \mathbf{r}_8 u_3 + \mathbf{p}_8 u_5 \\ \mathbf{r}_1 u_4 + \mathbf{p}_1 u_2 + \mathbf{r}_3 \mathbf{u}_4 + \mathbf{p}_3 u_1 \\ o \\ o \end{bmatrix} \tag{59}$$

In Fig. 8/a those currents are indicated, among which the sum of those belonging to one vertex supplies one element of column vector I_c'. Their direction is identical to the reference direction of the respective sections.

The branch currents forming I_c'' flow away from the individual nodes of the vertices, from which we obtain, upon arranging by vertices, the following column matrix I_c''.

$$I_c'' = \frac{1}{2}(\mathbf{A} - \mathbf{A}_i)\, I'' \tag{60}$$

Let us write also this for the discussed example.

$$I_c'' = \begin{bmatrix} \mathbf{p}_2 u_2 + \mathbf{r}_2 u_1 + \mathbf{p}_3 u_4 + \mathbf{r}_3 u_1 \\ \mathbf{p}_1 u_4 + \mathbf{r}_1 u_2 \\ \mathbf{p}_4 u_1 + \mathbf{r}_4 u_3 + \mathbf{p}_5 u_2 + \mathbf{r}_5 u_3 \\ \mathbf{p}_6 u_3 + \mathbf{r}_6 u_4 \\ \mathbf{p}_8 u_3 + \mathbf{r}_8 u_5 \\ \mathbf{p}_7 u_3 + \mathbf{r}_7 u_6 \end{bmatrix} \tag{61}$$

Currents in I_c flow towards that node of the vertex, from which the current corresponding to the elements of I_c' and I_c'' flows away. Thus the matrix form of the node rule, upon using (57), (58), and (60), further (52) and (55), is found to be

$$I_c' + I_c'' - I_c = \frac{1}{4}(\mathbf{A} + \mathbf{A}_i)\,[\mathbf{R}(\mathbf{A}^* + \mathbf{A}_i) + \mathbf{P}(\mathbf{A}^* - \mathbf{A}_i^*)\,]U_c +$$

$$+ \frac{1}{4}(\mathbf{A} - \mathbf{A}_i)\,[\mathbf{P}(\mathbf{A}^* + \mathbf{A}_i^*) + \mathbf{R}(\mathbf{A}^* - \mathbf{A}_i^*)]\, U_c + \mathbf{Y}_g U_c - \mathbf{Y}_g U_g = 0 \tag{62}$$

Upon arranging:

$$\left[\frac{1}{2}\mathbf{A}(\mathbf{R} + \mathbf{P})\,\mathbf{A}^* + \frac{1}{2}\mathbf{A}_i\,(\mathbf{R} - \mathbf{P})\,\mathbf{A}_i^* + \mathbf{Y}_g\right] U_c = \mathbf{Y}_g\, U_g \tag{63}$$

The multiplication factor of U_c in equation (63) will be termed the vertex admittance matrix.

$$\mathbf{Y}_c = \frac{1}{2}\mathbf{A}(\mathbf{R} + \mathbf{P})\,\mathbf{A}^* + \frac{1}{2}\mathbf{A}_i(\mathbf{R} - \mathbf{P})\mathbf{A}_i^* + \mathbf{Y}_g \tag{64}$$

Let us write its first two members for our example.

$$\frac{1}{2}\mathbf{A}(\mathbf{R} + \mathbf{P})\,\mathbf{A}^* + \frac{1}{2}\mathbf{A}_i(\mathbf{R} - \mathbf{P})\,\mathbf{A}_i^* = \tag{65}$$

$$= \begin{bmatrix} \mathbf{r}_2 + \mathbf{r}_3 + \mathbf{r}_4 & \mathbf{p}_2 & \mathbf{p}_4 & \mathbf{p}_3 & \mathbf{o} & \mathbf{o} \\ \mathbf{p}_2 & \mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_5 & \mathbf{p}_5 & \mathbf{p}_1 & \mathbf{o} & \mathbf{o} \\ \mathbf{p}_4 & \mathbf{p}_5 & \mathbf{r}_4 + \mathbf{r}_5 + \mathbf{r}_6 + \mathbf{r}_7 + \mathbf{r}_8 & \mathbf{p}_6 & \mathbf{p}_8 & \mathbf{p}_7 \\ \mathbf{p}_3 & \mathbf{p}_1 & \mathbf{p}_6 & \mathbf{r}_1 + \mathbf{r}_3 + \mathbf{r}_6 & \mathbf{o} & \mathbf{o} \\ \mathbf{o} & \mathbf{o} & \mathbf{p}_8 & \mathbf{o} & \mathbf{r}_8 & \mathbf{o} \\ \mathbf{o} & \mathbf{o} & \mathbf{p}_7 & \mathbf{o} & \mathbf{o} & \mathbf{r}_7 \end{bmatrix}$$

This hypermatrix is seen to be symmetrical. The main diagonal includes matrices $\mathbf{r}$ belonging to sections coincident with the vertex corresponding to the row (column). The other matrix elements are the matrices $\mathbf{p}$ pertaining to the section connecting the vertices corresponding to rows and columns. If the two vertices are not connected by a section, then the corresponding element of the hypermatrix is $\mathbf{o}$.

If one or more of the generators connected to the vertices are ideal ones, then the respective elements in $\mathbf{Y}_g$ are infinitely high. In this case it is advisable to rewrite equation (63) so that it includes $\mathbf{Z}_g = \mathbf{Y}_g^{-1}$

$$\left\{\frac{1}{2}\mathbf{Z}_g\left[\mathbf{A}(\mathbf{R}+\mathbf{P})\mathbf{A}^* + \mathbf{A}_i(\mathbf{R}-\mathbf{P})\mathbf{A}_i^*\right] + \mathbf{E}\right\} U_c = U_g \tag{66}$$

From these the required matrix U_c is

$$U_c = \left[\frac{1}{2}\mathbf{A}(\mathbf{R}+\mathbf{P})\mathbf{A}^* + \frac{1}{2}\mathbf{A}_i(\mathbf{R}-\mathbf{P})\mathbf{A}_i^* + \mathbf{Y}_g\right]^{-1}\mathbf{Y}_g U_g \tag{67}$$

and

$$U_c = \left\{\frac{1}{2}\mathbf{Z}_g\left[\mathbf{A}(\mathbf{R}+\mathbf{P})\mathbf{A}^* + \mathbf{A}_i(\mathbf{R}-\mathbf{P})\mathbf{A}_i^* + \mathbf{E}\right\}^{-1} U_g \tag{68}$$

In the knowledge of U_c currents I' and I'' can be calculated on the basis of (52) and (53) and thus the problem may be regarded as solved.

Summary

Electrical energy is distributed by transmission line systems. The individual sections of these transmission lines consist of coupled conductors. Processes taking place in them can be described by matrix functions. Relationships for the connected transmission line sections can be summarized by employing the graph theory in matrix equations. Thus the complete system can be characterized by a single equation containing hypermatrices.

References

1. SESHU, S.—REED, M. B.: Linear graphs and electrical networks. Addison—Wesley, London 1961.
2. SIMONYI, K.: Foundations of Electrical Engineering. Pergamon Press, London 1963.
3. GUILLEMIN, E. A.: Theory of linear physical systems. Wiley, New York 1963.
4. ВАГО, И.:; Расчет характеристики излучения горизонтальных проволочных антенн бегущей волны с потерями расположенных над землей. Periodica Polytechnica El. **8**, 47—63 (1964).
5. VÁGÓ, I.: The theory of transmission lines consisting of cylindrical leads, on the basis of the electromagnetic field. Periodica Polytechnica El. **8**, 251—264 (1964).
6. HEDMAN, D. E.: Propagation on overhead transmission lines. IEEE Trans. on Power, Apparatus and Systems **84**, 200—211 (March 1965).
7. FODOR, GY.—SIMONYI, K.—VÁGÓ, I.: Elméleti villamosságtan példatár. Tankönyvkiadó, Budapest 1967.
8. GESZTI, P. O.: Villamosművek. Tankönyvkiadó, Budapest 1967.
9. VÁGÓ, I.: The calculation of transmission networks by the graph theory. Periodica Polytechnica El. **13**, 155—168 (1969).

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