SOME THEOREMS FOR A NEW SYNTHESIS METHOD IN THRESHOLD LOGIC

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1. Introduction

During the recent developments in threshold logic several testing and synthesis methods have been proposed. From the viewpoint of practical application, however, many basic problems remain to be solved [1, 2]. Among them are the testing method of 1-realizability and the practical compound synthesis of nonrealizable functions. One of the problems seems to be the difficulty of mediating Boolean algebra and the theory of linear inequalities. To avoid this difficulty several authors use the complete monotonicity. Complete monotonicity was found to be a necessary condition for 1-realizability by PAULL and McCLUSKEY [3], MUROGA [4] and WINDER [5]. YAJIMA and IBARAKI introduced the concept of mutual monotonicity [6]. ELGOT [7] has proved that a function which is k-summable for any k is a threshold function, and vice versa.

This paper gives a new necessary condition for 1-realizability and some other theorems for a new testing and synthesis method.

2. Terminology

An arbitrary Boolean function $F(x_1 \ldots x_n)$ can be considered as a mapping from the set of 2^n input vectors x given by n bivalued input variables to the set of three elements. These are

- 1 logical ONE
- 0 logical ZERO
- — DON'T CARE (no restriction on the output value)

Input vectors, for which $F(x = (x_1 \ldots x_n))$ must have the value of 1, can be denoted by x^1.

Input vectors, for which $F(x = (x_1 \ldots x_n))$ must have the value of 0, can be denoted by x^0.

$$F(x^1) = 1; \qquad F(x^0) = 0$$

As is well known, an arbitrary Boolean function $F(x)$ of n binary variables is realizable by a single threshold element or 1-realizable if and only if it has a weight vector $w = (w_1 \ldots w_n)$, each component of which is a real number, and a real number T, called a threshold, such that

$$wx \geq T \text{ for every } x \text{ vector} \tag{1}$$

which equals any of the x^1 vectors

$$wx < T \text{ for every } x \text{ vector which} \tag{2}$$

equals any of the x^0 vectors where

wx is the scalar product of two vectors.

An angle φ can be defined between two n-dimensional vectors as follows

$$\varphi = \arccos \frac{wx}{|w| \cdot |x|}$$

where $|w|$ and $|x|$ are the absolute values of the vectors.

From the view-point of practical application the threshold T must be considered as a domain:

$$u \geq T > l$$

and the inequalities (1) and (2) can be rewritten as follows

$$wx \geq u \quad \text{for every } x \text{ vector} \tag{3}$$

which equals any of the x^1 vectors

$$wx \leq l \quad \text{for every } x \text{ vector} \tag{4}$$

which equals any of the x^0 vectors

$$\text{and} \qquad u - l > 0 \tag{5}$$

3. Assumptions

1) The Boolean function $F(x)$ is given by truth table.

2) It is not necessary for $F(x)$ to be completely specified.

3) The Boolean function $F(x)$ will be considered to be 1-realizable if and only if inequalities (3), (4) and (5) hold.

4. Theorems for vectors derived from the truth table of Boolean function $F(x)$

Let y^k denote the difference vector of one of the vectors x^1 and one of the vectors x^0. With this notation inequalities (3), (4) and (5) can be rewritten as follows:

$$wy^k > 0 \tag{6}$$

for every k^k derivable from the truth table of $F(x)$.

Thus, 1-realizability means that there exists at least one vector w the scalar products of which with every y^k are greater than zero.

Let m denote the number of vectors y^k derivable from the truth table of $F(x)$.

For the sum-vector s of all vectors y^k the next theorem holds.

Theorem 1: If $s = \sum_{k=1}^{m} y^k \neq 0$, then there exist no vectors linearly independent from s, with the absolute value $|s|$, the scalar products of which with each y^k are greater than the scalar product of s with the corresponding y^k.

Proof: If there exists a vector a, such that

$$a \neq s; \qquad |a| = |s|$$

and

$$sy^k < ay^k \text{ for each } k \, (1 \leq k \leq m),$$

then

$$s \sum_{k=1}^{m} y^k < a \sum_{k=1}^{m} y^k .$$

But

$$s = \sum_{k=1}^{m} y^k \text{ and } |a| = \left| \sum_{k=1}^{m} y^k \right|$$

thus

$$\sum_{k=1}^{m} y^k \sum_{k=1}^{m} y^k < a \sum_{k=1}^{m} y^k$$

$$\left| \sum_{k=1}^{m} y^k \right|^2 < \left| \sum_{k=1}^{m} y^k \right|^2 \cos \varphi_a$$

$$1 < \cos \varphi_a$$

where φ_a denotes the angle between the vectors a and $\sum_{k=1}^{m} y_k$.

Since the last inequality is a contradiction, Theorem 1 is proved.

With other words Theorem 1 means that for every vector a, such that $a \neq \sum_{k=1}^{m} y^k$ and $|a| = \sum_{k=1}^{m} y^k \neq 0$, a value of k can be found for which

$$y^k \sum_{k=1}^{m} y^k \geq ay^k .$$

The following theorem gives a sufficient condition for the sum-vector s to be different from zero.

Theorem 2: If there exists at least one vector y^i, the scalar products of which with every remaining vector y^k are not negative, then the sum-vector s of all y^k is different from zero.

Proof: According to the theorem

$$y^i \sum_{\substack{k=1 \\ k \neq i}}^{m} y^k \geq 0 \quad \text{holds}.$$

It means that

$$\sum_{\substack{k=1 \\ k \neq i}}^{m} y^k \neq -y^i$$

Thus

$$\sum_{k=1}^{m} y^k \neq 0$$

and the theorem is proved.

The next theorem gives necessary condition for a Boolean function to be 1-realizable.

Theorem 3: If for a Boolean function the sum-vector of all y^k is zero, then the function is not 1-realizable.

Proof: Suppose that a weight a vector w can be found, such that

$$wy^k > 0 \text{ holds for each } 1 \leq k \leq m.$$

Then the inequality

$$w \sum_{k=1}^{m} y^k > 0 \quad \text{would be satisfied.}$$

$$\text{But according to the theorem } \sum_{k=1}^{m} y^k = 0\,.$$

Thus the last inequality is a contradiction and the theorem is proved.

It can be shown that Theorem 3 is only a necessary condition for the 1-realizability, as follows from Theorem 6.

Theorem 4: If a Boolean function has a non-zero sum-vector s, then it is impossible for each $1 \leq i \leq m$, that the inequality

$$y^i \sum_{k=1}^{m} y^k \leq 0 \quad \text{holds}.$$

Proof: In the opposite case the inequality

$$\sum_{i=1}^{m} y^i \sum_{k=1}^{m} y^k \leq 0 \quad \text{would be satisfied.}$$

But it means that

$$s.s \leq 0$$

which is a contradiction and the proof is completed.

It is possible for a non-zero sum-vector $s = \sum_{k=1}^{m} y^k$ to be a weight vector of $F(x)$. The following theorem gives a sufficient condition for s to be a weight vector.

Theorem 5: If the scalar products of every pair of y^k vectors are non-negative, then the sum-vector $s = \sum_{k=1}^{m} y^k$ is a weight vector of the Boolean function $F(x)$ from the truth table of which the vectors y^k derive.

Proof: If the inequality

$$y^i y^j \geq 0 \text{ holds for each } 1 \leq i \leq m \text{ and } 1 \leq j \leq m$$

then there is no vector y^i for which

$$y^i \sum_{k=1}^{m} y^k \leq 0 \text{ holds.}$$

But it means that the sum-vector $s = \sum_{k=1}^{m} y^k$ with $w = s$ satisfies inequality (6) and so the proof is completed. The next theorem shows that Theorem 5 is only a sufficient condition.

Theorem 6: If among the vectors y^k there are vectors y^i for which the inequalities

$$y^i \sum_{k=1}^{m} y^k \leq 0 \quad \text{hold}$$

and the scalar products of every pair of these vectors y^i are not positive, and the number of these vectors y^i is

$$z > n - 1$$

where n is the number of input variables, then the Boolean function is not 1-realizable.

Proof: According to the theorem there are $z + 1 > n$ vectors in the n-dimensional space and there are angles not smaller than $\frac{\pi}{2}$ between every pair of these $z + 1$ vectors. It is not possible to find a weight-vector w in the n-dimensional space, because according to inequality (6) between the weight-vector and each of $z + 1$ vectors there must be angles smaller than $\frac{\pi}{2}$. Thus the theorem is proved. The following theorem also gives a necessary condition for a Boolean function to be 1-realizable.

Theorem 7: If among the vectors y^k at least one pair of vectors y^i and y^j can be found, such that

$$y^i y^j = -|y^i| \cdot |y^j|$$

then the Boolean function is not 1-realizable.

Proof: According to the theorem there is an angle π between the vectors y^i and y^j. But in that case no vector can be found the angles of which to y^i and to y^j are smaller than $\frac{\pi}{2}$. So inequality (6) cannot be satisfied and the proof is completed.

5. Conclusions

Theorem 3 is equivalent to k-summability [7], where the value of k depends on the truth table of the Boolean function.

The above theorems do not use at all the fact that the components of vectors y^k can have only three different values deriving from their definition. These values are: 0,1 and -1. Considering this restriction on the values of components some simplifications can be made in the synthesis procedure.

According to Theorem 5 there are cases where the sum-vector s may be considered as a weight vector. This is useful from the view-point of synthesis procedure, because adjusting the sum-vector a weight vector can be found in other cases, too.

Summary

In this paper some theorems are given for a testing and synthesis method which is under development and will be published later. The difference vectors y^k can be formed from the truth table of a Boolean function and the sum and the scalar products of these vectors will be used in the synthesis procedure.

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