SENSITIVITY INVESTIGATIONS BY ROOT-LOCI ON THE ANALOGUE MACHINE POLCOMP

By

B. Szűcs

Department for Automation, Polytechnical University, Budapest

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Presented by Prof. F. Csáki

Introduction

The paper suggests for sensitivity investigations of linear control systems the generalized root-locus method. In this case the effect of parameter-varieties of the open loop system can be immediately described with the variety of the dynamic characteristics of the closed loop system. The paper is generalizing the usual root-locus method. The method is presented for the cases when one of the time-constants or one of the damping ratios or one of the second order time-constants of the system changes within a wide range. The effect of the gain changes can be indicated by the usual root-loci. The paper presents examples for a quick adaptation of the suggested method on the special purpose analogue computer named POLCOMP built in Hungary.

1. The generalized root-locus method

Root-locus method to analyse gain sensitivity of a control system is a usual procedure. In this case the feedback is negative and the overall gain (or the gain of a block) changes for the range of positive values $0 \leq K_h \leq +\infty$.

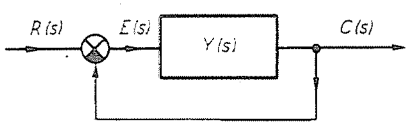

Fig. 1

In the cases when the parameter sensitivity of a time-constant or a damping ratio are investigated, it is necessary to extend the root-locus method for the positive feedback or the change of negative values of the parameter under investigation. This extension gives the generalized root-locus method. It can be applied for the determination of the inverse root-loci, too [1]. The generalization of the constructional rules of the usual root-locus method is helpful to estimate qualitatively the places of root-loci.

For the open loop system of Fig. 1, the transfer function is

$$Y(s) = K_h \frac{\prod_{k=1}^{Z}(s - z_k)}{\prod_{i=1}^{P}(s - p_i)} = K_h \frac{\prod_{k=1}^{Z} C_k \measuredangle \gamma_k}{\prod_{i=1}^{P} D_i \measuredangle \delta_i} \tag{1}$$

where $z_1, \ldots z_k, \ldots z_Z$ are the zeros and $p_1, \ldots p_i, \ldots p_P$ are the poles of the transfer function $Y(s)$ and K_h is the overall gain [2]. The constructional rules of root-loci for negative and positive feedback are contained in Table 1. The nomenclature of Equ. (1) and Table 1 is illustrated on Fig. 2.

Table 1

Type of feedback	Negative	Positive
Equation determining root-loci	$Y(s) = -1$	$Y(s) = 1$
Angle condition	$\sum_{i=1}^{P}\delta_i - \sum_{k=1}^{Z}\gamma_k = \pm N.180^\circ$ where $N = 1, 3, 5, \ldots$	$\sum_{i=1}^{P}\delta_i - \sum_{k=1}^{Z}\gamma_k = \pm M.360^\circ$ where $M = 0, 1, 2, \ldots$
Magnitude condition	$K_h = \frac{\prod_{i=1}^{P} D_i}{\prod_{k=1}^{Z} C_k}$	
The root-locus places on the real-axis if the joint number of poles and zeros on the right of the point under investigation is	odd number	even number
Angle δ_{om} of setting out from pole p_m	$\delta_{om} = \pm 180^\circ + \sum_{k=1}^{Z}\gamma_k - \sum_{i=1, i\neq m}^{P}\delta_i$	$\delta_{om} = \sum_{k=1}^{Z}\gamma_k - \sum_{i=1, i\neq m}^{P}\delta_i$
Angle γ_{om} of setting in into zero z_m	$\delta_{om} = \pm 180^\circ - \sum_{k=1, k\neq m}^{Z}\gamma_k - \sum_{i=1}^{P}\delta_i$	$\gamma_{om} = \sum_{k=1, k\neq m}^{Z}\gamma_k - \sum_{i=1}^{P}\delta_i$
Angle α of assimptotes of root-loci	$\alpha = \frac{\pm N.180^\circ}{P-Z}$ where $N = 1, 3, 5, \ldots$	$\alpha = \frac{\pm M.360^\circ}{P-Z}$ where $M = 0, 1, 2, \ldots$
Intersection of assimptotes on the real axis x_0	$x_0 = \frac{1}{P-Z}\left\{\sum_{i=1}^{P}p_i - \sum_{k=1}^{Z}z_k\right.$	
Intersection of root-loci on the real axis x	$\sum_{i=1}^{P}\frac{1}{x-p_i} - \sum_{k=1}^{Z}\frac{1}{x-z_k} = 0$	

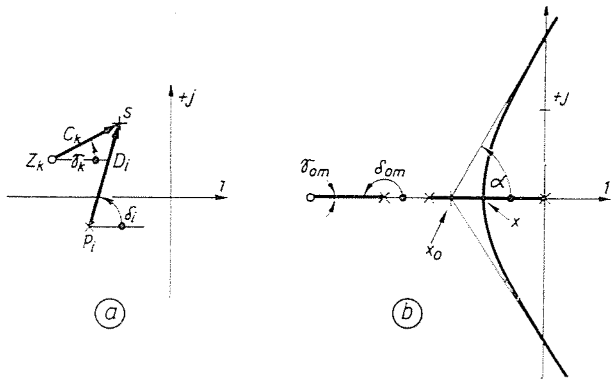

Fig. 2

2. Parameter sensitivity by root-locus method [3]

For the rigid feedback system of Fig. 1, the characteristic equation (i.e. Equ. for root-loci) can be written

$$Y(s) = \frac{G(s)}{H(s)} = -1 \tag{2}$$

The variable parameter under investigation can exist in a nonrepeated factor of $G(s)$ or $H(s)$. The results obtained for rigid feedback system can be generalized.

2.1. *Time-constant sensitivity* [4]

The variable time-constant may be written as

$$T_1 = T_{10} + \Delta T_1 \tag{3}$$

where T_{10} is the fixed component and ΔT_1 is the variable component of T_1. Assuming that T_1 occurs in the numerator:

$$G(s) = G_1(s) \cdot (1 + T_1 s) \tag{4}$$

where $G_1(s)$ is by definition the original term excluding factor $(1 + T_1 s)$. Substituting Equs. (3) and (4) into Equ. (2) and transscribing the latter

$$\frac{G_1(s)\, s \Delta T_1}{G_0(s) + H(s)} = -1 \tag{5}$$

where $G_0(s) = G_1(s) \cdot (1 + T_{10} s)$.

If denominator $H(s)$ contains the variable time-constant, then the equation of root-loci is

$$\frac{H_1(s)\, s\Delta T_1}{G(s) + H_0(s)} = -1 \tag{6}$$

Equs. (5) and (6) permit plotting of the root-loci as a function of ΔT_1 for the range of values $-\infty < \Delta T_1 < \infty$.

2.2. Second order time-constant sensitivity

The variable second order time-constant may be written as

$$T_2 = T_{20} + \Delta T_2 \tag{7}$$

where T_{20} is the fixed component and ΔT_2 is the variable component of T_2. Assuming that T_2 occurs in the numerator $G(s)$:

$$G(s) = G_2(s) \cdot (1 + 2\zeta T_2 s + T_2^2 s^2) \tag{8}$$

where $G_2(s)$ is by definition the original term excluding the factor containing T_2. Substituting Equ. (7) into Equ. (8)

$$G(s) = G_0(s) + G_2(s) \cdot \left(1 + \frac{T_{20}}{\zeta} s\right) 2\zeta s \Delta T_2 + G_2(s)\, \Delta T_2^2 s^2 \tag{9}$$

where

$$G_0(s) = G_2(s) \cdot (1 + 2\zeta T_{20} s + T_{20}^2 s^2) \tag{10}$$

Leaving out of consideration the final term of Equ. (9) which is possible if $|\Delta T_2| \leqq 0.1\ T_{20}$ is valid, the characteristic equation is the following:

$$\frac{G_2(s) \cdot \left(1 + \frac{T_{20}}{\zeta} s\right) 2\zeta\, s\Delta T_2}{G_0(s) + H(s)} = -1 \tag{11}$$

If denominator $H(s)$ contains the variable second order time-constant then the equation of root-loci is

$$\frac{H_2(s) \cdot \left(1 + \frac{T_{20}}{\zeta} s\right) 2\zeta\, s\Delta T_2}{G(s) + H_0(s)} = -1 \tag{12}$$

Equs. (11) and (12) permit plotting of the root-loci as a function of ΔT_2 for the range of values

$$-0{,}1\, T_{20} < \Delta T_2 < 0{,}1\, T_{20}\,. \tag{13}$$

2.3. *Damping ratio sensitivity*

The variable damping ratio may be written as

$$\zeta = \zeta_0 + \Delta\zeta \tag{14}$$

where ζ_0 is the fixed component and $\Delta\zeta$ is the variable component of ζ. Assuming that ζ occurs in the numerator $G(s)$ Equ. (8) can be applied. Substituting Equ. (14) into Equ. (8)

$$G(s) = G_0(s) + 2T_2\,\Delta\zeta\, sG_2(s) \tag{15}$$

where $G_0(s)$ is given according to Equ. (10). Substituting Equ. (15) into Equ. (2) the characteristic equation can be found as follows:

$$\frac{G_2(s)\, 2T_2\, s\Delta\zeta}{G_0(s) + H(s)} = -1 \tag{16}$$

If denominator $H(s)$ contains the variable damping ratio, then the equation of root-loci is

$$\frac{H_2(s)\, 2T_2\, s\Delta\zeta}{G(s) + H_0(s)} = -1 \tag{17}$$

Equs. (16) and (17) permit plotting of root-loci as a function of $\Delta\zeta$ for the range of values $-\infty < \Delta\zeta < \infty$ mathematically, but for practical cases the values $-\zeta_0 < \Delta\zeta < 5$ are satisfactory.

3. Application of the method by POLCOMP. Examples

A useful characteristic of POLCOMP is that by its application for tracing root-locus plots — which would otherwise require considerable calculating and designing work — can be carried out within a short time. The model of the transfer function of the examined control system can be built up according to Equ. (1) on the machine [1]. The picture of the POLCOMP is shown on

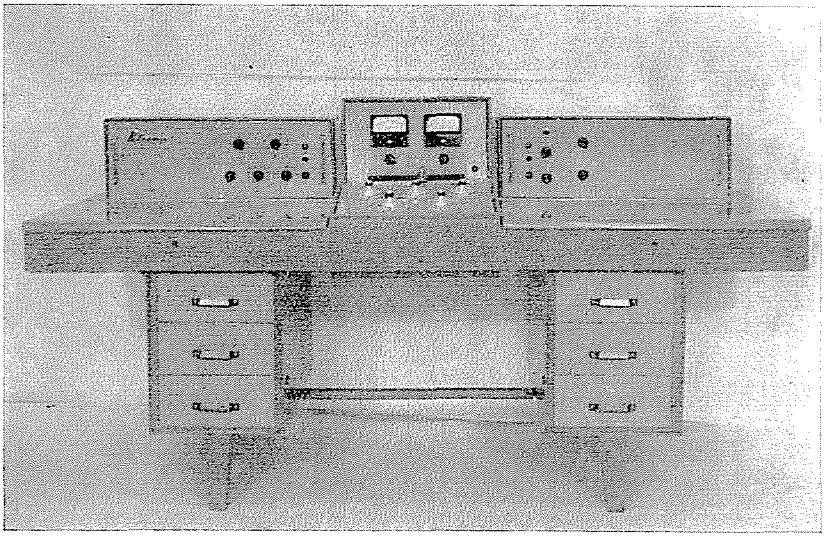

Fig. 3

Fig. 3. To illustrate the suggested method, let us consider a simple feedback system for which the open loop transfer function is

$$Y(s) = \frac{K(1 + T_1 s)}{1 + 2\zeta T_2 s + T_2^2 s^2} \tag{18}$$

where T_1, T_2, ζ may be expressed according to Equs. (3), (7), (14) and the fixed components are $K_0 = 90$, $T_{10} = 1$ sec, $T_{20} = 10$ sec, $\zeta_0 = 0.5$.

Gain (K) sensitivity of the system was determined according to equation

$$\frac{K(1 + T_{10} s)}{1 + 2\zeta_0 T_{20} s + T_{20}^2 s^2} = -1 \tag{19}$$

Equ. (19) was programmed and solved on the machine and the obtained root-locus plot is shown on Fig. 4 where poles of the closed loop system ($p_{1,2} = 0.95\underline{/180° \pm 58°}$) are marked at value K_0.

Time constant (T_1) sensitivity of the system was determined according to Equ. (5) as follows:

$$\frac{K_0 s \Delta T_1}{(1 + K_0)\left(1 - \frac{s}{p_1}\right) \cdot \left(1 - \frac{s}{p_2}\right)} = -1 \tag{20}$$

Equ. (20) was programmed and solved on POLCOMP and the resulting root-loci are shown on Fig. 5.

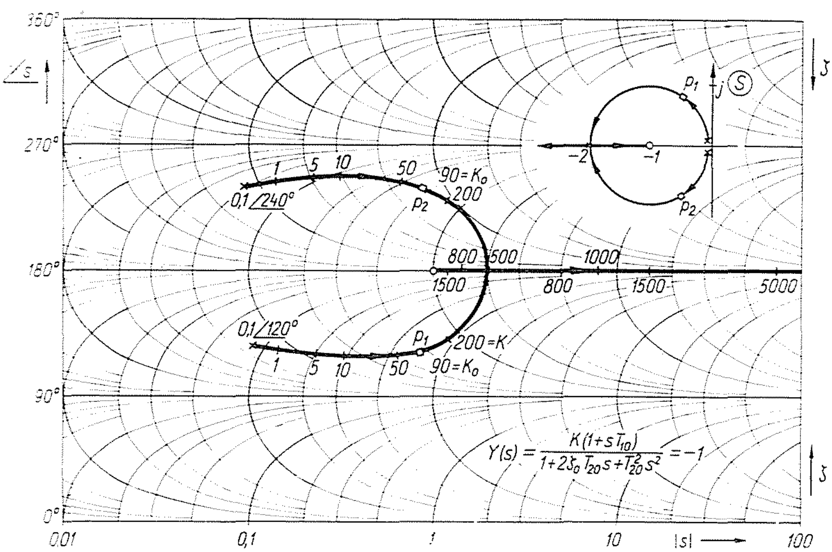

Fig. 4

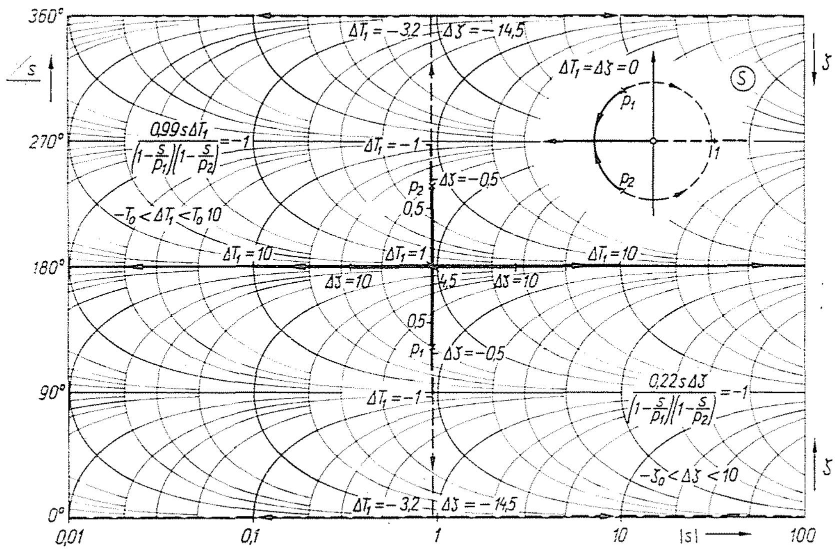

Fig. 5

Time-constant (T_2) sensitivity of the system was determined according to Equ. (12) as follows:

$$\frac{\left(1 + \frac{T_{20}}{\zeta_0} s\right) 2\zeta_0 \, s \Delta T_2}{(1 + K_0)\left(1 - \frac{s}{p_1}\right) \cdot \left(1 - \frac{s}{p_2}\right)} = -1 \tag{21}$$

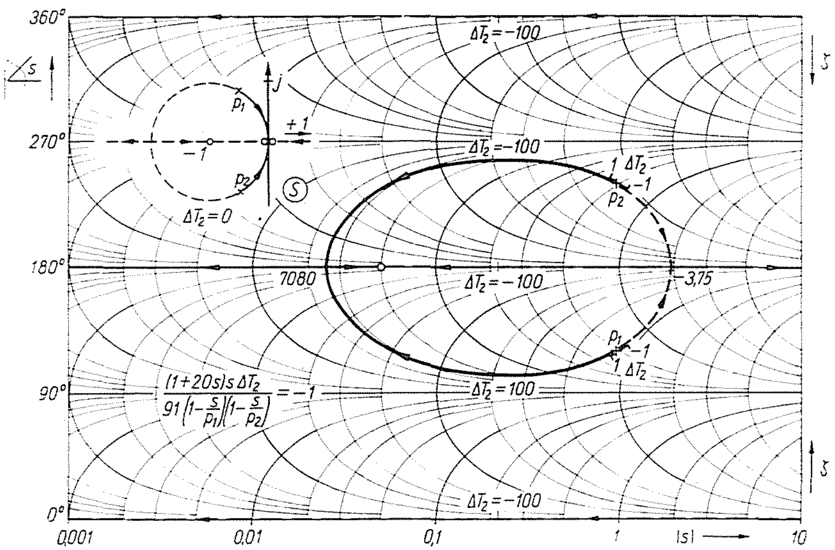

Fig. 6

Equ. (21) was solved on the machine and the obtained root-loci are shown on Fig. 6. Considering condition (13) the curves are marked belonging to values $-1 < \Delta T_2 < 1$ with thick lines.

Damping ratio (ζ) sensitivity of the system was determined according to Equ. (17) as follows:

$$\frac{2T_{20}\, s\Delta\zeta}{(1+K_0)\left(1-\dfrac{s}{p_1}\right)\cdot\left(1-\dfrac{s}{p_2}\right)} = -1 \qquad (22)$$

Equ. (22) was solved on POLCOMP and the resulting root-loci are shown on Fig. 5.

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Summary

The paper suggests the generalized root-locus method for sensitivity investigation of linear control systems. The varieties of dynamic characteristics of the closed loop system can be examined by the application of the described method when one of the time-constants or damping-ratios of the open loop system is changing. The paper presents examples for a quick application of the method on analogue machine POLCOMP.

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Béla Szücs; Budapest, XI. Egry József u. 18—20. Hungary