ANALYSIS OF PASSIVE ADAPTIVE SYSTEMS CONTAINING CONDITIONAL FEEDBACK

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(Received January 10, 1968)

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1. Introduction

In order to optimize discrete-working control systems the structure of the system and the control algorithm must be determined for a given process at known noises, disturbances, and prescribed control input. The system is considered to be ideal if the control input is followed without error, the controlled variable (or some kind of quality variable) is independent of the external disturbances as well as of the parameter variations of the controlled section.

The conventional control systems reduce the effect of the external disturbance by disturbance-compensation in case of *measurable disturbances*. Since disturbances external to the process are generally not measurable, the application of the disturbance-compensation is limited.

Although the closed loop reduces the effects of internal parameter variations, in case of great parameter variations there will be deviations not permissible in the controlled variable. The effects of parameter variations may be compensated by nonlinear *adaptive* systems designed by using *process identification*.

In case of a more complicated process the transfer coefficient, zero, and pole may be determined by an analog or digital computer [7]. Adaptive systems usually compensate the effect of the variation of a single parameter.

A control system based upon model feedback is discussed in this paper, with the structure proposed by the author, that makes possible to minimize the effects of the external disturbance, and of the internal parameter variations in addition to optimal following of the reference input. It is not necessary to measure the external disturbance beforehand, there is no need for computer-performed identification.

2. Conditions of analysis

The process is assumed to fulfil the following conditions:

1. The relation between the output and input variables of the process is known, the parameters may vary.

2. The effects of external disturbances acting upon the process may be taken into account with a single variable appearing at the output of the process.

3. The control unit for controlling the process affects the process by a single variable, the corrected variable. The minimalization of the deviation from the set point, the effects of the external disturbance, and of the parameter variation are performed by modifying the *components* of the corrected variable.

4. The noise, and disturbance appearing beside the control input are independent of each other, statistical, stationary, and ergodic signals with known autocorrelation function.

5. The optimum criterion of the process is the minimum value of the square-integral of the control error, or the disturbance-component of the output, or the components of the output resulting from parameter variations, or the minimum of the average value of the squared sum of the corresponding signal sets.

6. If the system parameters are unknown, the parameters are considered to be *random variables* with known distribution function. With that the process is described by real random variables, the process itself is a complex random variable. The optimum criterion be the minimum value of the average of the squared-sum of the signal sets, in short, the minimum of the *expected value* of the mean-square error. In case of parameters varying in time the transfer coefficient, zero, and pole of the process may be considered to be *random-type process varying in time*, with known autocorrelation functions.

3. Mathematical description of parameter variations using sensitivity function

The effect of small parameter variations may be taken into account with the *sensitivity function* proposed by BODE. By definition, if $W(z)$ is the pulse transfer function of the closed loop, p is the varying parameter, the sensitivity for parameter variations is:

$$S_p^W = \frac{\partial W}{\partial p} \cdot \frac{p}{W} = \frac{\partial \ln W}{\partial \ln p} \tag{1}$$

Our analysis is performed on the so-called *model feedback*, or conditional feedback control systems. A model, corresponding to the nominal value of the controlled section is inserted into the control loop. Beside the direct feedback, the difference of the controlled variable and the output of the model is fed back to the system input through a pulse controller (Fig. 1). A signal, proportional to $U(z)$ in case of external disturbance, and to $[G_S(s) - G_{SO}(s)]$ in case of parameter variation in the controlled section is fed back. The one-loop equivalent of the system is shown in Fig. 2.

Giving the set of input values of final control element, corresponding to the set of the difference variable values is performed by giving the *control*

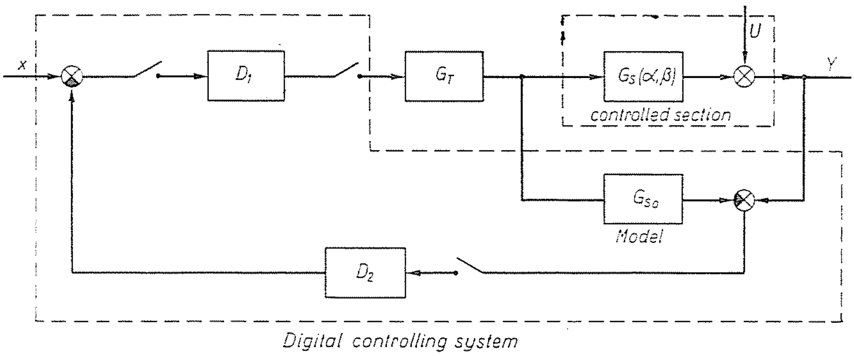

Fig. 1. Block diagram of passive adaptive system with model feedback

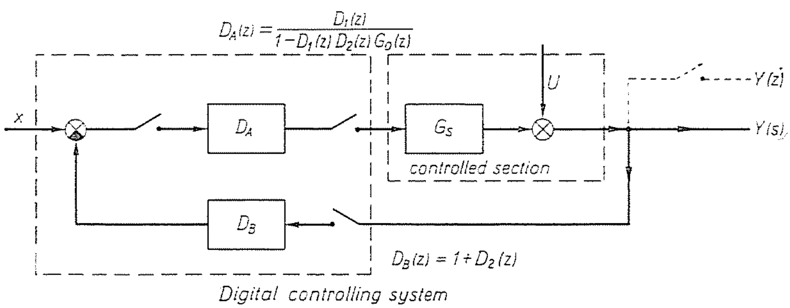

Fig. 2. Producing $M(z) = (X - D_B Y)D_A$, the input of the final control element from the reference input and the controlled variable

algorithm. The control algorithm can be given by the pulse transfer functions of the pulse controllers required for optimal operation. In case of computer control the direct or the iterative program may be written using the pulse transfer function.

The pulse transfer function of the closed loop is

$$W(z) = C(z)\, D_1(z)\, G(z) \tag{2}$$

where
$$C(z) = \frac{1}{1 + D_1(z)\, G(z) + D_1(z)\, D_2(z)\, [G(z) - G_0(z)]}$$

$$G(z) = Z[G_T(s)\, G_S(s)] \quad \text{and} \quad G_0(z) = Z[G_T(s)\, G_{SO}(s)]$$

and symbol Z indicates the $z = e^{sT}$ transformation.

The pulse transfer function corresponding to the disturbance is:

$$T(z) = \frac{Y_u(z)}{U(z)} = C(z)\left[1 - D_1(z)\, D_2(z)\, G_0(z)\right] \tag{3}$$

With substitution we get:

$$T(z) = 1 - [1 + D_2(z)\ W(z)] \tag{4}$$

Factoring $G(z)$ we get:

$$G(z) = Kz^{-m} \frac{\prod_i (1 + a_i z^{-1})}{\prod_j (1 + b_j z^{-1})} \tag{5}$$

The variable parameter p may stand for amplification coefficient K, zero (a_i) or pole (b_j). Since

$$S^W = \frac{\partial W}{\partial G} \cdot \frac{G}{W} = C(z)\left[1 - D_1(z)\ D_2(z)\ G_0(z)\right] \tag{6}$$

the sensitivity functions of the closed system's transfer function describing the effects of parameter variations may be written up in Table 1.

Table 1

Sensivity functions of the closed loop's transfer function describing the effects of parameter variations

Variable parameter	$\frac{\partial G}{G}$	$S_p^W = \frac{\partial W/W}{\partial p/p}$
$p = K$	$\frac{\partial K}{K}$	$S_K^W = S^W = C(z)\left[1 - D_1(z)\, D_2(z)\, G_0(z)\right]$
$p = a_i$	$\frac{\partial a_i}{a_i} \quad \frac{a_i}{z+a_i}$	$S_{a_i}^W = S^W \frac{a_i}{z+a_i}$
$p = b_j$	$-\frac{\partial b_j}{b_j} \quad \frac{b_j}{z+b_j}$	$S_{b_i}^W = -S^W \frac{b_i}{z+b_i}$

To express the sensitivity for parameter variation let us define a sensitivity function for $T(z)$ beside the sensitivity function S^W used in the literature:

$$S_p^T = \frac{\partial T}{\partial p} \frac{p}{T} = \frac{\partial \ln T}{\partial \ln p} \tag{7}$$

It can be proved that the two sensitivity functions are not independent of each other. Since

$$S^T(z) = \frac{\partial T}{\partial G} \cdot \frac{G}{T} = -C(z)\left[1 + D_2(z)\right] G(z) \tag{8}$$

the relation between the sensitivity functions is:

$$S^T(z) = \frac{\partial S^W}{\partial G} \cdot \frac{G}{S^W} = S^W(z) - 1 \tag{9}$$

The sensitivity functions describing the effects of the parameter variations of the pulse transfer function for noise are given in Table 2.

Table 2

Sensitivity functions of the pulse transfer function corresponding to disturbance, describing the effects of parameter variations

Variable parameter	$S_p^T = \frac{\partial T/T}{\partial p/p}$	
$p = K$	$S_K^T = S^T = -C(z)[1+D_2(z)]G(z)$	$S_K^T = S_K^W - 1$
$p = a_i$	$S_{a_i}^T = S^T \frac{a_i}{z+a_i}$	$S_{a_i}^T = S_{a_i}^W - \frac{a_i}{z+a_i}$
$p = b_j$	$S_{b_j}^T = -S^T \frac{b_j}{z+b_j}$	$S_{b_j}^T = S_{b_j}^W + \frac{b_j}{z+b_j}$

If the condition $S^W(z) = 0$ given in the literature, called full invariancy, could be fulfilled, namely $D_1(z)\, D_2(z)\, G_0(z) = 1$, then

$$S^T(z) = -1, \text{ that is } \partial T/T = -\partial G/G,$$

the parameter variation $\partial G/G\,(z)$ would appear entirely at the output.

The *passive adaptive* character of the system with model feedback is shown by the equivalent systems of Fig. 3. The fluctuations of the loop transfer function resulting from parameter variation are compensated by the system itself. The feedback in the loop is *conditional*, if $G = G_0$, $L(z) = D_1 G_0$, and the feedback branch is open. The characteristics of the system are given in Table 3. The suppression of external disturbance does not alter the stability characteristics, makes unnecessary the measurement of disturbance.

The *fluctuation of the controlled variable* resulting from parameter variations can be determined using the sensitivity functions. In case of small parameter variations the component of the fluctuation of output resulting from the reference input is:

$$H_x(z) = X(z)\left[W(z) - W_0(z)\right] = X(z)\,W_0(z)\,S^W(z)\,\frac{\partial G}{G} \tag{10}$$

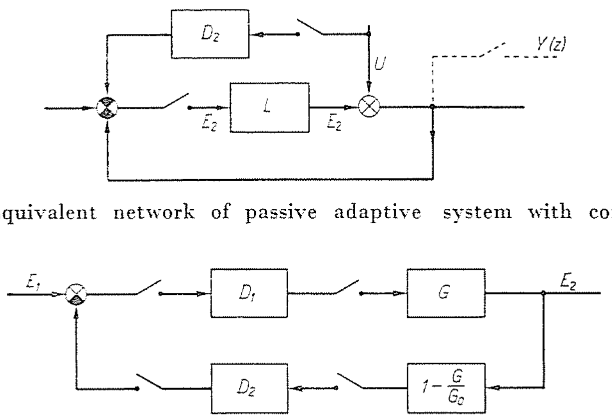

Fig. 3. (a) Equivalent network of passive adaptive system with conditional feedback

(b) Realization of the loop transfer function $L(z)$ in passive adaptive system

and its component resulting from the disturbance is:

$$H_u(z) = U(z)\left[T(z) - T_0(z)\right] = U(z)\,T_0(z)\,S^T(z)\,\frac{\partial G}{G} \tag{11}$$

where

$$W_0(z) = W(z) \quad \text{if} \quad G(z) = G_0(z) \ \text{and}$$
$$T_0(z) = T(z) \quad \text{if} \quad G(z) = G_0(z) \ \text{respectively.}$$

Recalling that $T(z) = S^W(z)$, we get

$$H_u(z) = U(z)\,S_0^W(z)\,S^T(z)\,\frac{\partial G}{G} \tag{12}$$

The fluctuations of the output against the variable parameter are given in Table 4. The total fluctuation of the output is:

$$H(z) = H_x(z) + H_u(z) \tag{13}$$

Optimizing the system, the average value of the squared sum of the signal set made of the disturbance component of the output $[y_u(kT)]$ or of the fluctuation of the output $[h(kT)]$ may be minimized.

Table 3

Characteristics of passive adaptive system

	$L(z)$	$W(z)$	$T(z)$	$\frac{\Delta L}{L_0}=\frac{L-L_0}{L_0}$	Note
$G=G_0$	D_1G_0	$\frac{D_1G_0}{1+D_1G_0}$	$\frac{1-D_1D_2G_0}{1+D_1G_0}$	0	The stability characteristics of the system are not influenced by the model feedback
absolute invariancy $D_1D_2G_0=1$ $\delta G\neq 0$	$D_1G_0=\frac{1}{D_2}$	$\frac{D_1G_0}{1+D_1G_0}$	0	0	D_2 is physically not realizable
$G=G_0+\delta G$ $D_2=0$	$D_1G_0\left(1+\frac{\delta G}{G_0}\right)$	$\frac{D_1(G_0+\delta G)}{1+D_1(G_0+\delta G)}$	$\frac{1}{1+D_1(G_0+\delta G)}$	$\frac{\delta G}{G}$	Conventional system with rigid feedback
$G+G_0+\delta G$ $D_2\neq 0$	$D_1G_0\frac{1+\frac{\delta G}{G_0}}{1+D_1D_2\delta G}$	$\frac{D_1(G_0+\delta G)}{1+D_1G_0+D_1(1+D_2)\delta G}$	$\frac{1-D_1D_2G_0}{1+D_1G_0+D_1(1+D_2)\delta G}$	$\frac{\delta G}{G}\,\frac{1-D_1D_2G_0}{1+D_1D_2\delta G}$	The variation of the loop transfer function decreases because of the passive adaptation.

Table 4

Fluctuation of the controlled variable as a function of the variable parameter

Variable parameter	$H_x(z)$	$H_u(z)$
$p = K$	$X(z)\, W_0(z)\, S^W(z) \dfrac{\partial K}{K}$	$U(z)\, T_0(z)\, S^T(z) \dfrac{\partial K}{K}$
$p = a_i$	$X(z)\, W_0(z)\, S^W(z) \dfrac{a_i}{z+a_i} \dfrac{\partial a_i}{a_i}$	$U(z)\, T_0(z)\, S^T(z) \dfrac{a_i}{z+a_i} \dfrac{\partial a_i}{a_i}$
$p = b_j$	$-X(z)\, W_0(z)\, S^W(z) \dfrac{b_j}{z+b_j} \dfrac{\partial b_j}{b_j}$	$-U(z)\, T_0(z)\, S^T(z) \dfrac{b_j}{z+b_j} \dfrac{\partial b_j}{b_j}$

4. Analysis of control circuits with respect to parameter variations

Any *parameter variation* appearing in the controlled section may be taken into consideration by the relative variation of the numerator and the denominator. If

$$G(z) = \frac{A(z)}{B(z)} = \frac{A_0(z)\,[1+\alpha(z)]}{B_0(z)\,[1+\beta(z)]} \tag{14}$$

where

$$\alpha(z) = \frac{A(z) - A_0(z)}{A_0(z)} \qquad \text{and} \qquad \beta(z) = \frac{B(z) - B_0(z)}{B_0(z)} \tag{15}$$

then the pulse transfer function of the closed system is

$$W(z) = W_0(z)\, V(z, \alpha, \beta) \tag{16}$$

where

$$V(z, \alpha, \beta) = \frac{1+\alpha}{1+\alpha(1-T_0)+\beta T_0} \tag{16a}$$

The pulse transfer function corresponding to noise is:

$$T(z) = T_0(z) \frac{1+\beta}{1+\alpha(1-T_0)+\beta T_0} \tag{17}$$

The sensitivity function is given as

$$S^W(z) = S_0(z) \frac{1+\beta}{1+\alpha(1-S_0)+\beta S_0} \qquad \text{and} \quad S_0(z) = T_0(z) \tag{18}$$

Table 5

Relative variation of the controlled section as a function of the variable parameter for an arbitrary parameter variation

Variable parameter	$\alpha(z)=\dfrac{A-A_0}{A_0}$	$\beta(z)=\dfrac{B-B_0}{B_0}$
$K=K_0+\delta K$	$\alpha(z)=\dfrac{\delta K}{K_0}$	—
$a_i=a_{i_0}+\delta a_i$	$\alpha(z)=\dfrac{\delta a_i}{a_{i_0}}\ \dfrac{a_{i_0}}{z+a_{i_0}}$	—
$b_j=b_{j_0}+\delta b_j$	—	$\beta(z)=\dfrac{\delta b_j}{b_{j_0}}\cdot\dfrac{b_{j_0}}{z+b_{j_0}}$
$K=K_0+\delta K$ $a_i=a_{i_0}+\delta a_i$	$\alpha(z)\cong\dfrac{\delta K}{K_0}+\dfrac{\delta a_i}{a_{i_0}}\ \dfrac{a_{i_0}}{z+a_{i_0}}$ $\delta K a_{i_0}+\delta a_{i_0} K_0 \gg \delta K \delta a_i$	—

The relative variations $\alpha(z)$ and $\beta(z)$ as a function of the variable parameter are given in Table 5.

5. Statistical synthesis of control circuits taking into consideration the parameter variations

5.1. *Determination of optimal disturbance compensation and optimal sensitivity function*

The synthesis is carried out for statistical signals. If $r(t)$ stands for the control input and $n(t)$ for the nondesirable noise being present together with the control input, then the input is: $x(t) = r(t) + n(t)$. The control input, noise, and disturbance $u(t)$ are stationary and ergodic signals with known statistical characteristics. For the sake of simplicity let us assume that there is no correlation between noise and disturbance.

As we have shown earlier [6] the average value of the squared sum of the disturbance appearing at the output may be minimized by controller $D_2(z)$. The synthesis carried out with this condition gives optimal value for the pulse transfer function corresponding to disturbance and for the sensitivity function S^W of the closed system.

Consider the parameter variations (α, β) of the controlled system as known statistically variable signals. The set of disturbance signals $y_u(kT)$, appearing at the output depends on statistical variables describing the parameter variations. Because of this the *optimum criterion* be the *expected value of the squared-sum average* of the set $y_u(kT)$. That is

$$M\{\overline{y_u^2(kT,\alpha,\beta)}\} = M\left\{\frac{1}{2\pi j}\oint_C \Phi_{y_u y_u}(z,\alpha,\beta)\frac{dz}{z}\right\} = \min \qquad (19)$$

where C : the unit circle on the z plane; to be followed in clockwise
$\Phi_{y_u y_u}$: the two-sided z transform of the autocorrelation function of the set $y_u(kT)$
$M\{\}$: symbol for forming the expected value

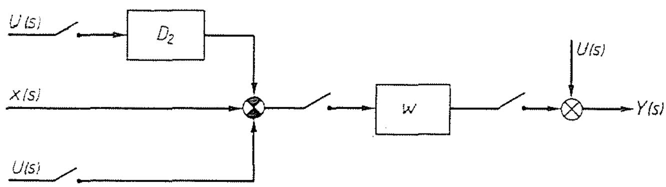

Fig. 4. Determination of the disturbance component of the output

It can be proved that the sequence of integrations can be interchanged, so

$$M\{\overline{y_u^2(kT,\alpha,\beta)}\} = \frac{1}{2\pi j}\oint_C M\{\Phi_{y_u y_u}(z,\alpha,\beta)\}\frac{dz}{z} = \min \qquad (20)$$

With respect to Fig. 4 it can be proved that

$$\Phi_{y_u y_u} = \Phi_{uu}[1 + \overline{W}W(1+\overline{D}_2 D_2) + \overline{W}W(\overline{D}_2 + D_2) - \\ - W(1+D_2) - \overline{W}(1+\overline{D}_2)] \qquad (21)$$

where Φ_{uu} : the two-sided z transform of the autocorrelation function of the disturbance, and

$$\overline{W} = \overline{W}(z) = W(z^{-1}); \qquad \overline{D}_2 = \overline{D}_2(z) = D_2(z^{-1})$$

Since

$$W(z) = W_0(z)V(z,\alpha,\beta) \quad \text{and} \quad \overline{W}W = \overline{W}_0 W_0 \overline{V}V$$

the solution of the Wiener—Hopff integral-equation known so far must be generalized for processes characterized by complex value statistical variables. It can be proved [6] that

$$M\{\overline{y_u^2(kT,\alpha,\beta)}\} = \min$$

is the necessary and sufficient condition for the

$$P(z) = M\{\overline{W}W\}(1 + D_2)\Phi_{uu} - M\{W\}\,\Phi_{uu} \tag{22}$$

expression's every pole to be outside the unit-circle.

Introduce the following symbols:

$$\begin{aligned}
&1.\ M\{W(z)\} = W_0(z)\,J_v(z) \qquad \text{where} \\
&\quad J_v(z) = M\{V(z, \alpha, \beta)\} \\
&2.\ M\{\overline{W}(z)W(z)\} = \overline{W}_0 W_0 J^+(z)J^-(z) \qquad \text{where} \\
&\quad J^+(z)J^-(z) = M\{\overline{V}V\} \\
&3.\ W_0(z^{-1})W_0(z) = L_0^+(z)L_0^-(z) \quad \text{and} \\
&4.\ \Phi_{uu}(z) = \Phi_{uu}^+(z)\Phi_{uu}^-(z)
\end{aligned} \tag{23}$$

In this decomposition every pole and zero of $J^+(z)$; $L_0^+(z)$; $\Phi_{uu}^+(z)$ are inside the unit-circle. The poles and zeros of $J^-(z)$; $L_0^-(z)$; $\Phi_{uu}^-(z)$ are outside the unit-circle.

From (22) we get the transfer function of the realizable pulse controller using the known method:

$$D_2(z) = \frac{1}{J^+(z)\,L_0^+(z)\,\Phi_{uu}^+(z)}\left[\frac{J_v^-(z)\,W_0^-(z)\,\Phi_{uu}^+(z)}{J^-(z)\,L_0^-(z)}\right]_+ - 1 \tag{24}$$

It can be shown that the average expected value of the squared sum of the set $y_n(kT)$ using (20), (21), and (24) is,

$$M\{y_u^2\} = \frac{1}{2\pi j}\oint_C M\{(1-W)(1-\overline{W}) - \overline{W}W D_2 D_2\}\,\Phi_{uu}\frac{dz}{z} \tag{25}$$

In case of total disturbance suppression — absolute invariancy — from (4) we get

$$D_2(z)W(z) = 1 - W(z) \quad \text{and} \quad M\{\overline{y_u^2}\} = 0 \tag{26}$$

The optimal pulse transfer function and sensitivity function for noise is

$$T(z) = S^W(z) = 1 - W(z) - [D_2(z)]_{\text{opt}}\,W(z) \tag{27}$$

5.2. *Determination of optimal transfer function of closed system*

Since the sensitivity for the parameter variation is minimized, $D_1(z)$ can be determined from (2) using the following expression:

$$D_1(z) = \frac{1}{G_0(z)}\,\frac{W_0(z)}{1 - W_0(z)} \tag{28}$$

if $G(z) = z^{-m} G_1(z) G_2(z)$ can be written up in this form, where:

- z^{-m} : the term representing the delay and time lag appearing in the controlled system
- $G_1(z)$: polynom, containing the zeros of the controlled system outside the unit-circle
- $G_2(z)$: the minimal-phase term of the controlled system, its zeros and poles are inside the unit-circle.

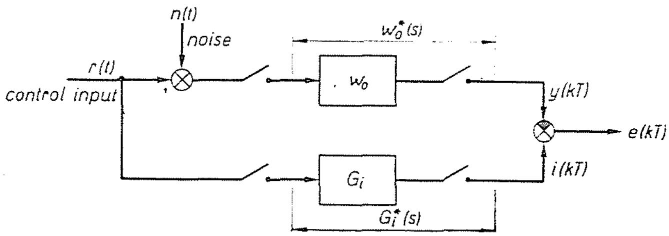

Fig. 5. Determination of $W_0(z)$ according to the $e^2(kT) = \min$ criterion, if $u(t) = 0$, $G(z) = G_0(z)$

Let us choose the optimum criterion so that the mean-square of the difference of the set $i(kT)$ made from the control input $r(t)$ by an ideal pulse transfer function $G_i(z)$ and of the output be minimum (Fig. 5). If $e(kT) = i(kT) - y(kT)$, then:

$$\overline{e^2(kT)} = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} e^2(kT) = \min \tag{29}$$

if $u(t) = 0$, and $G(z) = G_0(z)$.

The pulse transfer function of the network satisfying this can be determined by solving the Wiener–Hopff integral-equation [6]:

$$W(z) = \frac{1}{N(z)\,\Phi_{xx}^{+}(z)} \left[\frac{G_i(z)\,\Phi_{xr}(z)}{\overline{N}(z)\,\Phi_{xx}^{-}(z)} \right]_+ \tag{30}$$

where $\Phi_{xx} = \Phi_{xx}^{+}\,\Phi_{xx}^{-}$: two-sided z transform of the autocorrelation function of the input signal $x(t)$

Φ_{xr} : two-sided z transform of the cross-correlation function of the signals $x(t)$ and $r(t)$

$$N(z) = z^{-m} \frac{G_1(z^{-1})}{G_1(z)}$$

Satisfying (30) it can be proved that

$$\overline{e^2(kT)} = \frac{1}{2\pi j} \oint \left[G_i(z^{-1})\, G_i(z)\, \Phi_{rr}(z) - W(z^{-1})\, W(z)\, \Phi_{xx}(z) \right] \frac{dz}{z} \tag{31}$$

6. Synthesis of systems with time-dependent random parameters

Disregarding the external disturbance $U(z) = 0$, for the parameter variation $\Delta G(z)$ the following fluctuation of the controlled variable will appear:

$$H_x(z) = X(z) W_0(z) [1 - W_0(z)] [1 - D_1(z) D_2(z) G_0(z)] \frac{\Delta G}{G}(z) \tag{32}$$

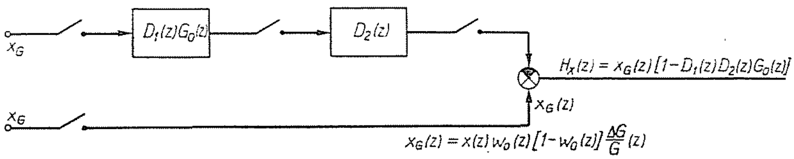

Fig. 6. Determination of the controlled variable's fluctuation in case of time-dependent random parameters

Be known $\Phi_{\Delta G \Delta G}(z)$, the two-sided z transform of the autocorrelation function describing $\Delta G(z)$ variation of the controlled section. Determine the value of $D_2(z)$ for given $W_0(z)$, $G(z)$, and $\Phi_{xx}(z)$ so, that the fluctuation of $h_x(kT)$ fulfil the condition $\overline{h_x^2(kT)} = \min$. This problem means the further generalization of the Wiener—Hopff integral-equation for systems with parameters varying stochastically.

From Fig. 6, it can be proved that:

$$H_X(z) = X_G[(z) 1 - D_1(z) D_2(z) G_0(z)] \tag{33}$$

where X_G is an auxiliary signal varying stochastically:

$$X_G(z) = X(z) W_0(z) [1 - W_0(z)] \frac{\Delta G(z)}{G(z)} \tag{34}$$

7. Possibilities of further generalization

The method given can be applied for control systems with arbitrary structures. The optimal transfer function of the closed system can be determined easily taking into account the parameter variations. It has advantages in synthetizing systems not containing disturbance compensation, and for systems supplemented with control from the input.

The synthesis may be performed for deterministic inputs as well, furthermore, the deterministic and statistical methods can be combined.

8. Numerical example

The continuous transfer function of the controlled section and the holding organ be

$$G_s(s) = \frac{K}{1+2sT}; \quad G_T(s) = \frac{1-e^{-sT}}{s}; \quad T = 1\,\text{sec}$$

The autocorrelation functions of the control input and the noise be

$$\varphi_{rr}(\tau) = r_0^2\, e^{-\vartheta|\tau|}; \quad \varphi_{nn}(\tau) = \begin{cases} 1 & \tau = 0 \\ 0 & \tau > 0 \end{cases}; \quad \varphi_{rn}(\tau) = 0$$

The autocorrelation function of the disturbance entering the system is

$$\varphi_{uu}(\tau) = u_0^2\, e^{-\mu|\tau|}; \quad \varphi_{xu}(\tau) = 0$$

In the reference input the power of the control input be 10 times that of the noise, thus $r_0^2 = 10$, and

$$U_0^2 = 45, \quad \nu T = 0.7, \quad \mu T = 0\cdot 2, \quad G_i(s) = e^{2sT}$$

The variation of the transfer coefficient of the controlled section around its nominal value be $\pm 20\%$ and consider $\alpha(z) = \delta K/K$ as *random variable with uniform distribution.*

With the data given above we get

$$G_i(z) = z^2; \quad G(z) = \frac{0.394\, K z^{-1}}{1 - 0.606 z^{-1}}$$

The two-sided z transforms of the autocorrelation functions of the control input, the noise, and the disturbance are:

$$\Phi_{rr}(z) = \frac{7.5}{(1-0.5\, z^{-1})(1-0.5\, z)}; \quad \Phi_{nn} = 1;$$

$$\Phi_{uu}(z) = \frac{16}{(1-0.8\, z^{-1})(1-0.8\, z)}$$

If $G_0(z) = D_2(z) = 0$, then the transfer coefficient $K = 4.07$ in the uncompensated system corresponding to the stability limit. The nominal value of the controlled section's transfer coefficient be $K_0 = 10$. $S_0\, D_1(z)$ *beside the optimalization is stabilizing the system also.*

The optimal transfer function of the closed system according to (30) is:

$$W(z) = \frac{0.116\, z^{-1}}{1 - 0.06\, z^{-1}}$$

Using (28) we get the transfer function of the pulse controller:

$$D_1(z) = 0.0295\, \frac{1 - 0.606\, z^{-1}}{1 - 0.176\, z^{-1}}$$

If there is no parameter variation,

$$\alpha = 0; \quad J^V = J^+ = 1; \quad L_0^+(z) = \frac{0.116}{1 - 0.006\, z^{-1}}$$

and so from (24)

$$D_2(z) = 5.9(1 - 0.07\, z^{-1})$$

If there is no disturbance compensation, when $D_2(z) = 0$, then according to (25) we get:

$$M\{\overline{y_u^2(kT)}\} = \frac{1}{2\pi j} \oint_C M\{(1 - W)(1 - \overline{W})\}\, \Phi_{uu} \frac{dz}{z} = 27.67$$

And for given $D_2(z)$

$$M\{\overline{y_u^2(kT)}\} = \frac{1}{2\pi j} \oint_C M\{(1 - W)(1 - \overline{W}) - \overline{W} W \overline{D}_2 D_2\}\, \Phi_{uu} \frac{dz}{z} = 7.17$$

With disturbance compensation a significant improvement may be achieved in $M\{\overline{y_u^2(kT)}\}$. According to (4), (6), and (9):

$$T_0(z) = S_0^W H(z) = 1 - 0.8\, z^{-1}; \quad S_0^T = -0.8\, z^{-1}$$

If $M\{\alpha\} = 0$ and $M\{\beta\} = 0$, and assume furthermore that there is no correlation between α and β, in other words $M\{\alpha\beta\} = M\{\bar{\alpha}\bar{\beta}\} = 0$, then using (23) and expanding into series leads to

$$J_v(z) \cong 1 - M\{\alpha^2\}\, T_0(1 - T_0) + M\{\beta^2\}\, T_0^2$$

$$J^+(z) J^-(z) \cong 1 - M\{\alpha^2\}\, T\,(1 - T_0) - M\{\bar{\alpha}^2\}\, \overline{T}_0(1 - \overline{T}_0) + M\{\alpha\bar{\alpha}\}\overline{T}_0 T_0$$

In case of parameter variation as in our case:

$$M\{\alpha^2\} = M\{\bar{\alpha}^2\} = M\{\alpha\bar{\alpha}\} \cong 0.01$$

by substitution

$$J_v = 1 - 0.008\, z^{-1} (1 - 0.8\, z^{-1})$$

$$J^+ J^- = (1 - 0.01\, z^{-1})(1 - 0.01\, z)$$

and so we get using (24):

$$D_2(z) = 5.9 \frac{1 - 0.068\, z^{-1}}{1 - 0.01\, z^{-1}}$$

Based upon Fig. 2 the *algorithm* of the control unit (direct program) is:

$$m(kT) = \sum_{i=0}^{2} a_i\, x(kT - iT) + \sum_{i=0}^{2} b_i\, y(kT - iT) - \sum_{l=1}^{2} c_i\, m(kT - iT)$$

where:

	a_i	b_i	c_i
$i = 0$	0.295	−2.04	—
$i = 1$	−0.182	0.300	−0.872
$i = 2$	0.0178	0.0734	+0.0670

Summary

A statistical synthesis method is given for invariant sampled data control systems with model feedback, taking into account the parameter variations of the control system. The passive adaptive control system with model feedback makes possible to minimize the effects of the external disturbance, and internal parameter variations in addition to following the reference input, without measuring the external disturbance. Introducing the sensitivity function the variation of the output in case of the variation of the transfer coefficient, zero or pole of the controlled section, is determined by the author. The relationship between the transfer function of the closed loop and the sensitivity of the transfer function corresponding to the disturbance is given. The expected value of the mean-square error is used as optimalization criterion. The parameter variation is considered to be of known statistical variation. The pulse transfer function of the controller required for the optimal disturbance compensation is determined. Generalization possibilities are given by the author. Finally the application of the statistical synthesis is illustrated by a numerical example. The method given may be considered as the improvement of the statistical synthesis assuming constant parameters.

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