STABILITY TEST OF LINEAR CONTROL SYSTEMS WITH DEAD TIME

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The relation between the input and output variables of control systems with finite delay may be characterized by the differential equation

$$\sum_{j=0}^{n} a_j \frac{d^j x_o(t)}{dt^j} = \sum_{k=0}^{m} b_k \frac{d^k x_i(t-\tau)}{dt^k}$$

where τ is the dead time characterizing the finite delay. In the case of such systems the number of roots determining stability is infinite in consequence of the exponential function appearing in the characteristic equation. A condition for the system stability is that all of these roots fall to the left side of the complex plane. As the direct determination of this criterion is not possible, different approximating methods have been worked out for testing the stability of control systems with dead time.

Soliman and Shaikh [3, 4] applied Pontryagin's method (see Appendix) for the stability test of a control system with first-order lag and finite delay, compensated by P, I, PI, PD and PID components.

The present paper investigates — as a generalization of the results obtained by [3] and [4] — the stability of the linear control systems with second-order lag and finite delay shown in Fig. 1 with a unit feedback, compensated generally speaking by a PID controller. First the Pontryagin method is used, then the Nyquist stability criterion is applied.

The transfer function of the plant is:

$$G(s) = \frac{\exp(-s\tau)}{1 + 2\zeta Ts + T^2 s^2}$$

where τ is the finite delay or dead time, T is the time constant of the second-order lag, ζ is the so-called damping factor, which is an arbitrary positive constant.

The transfer function of the PID controller is:

$$C(s) = K\left(1 + \frac{1}{T_i s} + T_d s\right)$$

where K is the loop gain, T_i is the integral time constant and T_d is the derivative time constant.

In a further paper diagrams computed by a digital computer are plotted, showing the critical loop gain versus the dead time for the values $0 < \tau \leq 10$ in the case of a proportional acting controller and for $0 < \tau < \infty$ in the case of an integral acting controller.

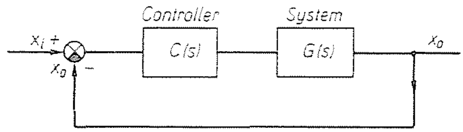

Fig. 1

1. Application of the state-space method for the stability test of control systems with dead time

The behaviour of any linear, time invariant continuous system with dead time may be described by the following first order difference-differential equation system:

$$\dot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t-\tau) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{x}(t-\tau) + \mathbf{f}(t) \tag{1}$$

and

$$\mathbf{x}_o(t) = \mathbf{D}\mathbf{x}(t)$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ are stationary matrices containing constant elements, $\mathbf{x}(t)$ is the state vector, $\mathbf{f}(t) = \mathbf{x}_i(t)$ is the vector of the input signal, τ is the dead time.

The stability condition of the homogeneous systems, i.e. for a system without external excitation, in the case $\mathbf{f}(t) = 0$ is the following: when $t \to \infty$ we must have $\mathbf{x}(t) \to 0$. This condition is relatively easy to determine. By taking the Laplace transforms of both sides of the homogeneous equation we obtain

$$\mathbf{X}(s) = -[\mathbf{S}(s)]^{-1}\,\mathbf{X}(0).$$

By the way the inverse Laplace transform of $-[\mathbf{S}(s)]^{-1}$ is the so-called transition matrix. The condition of the stability is that all eigenvalues of the equation

$$\det \mathbf{S}(s) = 0 \tag{2}$$

should have negative real parts. Here

$$[\mathbf{S}(s)]^{-1} = [\mathbf{A} + \mathbf{B}\exp(-s\tau) - s\mathbf{C}\exp(-s\tau) - s\mathbf{I}]^{-1} \times [\,\mathbf{C}\exp(-s\tau) + \mathbf{I}] \tag{3}$$

where $\mathbf{I}$ is the unit matrix.

In the case of systems with dead time det **S** is an exponential polynomial having an infinite number of roots. With the help of the PONTRYAGIN method the critical loop gain and thus the region of stability may be determined without the knowledge of the roots' positions in the complex plane. The method and the criteria are described in the Appendix.

2. A general stability test of a linear unit feedback control system with second-order lag and dead time compensated by a PID controller

2.1 *Pontryagin's method*

From the resultant transfer function of the open loop control system shown in Fig. 1 the relation between the Laplace transforms of the output variable $\mathbf{x}_o(t)$ and the input variable $\mathbf{x}_i(t)$ — if the dead time lag is considered separately — is:

$$X_o(s) = \left[X_i(s)\frac{K}{T^2}\frac{1}{T_i}\right]\frac{1}{s^3} + \left[X_i(s)\frac{K}{T^2} - X_o(s)\frac{1}{T^2}\right]\frac{1}{s^2} + \left[X_i(s)\frac{K}{T^2}T_d - X_o(s)\frac{2\zeta}{T}\right]\frac{1}{s}.$$

With this we can determine the state equation of the system on the basis of the analogue model shown in Fig. 2. The sign reversion of the integrators to the contrary of the usual practice is disregarded. Taking the dead time also into account the state-space equations will be:

$$\dot{x}_1(t) = -\frac{2\zeta}{T}x_1(t) + x_2(t) + \frac{K}{T^2}T_d\,x_i(t) - \frac{K}{T^2}T_d\,x_1(t-\tau)$$

$$\dot{x}_2(t) = -\frac{1}{T^2}x_1(t) + x_3(t) + \frac{K}{T^2}x_i(t) - \frac{K}{T^2}x_1(t-\tau)$$

$$\dot{x}_3(t) = \frac{K}{T^2}\frac{1}{T_i}x_i(t) - \frac{K}{T^2}\frac{1}{T_i}x_1(t-\tau).$$

In the case of a system without external excitation corresponding to equation system (1) is:

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\end{bmatrix} = \begin{bmatrix}-\frac{2\zeta}{T} & 1 & 0\\ -\frac{1}{T^2} & 0 & 1\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix} + \begin{bmatrix}-\frac{K}{T^2}T_d & 0 & 0\\ -\frac{K}{T^2} & 0 & 0\\ -\frac{K}{T^2}\frac{1}{T_i} & 0 & 0\end{bmatrix}\begin{bmatrix}x_1(t-\tau)\\ x_2(t-\tau)\\ x_3(t-\tau)\end{bmatrix}. \qquad (4)$$

With the use of matrices **A, B, C** and utilizing equations (3) and (2), we have:

$$\det \mathbf{S} = s^3 + \frac{2\zeta}{T} s^2 + \frac{K}{T^2} T_d s^2 \exp(-s\tau) + \frac{s}{T^2} + \frac{K}{T^2} s \exp(-s\tau) +$$
$$+ \frac{K}{T^2} \frac{1}{T_i} \exp(-s\tau) = 0. \tag{5}$$

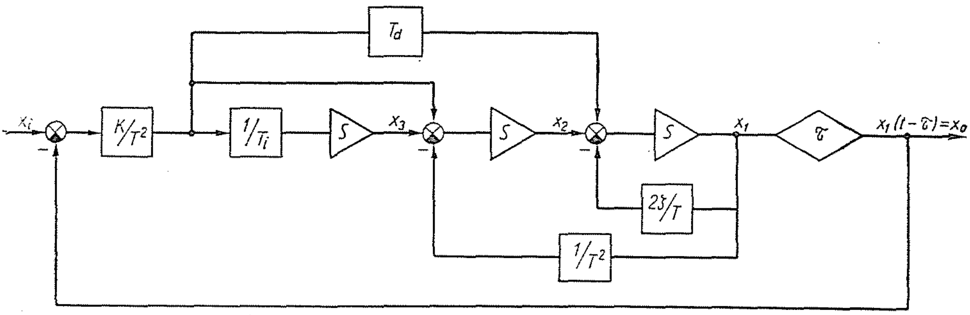

Fig. 2

This is evidently a polynomial of the form

$$\det \mathbf{S} = F[s, \exp(-s\tau)]$$

On applying the PONTRYAGIN method let us multiply the equations by the expression ($\exp s\tau$). After algebraic arrangements we have:

$$F(s, \exp s\tau) = T^2 T_i s^3 \exp(s\tau) + 2\zeta T T_i s^2 \exp(s\tau) + K T_d T_i s^2 + T_i s \exp(s\tau) +$$
$$+ K T_i s + K. \tag{6}$$

The principal term $T^2 T_i s^3 \exp(s\tau)$ does exist, therefore the necessary condition of the system stability is fulfilled.

Performing the substitution $s = j\omega$ we have:

$$F[j\omega, \exp(j\omega\tau)] = -2\zeta T T_i \omega^2 \cos\omega\tau + (T^2 T_i \omega^3 - \omega T_i) \sin\omega\tau -$$
$$- K\omega^2 T_d T_i + K + j[(\omega T_i - T^2 T_i \omega^3) \cos\omega\tau - 2\zeta T T_i \omega^2 \sin\omega\tau + \omega K T_i]. \tag{7}$$

From this form of

$$F[j\omega, \exp(j\omega\tau)] = P(\omega) + jQ(\omega) \tag{8}$$

the stability region may be determined with the help of any criterion of PONTRYAGIN (see Appendix). With the application of criterion 3 — which is generally the most easily evaluable — by using Eq. (8) the smallest critical loop gain K_{kr} may be obtained. The critical loop gain — after the evaluation with an arbitrary accuracy by iteration of the corresponding angular frequency ω_{kr} in radians — may be determined from the transcendental equation system of

$$Q(\omega) = 0$$

$$P(\omega)\dot{Q}(\omega) = 0.$$

In the case of $\dot{Q}(\omega) \neq 0$ the equation system reduces to

$$Q(\omega) = 0$$

$$P(\omega) = 0.$$

With the application of (7) and (8) to the considered system we have:

$$P(\omega) = -2\,\zeta\, TT_i\,\omega^2 \cos\omega\tau + (T^2 T_i\,\omega^3 - \omega T_i) \sin\omega\tau - \\ - K\omega^2\, T_d\, T_i + K = 0 \tag{9}$$

$$Q(\omega) = (\omega T_i - T^2 T_i\,\omega^3)\cos\omega\tau - 2\,\zeta\, TT_i\,\omega^2 \sin\omega\tau + \omega\, KT_i = 0. \tag{10}$$

From equation system (9), (10) the transcendent equation to be solved for ω, after the elimination of K, will be:

$$(T^2 T_i\,\omega^3 - \omega T_i + 2\,\zeta\, T\omega - 2\,\zeta\, T\,\omega^3\, T_d\, T_i)\,\mathrm{tg}\,\omega\tau = \\ = 2\,\zeta\, TT_i\,\omega^2 - T^2\omega^2 + 1 + \omega^4\, T_d\, T_i\, T^2 - \omega^2\, T_d\, T_i. \tag{11}$$

In the knowledge of $\omega = \omega_{kr}$ obtained from (11) by iteration the loop gain $K = K_{kr}$ is easily evaluated.

2.2 *The application of the Nyquist stability criterion*

Let us examine the method of determination of the critical loop gain concerning the control shown in Fig. 1 with the help of the better known Nyquist stability criterion.

The loop transfer function of the open loop system shown in Fig. 1, taking the dead time lag also into account, is:

$$Y(s) = KY_1(s) = K\left(1 + \frac{1}{T_i\, s} + T_d\, s\right) \frac{\exp(-s\tau)}{1 + 2\zeta Ts + T^2 s^2}.$$

After performing the substitution $s = j\omega$ and after algebraic arrangements we have the following frequency function:

$$Y(j\omega) = \frac{K}{j\omega T_i} \exp(-j\omega\tau) \frac{1 - T_d T_i \omega^2 + j\omega T_i}{1 - T^2 \omega^2 + j2\zeta T\omega} = K |Y_1(j\omega)| \exp[j\varphi(\omega)]. \tag{12}$$

With the critical loop gain the resultant phase angle must be $\varphi(\omega) = -\pi$. Thus,

$$-\pi = -\frac{\pi}{2} - \omega\tau + \tan^{-1} \frac{\omega T_i}{1 - T_d T_i \omega^2} - \tan^{-1} \frac{2\zeta T\omega}{1 - T^2 \omega^2} \tag{13}$$

i.e.

$$\tan^{-1} \frac{2\zeta T\omega}{1 - T^2 \omega^2} - \tan^{-1} \frac{\omega T_i}{1 - T_d T_i \omega^2} = \frac{\pi}{2} - \omega\tau.$$

Taking the tangents of both sides and utilizing the well known trigonometric formula

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha \tan\beta}$$

we have:

$$\frac{\dfrac{2\zeta T\omega}{1 - T^2\omega^2} - \dfrac{T_i\omega}{1 - T_d T_i \omega^2}}{1 + \dfrac{2\zeta T\omega}{1 - T^2\omega^2} \dfrac{T_d\omega}{1 - T_d T_i \omega^2}} = \text{cotan}\,\omega\tau = \frac{1}{\tan\omega\tau}.$$

From this we obtain for ω_{kr} a transcendent equation identical with (11). In the knowledge of the latter the critical loop gain

$$K_{kr} = \frac{1}{|Y_1(j\omega_{kr})|} \tag{14}$$

may be evaluated.

So with the use of the NYQUIST stability criterion we obtained the same result in a much shorter way. Consequently it has no reason to extend the state-space method to the stability tests of linear, constant parameter controls in the case of systems with a dead time either.

A further advantage of the use of the NYQUIST stability criterion is that with its help a transcendent equation for the determination of the frequency corresponding to an arbitrary phase margin may be easily derived from (13).

Other methods for the stability test of control systems with dead time are also known [5], [6]. But this paper does not wish to describe these different methods or to compare them, rather in the continuation it will examine also numerically — by utilizing the above said and with the help of data obtained by a digital computer — the stability regions in function of the dead time and the system time constants, of the control system shown in Fig. 1.

Appendix

Stability test of linear, constant coefficient difference-differential equation by Pontryagin's method

a) Let us multiply the polynomial of form $\det \mathbf{S} = F[s, \exp(-s\tau)]$, derived from equation (1) by such a high power of ($\exp s\tau$), that only positive powers of ($\exp s\tau$) should stay in the polynomial.

b) Let us find the term in which the highest powers of s and of ($\exp s\tau$) appear. This term will be called the principal term.

The first result of PONTRYAGIN's method is the instability criterion. According to this criterion, if a polynomial of form $F(s, \exp s\tau)$ does not contain a principal term, then the system is unstable, whatever the values of its coefficients.

Yet the presence of a principal term is only a necessary, but not a sufficient condition of the stability.

c) In order to test, whether the real parts of all the eigenvalues of the polynomial $F[s, \exp(-s\tau)]$ are negative, let us perform the substitution $s = j\omega$

$$[F(s, \exp s\tau)]_{s=j\omega} = F[j\,\omega, \exp(j\,\omega\tau)] = P(\omega) + jQ(\omega) \tag{15}$$

where $P(\omega)$ and $Q(\omega)$ are the real and imaginary parts of $F[j\omega, \exp(j\omega\tau)]$. With this the stability theorem of PONTRYAGIN is as follows:

A system described by equation (1) is stable then and only then, when the polynomial $F(s, \exp s\tau)$ contains a principal term and one of the following statements holds:

1. All roots of $P(\omega)$ and $Q(\omega)$, respectively, are real, single, alternative to each other and at least one value of ω satisfies the inequality:

$$P(\omega)\dot{Q}(\omega) - \dot{P}(\omega)Q(\omega) > 0. \tag{16}$$

2. All ω_p roots of $P(\omega)$ are real, single and all of these roots satisfy the inequality:

$$\dot{P}(\omega_p)Q(\omega_p) < 0. \tag{17}$$

3. All ω_q roots of $Q(\omega)$ are real, single and all of these roots satisfy the inequality:

$$P(\omega_q)\dot{Q}(\omega_q) > 0. \tag{18}$$

The stability boundaries may be determined by any of the conditions (16)—(18), if we change the symbol of inequality to that of equality.

Summary

This paper wishes to give an insight into the stability test of linear control systems with dead time. First PONTRYAGIN's method is used then the NYQUIST stability criterion is applied for a system with second-order lag and dead time with a unit feedback, compensated by a PID controller. It is shown also that the NYQUIST stability criterion gives the same result as PONTRYAGIN's method but in a much shorter way.

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