A NEW SYNTHESIS METHOD FOR ASYNCHRONOUS SEQUENTIAL CIRCUITS, I.

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1. Introduction

In an earlier paper [3] UNGER showed that a normal mode flow table cannot be realized by an asynchronous sequential circuit without inserted delay elements if the flow table contains essential hazard.

Even if delay elements are inserted, the synthesis method should attain as fast network response as possible. For this reason all the transitions should be completed in one step. Examples of this sort of realizations are the general state assignment method of HUFFMAN [2] which uses single internal variable changes to realize the transitions and the noncritical race state assignment methods of LIU [8] and TRACEY [9].

This paper gives a general state assignment method for this type of realization requiring fewer internal variables. Also a new synthesis method is suggested by using both the delayed and undelayed versions of the internal variables and this method is shown to be more economical in terms of the number of internal variables.

1.1. *Terminology*

The internal variables of an asynchronous sequential circuit will be denoted by y_i and the delayed versions of the internal variables will be denoted by Y_i. $\mathbf{x}$ will be written for the vector of the x_i input variables, $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, similarly $\mathbf{y} = (y_1, y_2, \ldots, y_m)$, $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_m)$ and $\mathbf{f} = (f_1, f_2, \ldots, f_m)$, where $f_i = f_i(\mathbf{x}, \mathbf{y}, \mathbf{Y})$ or $f_i = f_i(\mathbf{x}, \mathbf{Y})$ are the "next" values of y_i.

The realization of Fig. 1 will be called $\mathbf{f}(\mathbf{x}, \mathbf{Y})$ type realization, and the realization shown in Fig. 2 will be called $\mathbf{f}(\mathbf{x}, \mathbf{y}, \mathbf{Y})$ type realization.

A cube defined by $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ is the set of all vectors $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ such that $\min(a_i, b_i) \leq x_i \leq \max(a_i, b_i)$ for $i = 1, 2, \ldots, n$ This cube will be denoted by $[\mathbf{a}, \mathbf{b}]$.

If $\mathbf{A}$ and $\mathbf{B}$ are cubes and $\mathbf{A} \cap \mathbf{B} = \Phi$, $\mathbf{A} * \mathbf{B}$ will be written.

P_i denotes statements, e.g. $P_1(\mathbf{a} \in [\mathbf{b}, \mathbf{c}])$ has the truth value: "true" (1) if $\mathbf{a} \in [\mathbf{b}, \mathbf{c}]$ and false (0) otherwise.

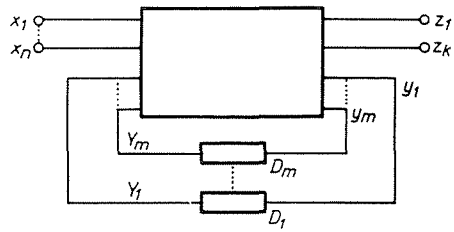

Fig. 1

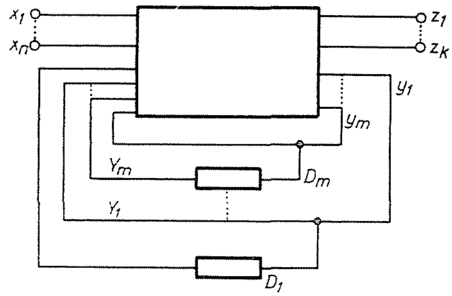

Fig. 2

k_m denotes the minimum number of internal variables necessary for the coding of an r row flow table (k_m is the smallest integer such that $2k_m \geq$ r)

1.2. *Assumptions*

1. The terminal characteristics of the circuit to be designed are described by a normal mode flow table, i.e. no input change leads to more than one state change.

2. The flow table will be realized by an asynchronous sequential circuit operating in fundamental mode [4], i.e. the inputs are never changed unless the circuit is stable internally.

3. The combinational networks are built of gate type elements.

4. Both line (wiring) and gate delays are taken into account, since the input delay model [7] will be used to describe the possible effects of the stray delays in the network and all the stray delays are assumed to be bounded $\delta_i \leq \lambda$.

5. The inserted delays (D_i) are assumed to be *intertial* delays and $D_i \gg \lambda$.

6. The combinational networks are free of logical hazard [6].

2. f (x, y, Y) type realization

2.1. *Speed independent state transition*

Since in any stable state $y_i = Y_i$ ($i = 1, 2, \ldots, n$) for the $(\mathbf{x}^1, \mathbf{y}^1, \mathbf{Y}^1) \rightarrow$ $\rightarrow (\mathbf{x}^2, \mathbf{y}^2, \mathbf{Y}^2)$ state transition $(\mathbf{x}^1, \mathbf{y}^1) \rightarrow (\mathbf{x}^2, \mathbf{y}^2)$ will be written.

f	x	y	Y
y^1	x^1	y^1	y^1
-	$[x^1 x^2]$	-	y^1
y^2	x^2	-	y^1
y^2	x^2	y^2	$[y^1 y^2]$

Fig. 3

	f	x	y	Y
1	y^1	x^1	y^1	y^1
2	-	$[x^1 x^2]$	-	y^1
3	y^2	x^2	-	y^1
4	y^2	x^2	y^2	y^1
5	y^2	x^2	y^2	$[y^1 y^2]$
6	y^2	x^2	y^2	y^2

Fig. 4

$(\mathbf{x}^1, \mathbf{y}^1) \rightarrow (\mathbf{x}^2, \mathbf{y}^2)$ state transition is speed independent, i.e. the circuit reaches the final state independently from the actual values of the stray delays, if the possible values of f_i are restricted the following way (Fig. 3).

To be able to prove the speed independence of this transition, we write it in a more detailed form. The first row in Fig. 4 represents the initial stable state, $\mathbf{x}^1\mathbf{y}^1$. In the second and third rows the input variables and the undelayed versions of the internal variables are changing. In the fourth row **y** is in the final and **Y** in the initial state. In the fifth row the Y_i variables are changing and the sixth row represents the final stable state $\mathbf{x}^2\mathbf{y}^2$.

The speed independence of the first part of the transition (rows 1, 2, 3 and 4) can be proved by using the analysis procedure for asynchronous circuits given by HALL [7]. This proof is given in the Appendix.

Because of assumptions 4 and 5, **Y** starts to change only after the network's stabilization in $\mathbf{x}^2\mathbf{y}^2\mathbf{Y}^1$. This part of the transition is obviously speed independent, for **f** does not change any more and the combinational networks are assumed to be free of logical hazard.

This is not the only possible way to define speed independent transitions between two stable states, but the following results are based on this definition.

2.2. *Expanded state table*

The expanded state table can be derived by plotting the values of $\mathbf{f}(\mathbf{x}, \mathbf{y}, \mathbf{Y})$ on a table, the rows of which are defined by **y** and the columns by **x** and **Y**. As an example the expanded state table is given for a circuit, the terminal characteristics of which are described by the flow table of Fig. 5.

If the next state functions are:

$$f_1(\mathbf{x}, \mathbf{y}, \mathbf{Y}) = \bar{x} \cdot Y_2 + x \cdot y_1$$

$$f_2(\mathbf{x}, \mathbf{y}, \mathbf{Y}) = \bar{x} \cdot Y_2 + x \cdot \bar{y}_1$$

	0	1
a	a	b
b	c	b
c	c	d
d	a	d

Fig. 5

x	0				1			
$y_1 y_2$ \ $Y_1 Y_2$	00	01	11	10	00	01	11	10
00	00	11	11	00	01	01	01	01
01	00	11	11	00	01	01	01	01
11	00	11	11	00	10	10	10	10
10	00	11	11	00	10	10	10	10

Fig. 6

then the corresponding expanded state table is shown in Fig. 6. The stable states are encircled on the expanded state table. Obviously in any part of the map defined by one of the input combinations, any row and any column can contain at most one stable state.

2.3. *Design procedure*

The design of the f(x, y, Y) type realization of a given flow table has two steps:

1. to assign single internal states to the rows of the flow table
2. to specify all the transitions of the flow table according to the definition of the speed independent transition.

	00				01				10				11			
	00	01	10	11	00	01	10	11	00	01	10	11	00	01	10	11
00	00				—				—				11			
01	—				—				—				11			
10	—				—				—				11			
11	—				—				—				11	11	11	11

Fig. 7

The first problem is the more serious one, since before making a proper state assignment we must be able to characterize those ones in which every specified transition can be made speed independent. This problem will be attacked in the next few paragraphs.

The second part of the design procedure can be performed on the expanded state tables. As an example, the transition (00, 00) → (11, 11) will be specified on the expanded state table of Fig. 7.

2.4. *Interaction between two transitions*

In this section two arbitrary transitions will be considered (Fig. 8) and conditions will be developed which must be satisfied to be able to specify both transitions as being speed independent.

	f	x	y	Y
1	y^1	x^1	y^1	y^1
2	-	$[x^1 x^2]$	-	y^1
3	y^2	x^2	-	y^1
4	y^2	x^2	y^2	$[y^1 y^2]$
5	y^3	x^3	y^3	y^3
6	-	$[x^3 x^4]$	-	y^3
7	y^4	x^4	-	y^3
8	y^4	x^4	y^4	$[y^3 y^4]$

Fig. 8

	$x^1=x^4$	x^2	x^3
$y^1=y^3$	$y^1 \neq y^4$	y^2	y^3
y^2		y^2	
y^4	y^4		

Fig. 9

Two transitions cannot be speed independent at the same time if there is at least one combination defined by x, y, and Y, where the restrictions on the values of f(x, y, Y) cannot be satisfied simultaneously. This situation will be called interaction between the transitions.

Speed independent state transitions are defined in four parts (four rows of Fig. 8) and in three of them there are restrictions on the values of f(x, y, Y), so there are 9 possibilities of interaction between two transitions. Interaction can exist between rows 1—5, 1—7, 1—8, 3—5, 3—7, 3—8, 4—5, 4—7 and 4—8 of Fig. 8.

Rows 1 and 5 interact if and only if $P_1(y^1 \neq y^3) \cdot P_2(x^1 = x^3) \cdot P_3(y^1 = y^3) \cdot P_4(y^1 = y^3) = 1$ but this is impossible, for $P_1(y^1 \neq y^3) \cdot P_3(y^1 = y^3) = 0$ independently from y^1 and y^3.

For similar reasons there can be no interaction between rows 1—8, 4—5, and 4—8.

Rows 1 and 7 interact if and only if $P_1(y^1 \neq y^4) \cdot P_2(x^1 = x^4) \cdot P_3(y^1 = y^3) = 1$, but this is an impossible situation according to the flow table of Fig. 9.

	x^1	$x^2 = x^3$	x^4
$y^1 = y^3$	y^1	$y^3 \neq y^2$	y^4
y^2		y^2	
y^4			y^4

Fig. 10

	x^1	x^2 x^4	x^3
$y^1 = y^3$	y^1	$y^2 \neq y^4$	y^3
y^2		y^2	
y^4		y^4	

Fig. 11

Rows 3 and 5 interact if and only if $P_1(y^2 \neq y^3) \cdot P_2(x^2 = x^3) \cdot P_3(y^1 = y^3) = 1$, but this is an impossible situation according to the flow table of Fig. 10.

Rows 3 and 7 interact if and only if $P_1(y^2 \neq y^4) \cdot P_2(x^2 = x^4) \cdot P_3(y^1 = y^3) = 1$, but this is also an impossible situation according to the flow table of Fig. 11.

Rows 3 and 8 interact if and only if $P_1(y^2 \neq y^4) \cdot P_2(x^2 = x^4) \cdot P_3(y^1 \in [y^3 y^4]) = 1$.

Rows 4 and 7 interact if and only if $P_1(y^2 \neq y^4) \cdot P_2(x^2 = x^4) \cdot P_3(y^3 \in [y^1 y^2]) = 1$.

Rows 3 and 8, and 4 and 7 obviously can interact under certain conditions. Summarizing these conditions: two transitions $x^1y^1 \rightarrow x^2y^2$ and $x^3y^3 \rightarrow x^4y^4$ can interact only if the final internal states belong to the same input combination ($x^2 = x^4$), these final internal states are different ($y^2 \neq y^4$) and at least one of the conditions $y^1 \in [y^3y^4]$ and $y^3 \in [y^1y^2]$ is satisfied.

Now an important theorem can be stated:

Theorem 2.1.: Transitions $x^1y^1 \rightarrow x^2y^2$ and $x^3y^3 \rightarrow x^4y^4$ cannot interact in case of an f(x, y, Y) type realization, if conditions $y^1 \notin [y^3y^4]$ and $y^3 \notin [y^1y^2]$ are satisfied simultaneously.

Proof: Interaction can exist between rows 3—8 and 4—7 only. Satisfying conditions $P_1(\mathrm{y}^1 \in [\mathrm{y}^3\mathrm{y}^4]) = 0$ and $P_2(\mathrm{y}^3 \in [\mathrm{y}^1\mathrm{y}^2]) = 0$ simultaneously there can be no interaction between rows 3—8 and 4—7.

2.5. *General state assignment scheme*

To construct a general state assignment which can be used for the coding of any flow table, we must consider the realization of a flow table, where all the possible state transitions occur. Every transition can be made speed independent, if there is no interaction between them and this implies that the conditions of Theorem 2.1. must be satisfied for all the possible pairs of transitions.

If y^1, y^2, y^3 and y^4 are codes for arbitrary four states, then any of these codes must be disjoint to the cubes formed by any pair of the other three codes. So the coding of y^1, y^2, y^3 and y^4 must contain the columns determined from Fig. 12.

With the notations of Fig. 12 for the columns (I, A, B, AB . . .) which are used for later convenience, the conditions of the realizability of any transition between y^1, y^2, y^3 and y^4 are described by the following Boolean expression:

$$\begin{aligned}
&(AB + ab + ABC + abc) \cdot (AB + ab + C + c) \cdot (AB + ab + B + b) \cdot \\
&\cdot (AB + ab + A + a) \cdot (AC + ac + ABC + abc) \cdot (AC + ac + C + c) \cdot \\
&\cdot (AC + ac + B + b) \cdot (AC + ac + A + a) \cdot (BC + bc + ABC + abc) \cdot \\
&\cdot (BC + bc + C + c) \cdot (BC + bc + B + b) \cdot (BC + bc + A + a) = \\
&(A + a) \cdot (B + b) \cdot (C + c) \cdot (ABC + abc) + (AB + ab) \cdot (AC + ac) \cdot \\
&\cdot (BC + bc)
\end{aligned}$$

The 16 columns form a 16th order commutative group G, if an operation is defined between them, the componentwise mod 2 sum. $N = \{I, i\}$ is a proper subgroup of G and the quotient group G/N has the elements $\{I, i\}$, $\{A, a\}$, $\{B, b\}$, $\{C, c\}$, $\{AB, ab\}$, $\{AC, ac\}$, $\{BC, bc\}$, and $\{ABC, abc\}$. They represent the following partitions:

$$\begin{array}{llll}
\{I, i\} & \rightarrow \{\overline{\mathrm{y}^1\mathrm{y}^2\mathrm{y}^3\mathrm{y}^4}\} & \{C, c\} & \rightarrow \{\overline{\mathrm{y}^1\mathrm{y}^3\mathrm{y}^4},\ \overline{\mathrm{y}^2}\} \\
\{A, a\} & \rightarrow \{\overline{\mathrm{y}^1\mathrm{y}^2\mathrm{y}^3},\ \overline{\mathrm{y}^4}\} & \{AC, ac\} & \rightarrow \{\overline{\mathrm{y}^1\mathrm{y}^3},\ \overline{\mathrm{y}^2\mathrm{y}^4}\} \\
\{B, b\} & \rightarrow \{\overline{\mathrm{y}^1\mathrm{y}^2\mathrm{y}^4},\ \overline{\mathrm{y}^3}\} & \{BC, bc\} & \rightarrow \{\overline{\mathrm{y}^1\mathrm{y}^4},\ \overline{\mathrm{y}^2\mathrm{y}^3}\} \\
\{AB, ab\} & \rightarrow \{\overline{\mathrm{y}^1\mathrm{y}^2},\overline{\mathrm{y}^3\mathrm{y}^4}\} & \{ABC, abc\} & \rightarrow \{\overline{\mathrm{y}^2\mathrm{y}^3\mathrm{y}^4},\ \overline{\mathrm{y}^1}\}
\end{array}$$

Introducing the $\{I, i\} \rightarrow I$

$\{A, a\} \rightarrow A$

$\{ABC, abc\} \rightarrow ABC$ notations,

y^1	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
y^2	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
y^3	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
y^4	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
	I	A	B	AB	C	AC	BC	ABC	abc	bc	ac	c	ab	b	a	i
$y^1_*\ [y^3 y^4]$				X				X	X				X			
$y^2_*\ [y^3 y^4]$				X	X							X	X			
$y^1_*\ [y^2 y^4]$						X		X	X		X					
$y^3_*\ [y^2 y^4]$			X			X					X					
$y^1_*\ [y^2 y^3]$							X	X	X	X						
$y^4_*\ [y^2 y^3]$		X					X			X					X	
$y^2_*\ [y^1 y^4]$					X		X			X		X				
$y^3_*\ [y^1 y^4]$			X				X			X				X		
$y^2_*\ [y^1 y^3]$					X	X					X	X				
$y^4_*\ [y^1 y^3]$		X				X					X				X	
$y^3_*\ [y^1 y^2]$			X	X									X	X		
$y^4_*\ [y^1 y^2]$		X		X									X		X	

Fig. 12

	I_1	I_2	I_3	I_4
a	a	b	a	e
b	c	b	e	c
c	c	d	a	c
d	a	d	a	e
e	c	b	e	e

Fig. 13

the conditions of the realizability of any transition between y^1, y^2, y^3 and y^4 has the form

$$A \cdot B \cdot C \cdot ABC + AB \cdot AC \cdot BC \tag{2.5.—1}$$

If we assign arbitrary but distinct codes to y^1, y^2, y^3 and y^4, they define the partition $\{\overline{y^1}, \overline{y^2}, \overline{y^3}, \overline{y^4}\}$ so their coding must contain at least either XY and YZ, or X, Y and YZ, or X, Y and Z type columns, where X, Y and Z can be any three of A, B, C and ABC. For those y^1, y^2, y^3 and y^4, the coding of which contains only the columns defined by one of the complexes $\{XY, YZ\}$, $\{X, Y, YZ\}$, or $\{X, Y, Z\}$, the condition (2.5.—1) cannot be satisfied.

According to the Lagrange theorem, G/N can have proper subgroups of order 2 and 4 only. The 2nd order subgroups are: $\{I, A\}$, $\{I, B\}$, $\{I, C\}$, $\{I, AB\}$, $\{I, AC\}$, $\{I, BC\}$, $\{I, ABC\}$, and the 4th order subgroups are $\{I, A, B, AB\}$, $\{I, A, C, AC\}$, $\{I, B, C, BC\}$ $\{I, A, BC, ABC\}$ $\{I, B, AC, ABC\}$ $\{I, C, AB, ABC\}$, $\{I, AB, AC, BC\}$. Any of these subgroups have the form $\{I, X\}$, $\{I, XY\}$, $\{I, X, Y, XY\}$ or $\{I, XY, XZ, YZ\}$, respectively.

Obviously, none of the complexes mentioned before are subgroups (i.e. closed under the group operation) so that new elements can be generated by carrying out all the possible pairwise multiplications between the elements of the complex under consideration. Since

$$\begin{aligned} \{XY, YZ\} &\times \{XY, YZ\} = \{I, XY, XZ, YZ\} \\ \{X, Y, YZ\} &\times \{X, Y, YZ\} = \{I, X, Y, Z, XYZ, XY, YZ\} \\ \{X, Y, Z\} &\times \{X, Y, Z\} = \{I, X, Y, Z, XY, XZ, YZ\} \end{aligned}$$

the original elements of any of these complexes and the new elements generated by their pairwise multiplications together are sufficient for the realization of any transition between them.

Since k_m variables are always enough for a distinct coding of a flow table and $\binom{k_m}{2}$ is the number of the possible pairs, the proof of the following theorem is finished:

Theorem 2.2.: $k_m + \binom{k_m}{2}$ variables are always sufficient for the speed independent **f(x, y, Y)** type realization of an arbitrary flow table.

As an example a general state assignment for an eight row flow table is given:

K	L	M	$K \oplus L$	$K \oplus M$	$L \oplus M$
0	0	0	0	0	0
0	0	1	0	1	1
0	1	0	1	0	1
0	1	1	1	1	0
1	0	0	1	1	0
1	0	1	1	0	1
1	1	0	0	1	1
1	1	1	0	0	0

Columns K, L and M represent a distinct coding with the minimum number of variables, $K \oplus M$, $L \oplus M$ and $L \oplus K$ are the possible mod 2 sums of the original columns.

2.6. *State assignment procedure*

The general state assignment scheme is very easy to use, since it is completely independent of the flow table structure, but it has the disadvantage of requiring more than the necessary number of internal variables in most cases.

To make a state assignment for the f(x, y, Y) type realization of a given flow table the following method is suggested.

According to Theorem 2.1. for every pair of transitions $x^1y^1 \to x^2y^2$ and $x^3y^3 \to x^4y^4$ such that $x^2 = x^4$ and $y^2 \neq y^4$, $y^1 \divideontimes [y^3y^4]$ and $y^3 \divideontimes [y^1y^2]$ conditions have to be satisfied. The condition $y^k \divideontimes [y^my^n]$ can be represented by an incompletely specified Boolean vector, where arbitrarily y^k is coded by 1, $[y^my^n]$ is coded by 0 and the remaining elements are unspecified (—).

By listing these conditions for all the specified transitions an incompletely specified Boolean matrix is defined. The problem of reducing these matrices was considered by DOLOTTA and McCLUSKEY [10] and TRACEY [9]. Any of their methods can be used to find a reduced matrix representing a state assignment, where all the specified transitions can be made speed independent.

An example is given by finding a state assignment for the flow table of Fig. 13. The incompletely specified Boolean matrix for this case:

		a	b	c	d	e
I_1	*da, bc*	—	0	0	1	—
		0	1	—	0	—
	da, ec	—	—	0	1	0
		0	—	—	0	1
I_2	*cd, ab*	0	0	1	—	—
		1	—	0	0	—
	cd, eb	—	0	1	—	0
		—	—	0	0	1
I_3	*be, ca*	0	1	0	—	—
		—	0	1	—	0
	be, da	0	1	—	0	—
		—	0	—	1	0
I_4	*ae, bc*	1	0	0	—	—
		0	1	—	—	0
	ae, dc	1	—	0	0	—
		0	—	—	1	0

The "Matrix Reduction Algorithm 2" of [9] results in a state assignment with the minimum number of internal variables:

a	0	0	0
b	0	1	1
c	1	1	0
d	1	0	1
e	0	1	0

Summary

In this paper a new synthesis method is suggested for asynchronous (fundamental mode) circuits by using both the delayed and undelayed versions of the internal variables and this method is shown to be more economical in terms of the number of internal variables than the existing synthesis methods having the same capabilities.

In the second part of the paper (to be published in the next issue) a new general state assignment method resulting in a single transition time state assignment is given for asynchronous (fundamental mode) circuits which requires fewer internal variables than the existing state assignment methods of Huffman, Liu and Friedman.

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