WAVE PROPAGATION IN INHOMOGENEOUS, ANISOTROPIC, TIME-VARYING MEDIUM

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(Received May 31, 1968)

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The necessity to investigate wave propagation in media characterized by general parameters arose as a consequence of space research, and the development of special, particularly microwave devices. As the original aim of the research work was — with respect to a possibility to treat the Doppler-effect more exactly — to investigate propagation in ionized, in time and space varying, anisotropic media encountered in space research, the restrictions used must be adjusted thereto.

It is assumed that the space charge of the medium: $\varrho = 0$, that is, it is electrically neutral. A solution for the Maxwell-equations of the following form:

$$\bar{E} = \bar{E}_0 \, e^{j(\omega t - \varphi)} \tag{1}$$

is sought for while the abrupt discontinuities must be excluded, that is,

$$\frac{1}{E_{0i}} \frac{\partial E_{0i}}{\partial x_i} \ll \operatorname{grad}_i \varphi$$

$$\frac{1}{\omega - \dfrac{\partial \varphi}{\partial t}} \sum_{i=1}^{3} \frac{1}{E_{0i}} \frac{\partial E_{0i}}{\partial t} \ll 1 \tag{2}$$

Discussing the assumption (2) reveals that if the solutions have the form of (1), these restrictions are not very severe.

The effect of the energy exchange between the medium and the electromagnetic wave was not taken into account, that is, the energy of the "generator" governing the parameters of the medium, was considered as infinitely large. This is admissible so far the energy emitted or absorbed by the electromagnetic wave is negligible, as compared to the energy content of the media, or if the energy exchange can be steadily compensated (e.g. by a pump-source).

1. Solution by using standard formalism

Assume that the medium is characterized by the relative permittivity, $\overline{\overline{\varepsilon}}(\bar{r}, t)$ and the relative permeability $\overline{\overline{\mu}}(\bar{r}, t)$, where $\overline{\overline{\varepsilon}}$ contains the conductivity in the well-known manner. The desired form of the solution is $\bar{E} = \bar{E}_0(\bar{r}, t) \cdot e^{j[\omega t - \varphi(\bar{r}, t)]}$ where the restrictions (2) are evaluable. Only such solutions are searched for.

Maxwell's equations are:

$$\operatorname{rot} \bar{H} = \frac{\partial}{\partial t} (\varepsilon_0 \overline{\overline{\varepsilon}} \bar{E})$$

$$\operatorname{rot} \bar{E} = - \frac{\partial}{\partial t} (\mu_0 \overline{\overline{\mu}} \bar{H})$$

$$\operatorname{div} (\mu_0 \overline{\overline{\mu}} \bar{H}) = 0 \tag{3}$$

$$\operatorname{div} (\varepsilon_0 \overline{\overline{\varepsilon}} \bar{E}) = 0$$

where ε_0 and μ_0 are the permittivity and permeability of the vacuum, respectively, $\bar{r}$ is the position vector, t is time, $\overline{\overline{\varepsilon}}$ and $\overline{\overline{\mu}}$ are the relative permittivity and permeability of the medium, $\bar{E}$ is the electrical field, $\bar{H}$ the magnetic field, ω the radian frequency of the signal, φ the total phase, and

$$\bar{D} = \varepsilon_0 \overline{\overline{\varepsilon}} \bar{E}, \quad \bar{B} = \mu_0 \overline{\overline{\mu}} \bar{H} \tag{4}$$

using $\bar{D}$ as electric flux density and $\bar{B}$ as magnetic flux density.

Introducing here the

$$\bar{K} = \operatorname{grad} \varphi$$

and

$$\omega^* = \omega - \frac{\partial \varphi}{\partial t} \tag{5}$$

symbols, the following equation system equivalent to (3) in every respect can be obtained instead of Eqs (3):

$$\bar{K} \times \bar{H} = - \omega^* \varepsilon_0 \overline{\overline{\in}} \bar{E}$$

$$\bar{K} \times \bar{E} = \omega^* \mu_0 \overline{\overline{\mathrm{M}}} \bar{H}$$

$$\mu_0 \bar{K} \overline{\overline{\mathrm{M}}} \bar{H} = 0 \tag{6}$$

$$\varepsilon_0 \bar{K} \overline{\overline{\in}} \bar{E} = 0$$

where

$$\bar{\bar{\epsilon}} = \varepsilon - j \frac{1}{\omega^*} \frac{\partial \varepsilon}{\partial t} = \bar{\bar{1}} + \bar{\bar{e}}^*$$

$$\bar{\bar{\mathrm{M}}} = \bar{\bar{\mu}} - j \frac{1}{\omega^*} \frac{\partial \bar{\bar{\mu}}}{\partial t} = \bar{\bar{1}} + m^* \tag{7}$$

The form of Eqs (6) is analogous with the wave equations for plane waves which propagate in homogeneous, stationary media, so let the common method be used: after multiplying repeatedly with $\bar{K}$, one of the equations can be substituted into the other, and, therefore, it is possible to keep either $\bar{E}$ or $\bar{H}$ only.

The symbols used are:

$$\bar{\bar{A}} = \bar{K} \times (\bar{K} \times \ldots)$$

$$\bar{\bar{P}}^* \ldots = \bar{K} \times (\bar{\bar{e}}^* \ldots) \tag{8}$$

$$\bar{\bar{M}}^* \ldots = \bar{K} \times (\bar{\bar{m}}^* \ldots)$$

$$(\bar{\bar{A}} + \omega^{*2} \varepsilon_0 \mu_0 \bar{\bar{\mathrm{M}}}) \bar{H} + \omega^* \varepsilon_0 \bar{\bar{P}}^* \bar{E} = 0$$

$$- \omega^* \mu_0 \bar{\bar{M}}^* \bar{H} + (\bar{\bar{A}} + \omega^{*2} \varepsilon_0 \mu_0 \bar{\bar{\epsilon}}) \bar{E} = 0 \tag{9}$$

The non-trivial solution of the homogeneous linear equation system (9) exists only if the determinant is equal to zero whereby dispersion equation well known in homogeneous case is obtained. This can be written in two advantageous equivalent forms. Let us use $\alpha^2 = \omega^{*2} \varepsilon_0 \mu_0$ for cancellation:

$$\left| \left(\frac{\bar{\bar{A}}}{\alpha} + \alpha \bar{\bar{\epsilon}} \right) + \bar{\bar{M}}^* \left(\frac{\bar{\bar{A}}}{\alpha} + \alpha \bar{\bar{\mathrm{M}}} \right)^{-1} \bar{\bar{P}}^* \right| = 0 \tag{10}$$

or

$$\left| \left(\frac{\bar{\bar{A}}}{\alpha} + \alpha \bar{\bar{\mathrm{M}}} \right) + \bar{\bar{P}}^* \left(\frac{\bar{\bar{A}}}{\alpha} + \alpha \bar{\bar{\epsilon}} \right)^{-1} \bar{\bar{M}}^* \right| = 0 \tag{11}$$

These two equations give $\varphi(\bar{r}, t)$ resulting under propagation and, at the same time, the propagation path $\bar{r}$ and the connected time t, whereby $\bar{E}$ and $\bar{H}$ satisfying the given boundary conditions and now existing necessarily can be obtained from Eqs (9).

Eqs (10) or (11) contain the known cases of wave propagation — under the restrictions mentioned in the introduction (!) — and open the way for further general investigations.

2. Verification of the general applicability of the solution, the way of calculation

2.1. *Propagation in vacuum*

Now $\bar{\bar{\varepsilon}} = \bar{\bar{\mu}} = \bar{\bar{1}}$, and $\partial\varphi/\partial t = 0$, as no parameters vary in time. The (10) or (11) form of the dispersion equation is

$$|\bar{\bar{A}} + \omega^2 \varepsilon_0 \mu_0 \cdot \bar{\bar{1}}| = 0$$

$$\mathrm{grad}^2 \varphi = \omega^2 \varepsilon_0 \mu_0 = k_0^2 \quad \text{and} \quad \varphi = k_0 \cdot \bar{n} \cdot \bar{r} \tag{12}$$

where $\bar{n}$ is an arbitrary unit vector. The result is well known.

2.2. *Homogeneous ferrite or plasma*

If the characteristic parameters are $\bar{\bar{\mu}}$ = constant and $\bar{\bar{\varepsilon}} = \bar{\bar{1}}$, in stationary magnetic field the medium is ferrite, if $\bar{\bar{\varepsilon}}$ = constant and $\bar{\bar{\mu}} = \bar{\bar{1}}$, the medium is plasma. In case of ferrite it is advisable to use (11), in case of plasma rather (10). The resulting dispersion equations:

$$|\bar{\bar{A}} + \omega^2 \varepsilon_0 \mu_0 \bar{\bar{\mu}}| = 0 \qquad \text{— in case of ferrite}$$

and

$$|\bar{\bar{A}} + \omega^2 \varepsilon_0 \mu_0 \bar{\bar{\varepsilon}}| = 0 \qquad \text{— in case of plasma} \tag{13}$$

are also well known.

2.3. *Stationary, isotropic, inhomogeneous medium*

Assume $\bar{\bar{\varepsilon}} = \varepsilon(\bar{r})\, \bar{\bar{1}}$, $\bar{\bar{\mu}} = \bar{\bar{1}}$, and let the variation in time be omitted. The new form of (10) is

$$|\bar{\bar{A}} + \omega^2 \varepsilon_0 \mu_0 \varepsilon(\bar{r})\, \bar{\bar{1}}| = 0$$

and after manipulation

$$K^2 = \omega^2 \varepsilon_0 \mu_0 \varepsilon(\bar{r})$$

can be written.

This is the basic equation of the geometrical optics, the so-called Eikonal-equation, again well known:

$$\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2 + \left(\frac{\partial\varphi}{\partial z}\right)^2 = \omega^2 \varepsilon_0 \mu_0 \varepsilon(\bar{r}) \tag{14}$$

The following method of calculation is well adaptable in other inhomogeneous cases as well, so let it be examined in detail. Let the common method of characteristics excluding the discontinuities be used.

Eq. (14) is of the following form:

$$\mathscr{H}\left(\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z}, x, y, z\right) = 0$$

It is advisable to introduce here the parameter σ instead of the length of arc, defined by the equation

$$\frac{ds}{d\sigma} = \sqrt{\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2 + \left(\frac{\partial\varphi}{\partial z}\right)^2} = K$$

In this case the propagation path $\bar{r} = x(\sigma)\bar{i} + y(\sigma)\bar{j} + z(\sigma)\bar{k}$ can be obtained and, among all the possibilities, the actual one can be selected by the boundary conditions. The total phase shift between two given points is

$$\varphi_{12} = k_0^2 \int_{\sigma_1}^{\sigma_2} \varepsilon[\bar{r}(\sigma)]\, d\sigma \tag{15}$$

and, with the upper limit of the integral made variable, the desired $\varphi[\bar{r}(\sigma)]$ can be obtained. If, for instance, the Doppler-shift of an electromagnetic wave is to be determined, the transmitter and receiver approach one another, that is, $\sigma_1 = \sigma_1(t)$ and $\sigma_2 = \sigma_2(t)$, and the Doppler-shift is

$$\Delta\omega_D = -\frac{d\varphi_{12}}{dt}$$

In solving an actual task, the integrals can be evaluated by computers.

2.4. *Isotropic, inhomogeneous, quasi-stationary medium*

Assume that $\bar{\bar{\varepsilon}} = \varepsilon(\bar{r}, t)\,\bar{\bar{I}}$, $\bar{\bar{\mu}} = \bar{\bar{I}}$ and $\frac{1}{\omega}\frac{\partial\varepsilon}{\partial t} \ll \varepsilon$. In that case the actual form of (10) is:

$$K^2 = \omega^{*2}\varepsilon_0\,\mu_0\,\varepsilon(\bar{r}, t) \tag{16}$$

Using the method of characteristics, we get

$$\bar{r} = x(\sigma)\,\bar{i} + y(\sigma)\,\bar{j} + z(\sigma)\,\bar{k}$$

and

$$t = t(\sigma),$$

where the σ parameter differs from the previous one, and

$$\varphi = k_0^2 \int \left(\frac{p_t(\sigma)}{\omega} - 1 \right)^2 \varepsilon d\sigma + \\ + \frac{k_0^2}{\omega} \iint \left(\frac{p_t(\sigma)}{\omega} - 1 \right)^2 \frac{\partial \varepsilon}{\partial t} \left[\frac{1}{2} - k_0^2 \varepsilon \left(\frac{p_t(\sigma)}{\omega} - 1 \right) \right] d\sigma\, d\sigma \qquad (17)$$

where $p_t(\sigma) = \partial\varphi/\partial t$, and its value will be obtained as a result of the process. If it is assumed that the phenomena do not depend on time, (17) gives (15).

2.5. *Stationary, anisotropic, inhomogeneous medium*

Let us have, for instance, $\bar{\bar{\varepsilon}} = \bar{\bar{\varepsilon}}(\bar{r})$, $\bar{\bar{\mu}} = \bar{\bar{1}}$ and a stationary medium. So, if $\bar{\bar{\varepsilon}}$ is symmetric, we have the crystal optics, and if it is antisymmetric, the plasma. (If $\bar{\bar{\varepsilon}} = \bar{\bar{1}}, \bar{\bar{\mu}} = \bar{\bar{\mu}}(\bar{r})$ it is possible to discuss inhomogeneous ferrites, etc. The results are formally equivalent to those represented here.)

Eq. (10) can be written in the following form:

$$K^2 [K_x^2 \varepsilon_{11} + K_y^2 \varepsilon_{22} + K_z^2 \varepsilon_{33}] + k_0^2 [A(K^2 - K_x^2) + B(K^2 - K_y^2) + \\ + C(K^2 - K_z^2)] + K_x K_y [k_0^2 D + (K^2 - k_0^2 \varepsilon_{33}) F] + K_y K_z [k_0^2 G + \\ + (K^2 - k_0^2 \varepsilon_{11}) J] + K_x K_z [k_0^2 L + (K^2 - k_0^2 \varepsilon_{22}) M] + k_0^4 R = 0 \qquad (18)$$

where A, B, C, D, F, G, J, L, M and R are given functions consisting only of the components of $\bar{\bar{\varepsilon}}$.

Suppose now that $\bar{\bar{\varepsilon}}$ is homogeneous and antisymmetric; this results in the well known dispersion equation of the homogeneous plasma, and (18) can be solved by the method of characteristics. The discussion of the dispersion equation gives information on the mode of propagation.

Take $N = K/k_0$, and let us introduce as new variables the spherical coordinates, ϑ (the angle to the z-axis) and φ (that to the x-axis). Now (18) can be written in the following new form:

$$N^4 + aN^2 + b = 0,$$

where a and b are functions of ϑ and φ in every point of the space.

There is no propagation if $N^2 < 0$ and real, but if $N^2 > 0$ and real, the propagation is ideal. In other cases there is an attenuated propagation. So the range (surface or surface-system) can be outlined in every point, along which an ideal propagation exists: the $F_i(\vartheta, \varphi)$ as the solution of

$$\left| -\frac{a}{2} \pm \sqrt{\left(\frac{a}{2}\right)^2 - b} \right| = -\frac{a}{2} \pm \sqrt{\left(\frac{a}{2}\right)^2 - b}$$

and $F_n(\vartheta, \varphi)$, along which there is no propagation, as the solution of

$$\left| -\frac{a}{2} \pm \sqrt{\left(\frac{a}{2}\right)^2 - b} \right| = -\left(-\frac{a}{2} \pm \sqrt{\left(\frac{a}{2}\right)^2 - b} \right)$$

2.6. *In-time fast varying, isotropic medium*

Let us suppose that $\bar{\bar{\varepsilon}} = \varepsilon(\bar{r}, t)\bar{\bar{I}}$, $\bar{\bar{\mu}} = \bar{\bar{I}}$, then according to (10)

$$\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2 + \left(\frac{\partial \varphi}{\partial z}\right)^2 - \left(\frac{k_0}{\omega}\right)^2 \varepsilon \left(\frac{\partial \varphi}{\partial t}\right)^2 +$$

$$+ \frac{k_0^2}{\omega} \left(2\varepsilon - j \frac{1}{\omega} \frac{\partial \varepsilon}{\partial t}\right) \frac{\partial \varphi}{\partial t} - k_0^2 \varepsilon + j \frac{k_0^2}{\omega} \frac{\partial \varepsilon}{\partial t} = 0 \qquad (19)$$

which determines φ. Since φ is complex, the ideal isotropic medium attenuates or amplifies (!) the signal.

As the result is very interesting, let us briefly discuss the special results of (19) concerning in-space homogeneous media.

The new form of Eq (19) is:

$$\left(\frac{\partial \varphi}{\partial x}\right)^2 = \left(\frac{\partial \varphi}{\partial t}\right)^2 \varepsilon_0 \mu_0 \varepsilon(t) - j \frac{d\varepsilon(t)}{dt} \varepsilon_0 \mu_0 \frac{\partial \varphi}{\partial t} \qquad (20)$$

In case of an ideal propagation $Im \dfrac{\partial \varphi}{\partial x} = 0$, and

$$\bar{E}_1 = \frac{\bar{E}_0}{\sqrt{\varepsilon(t)}} e^{j(\omega t - Re \varphi)}$$

is a modulated signal, or a static electrical field:

$$\bar{E}_2 = \bar{E}_0 e^{b + ja}$$

$a = \text{const.}$, $b = \text{const.}$, may arise.

There is no propagation in the medium, if $\dfrac{\partial (Re\, \varphi)}{\partial x} = 0$. A special case of this can be obtained if $Re\, \varphi = 0$. Then it is necessary, to have

$$\varepsilon = \frac{\beta^2 - 1}{4} e^{j 2\omega (t + t_0)} + \frac{1}{2} \cos 2\omega (t + t_0), \quad t_0 = \text{const.}$$

Under the given restrictions ($Re\ \varphi = 0$) only $\beta = \pm 1$ can occur. So

and

$$Im\ \varphi = \ln \frac{1}{\sqrt{\frac{1}{2} \cos 2\omega (t + t_0)}} \pm x + \gamma, \quad \gamma = \text{const.}$$
$$\bar{E} = 2E_0 \cdot \left(\sqrt{\frac{2}{\cos 2\omega (t + t_0)}} + \gamma \right) \cdot e^{j\omega t} \tag{21}$$

describes the phenomena in degenerated, ideal parametric amplifiers with distributed parameters. The magnetic parametric amplifiers can be investigated in a similar manner etc.

It can be seen that (10) or (11) contains the hitherto well known results of which several were represented here. Solving the problem under other suitable conditions may result in completely new phenomena.

3. Relativistic formalism

The introduction of $\tau = jct$ does not involve any theoretical novelty; as its only consequence, our equations are considerably simplified by using the natural coordinates of Maxwell's equations. Under the assumptions made at the beginning,

$$\bar{E} = \bar{E}_0 \cdot e^{j\Phi} \tag{22}$$

In applying the symbols

$$K = \frac{\partial \varphi}{\partial x} \bar{i} + \frac{\partial \varphi}{\partial y} \bar{j} + \frac{\partial \varphi}{\partial z} \bar{k}$$
$$\bar{\bar{\xi}} = \bar{\bar{\varepsilon}} - j \frac{\partial \bar{\bar{\varepsilon}}/\partial \tau}{\partial \Phi/\partial \tau} = \bar{\bar{\varepsilon}} - j \frac{1}{\varkappa} \frac{\partial \bar{\bar{\varepsilon}}}{\partial \tau}$$
$$\mathcal{M} = \bar{\bar{\mu}} - j \frac{\partial \bar{\bar{\mu}}/\partial \tau}{\partial \Phi/\partial \tau} = \bar{\bar{\mu}} - j \frac{1}{\varkappa} \frac{\partial \bar{\bar{\mu}}}{\partial \tau} \tag{23}$$
$$\varkappa = \frac{\partial \Phi}{\partial \tau}$$

and keeping the other symbols in accordance with their meaning, Eqs (10) and (11) will remain invariably valid.

Then, for instance, the dispersion equation of the task in 2.4. can be written as

$$\mathrm{grad}_{\square}^{2}\,\Phi = -\,e\left(\frac{\partial \Phi}{\partial \tau}\right)^{2}, \quad \varepsilon = 1 + e$$

which is better arranged and more advantageous for calculation, etc.

A further task of generalization is to find a treatment whereby the abrupt discontinuities may be reckoned with in the same general manner.

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Acknowledgement is due to Prof. Dr. K. Simonyi and Dr. Á. Csurgay for their valuable assistance in this work.

Summary

The paper wants to find a general way for the discussion of wave propagation in inhomogeneous, anisotropic, time varying media. The abrupt discontinuities will be excluded from the discussion, that is, "plane-wave" solution is sought for. After the general solution of the problem, the paper suggests a calculation method, and the different well-known wave propagation equations (e.g. the Eikonal-equation) are shown to result from the general solution as special cases. Finally, the method of the relativistic rewriting is presented.

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