NONLINEAR CONTROL SYSTEM DYNAMICS CHARACTERIZATION BY ROOT-LOCUS CURVE

By

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Notations

For the notation of the functions the system of symbols used in mathematics is applied: generally capital letters for time functions of the physical variables and small letters for those of the variations at the workpoint. For example $S(t)$ denotes the time function of the controlled variable, $s(t) = S(t) - S_0$ is the same with respect to S_0.

The subscript o with some variables refers to the value at the workpoint 0, where the system is in stationary equilibrium.

t time

s Laplace operator

$\lambda = \frac{d}{dt}$ differential operator

$W(s) = \frac{G(s)}{H(s)}$ open-loop transfer function of linear control system

$H(s) = \sum_0^n{}' T_i^i s^i$ denominator of $W(s)$, a polynomial of nth order

$G(s) = K \sum_0^m{}' \tau_i^i s^i$ numerator of $W(s)$, a polynomial of mth order

i the number of order of the differentiation with respect to time

$S(t)$ controlled variable

$M(t)$ modified variable

$Z_q(t)$ qth disturbing variable

$S_a(t)$ basic value, the required value of $S(t)$

μ number of the disturbing variables

$$\left.\begin{aligned} s(t) &= S(t) - S_0 \\ m(t) &= M(t) - M_0 \\ z_q(t) &= Z_q(t) - Z_{q_0} \\ s_a(t) &= S_a(t) - S_{a_0} \end{aligned}\right\}$$

variations relative to workpoints S_0, M_0, Z_{q_0}, S_{a_0} of the controlled variable, modified variable, disturbing variable, basic value

$$\bar{\lambda} = \begin{bmatrix} \lambda^i \\ \lambda^{i-1} \\ \vdots \\ \lambda \\ 1 \end{bmatrix}$$ differential operator vector

$\bar{S}_a(t) = \bar{\lambda}_i \cdot S_a(t)$ controlled variable vector (in g: $i = 0, \ldots, k$; in f: $i = 0, \ldots, m$)

$\overline{\mathbf{S}}_a(t) = \overline{\lambda}_i(t) \cdot S_a(t)$	basic value vector ($i = 0, \ldots, \varepsilon$)
$\overline{\mathbf{Z}}_q(t) = \overline{\lambda}_i \cdot Z_q(t)$	qth disturbing variable vector ($i = 0, \ldots, Z_q$)
$\overline{\mathbf{M}}(t) = \overline{\lambda}_i \cdot M(t)$	modified variable vector (in g: $i = 0, \ldots,$ j; in f: $i = 0, \ldots, m$)
g	controller function
f	plant function

$$\frac{\partial \overline{\mathbf{f}}}{\partial \mathbf{S}}\bigg|_0 = \left[\frac{\partial f}{\partial S^{(i)}}\bigg|_0 \frac{\partial f}{\partial S^{(i-1)}}\bigg|_0 \quad \ldots\ldots \quad \frac{\partial f}{\partial S}\bigg|_0\right]$$

$$\frac{\partial \overline{\mathbf{f}}}{\partial \mathbf{Z}}\bigg|_0 = \left[\frac{\partial f}{\partial Z^{(i)}}\bigg|_0 \frac{\partial f}{\partial Z^{(i-1)}}\bigg|_0 \quad \ldots\ldots \quad \frac{\partial f}{\partial Z}\bigg|_0\right]$$

$$\frac{\partial \overline{\mathbf{f}}}{\partial \mathbf{M}}\bigg|_0 = \left[\frac{\partial f}{\partial M^{(i)}}\bigg|_0 \frac{\partial f}{\partial M^{(i-1)}}\bigg|_0 \quad \ldots\ldots \quad \frac{\partial f}{\partial M}\bigg|_0\right]$$

Taylor coefficient vectors of the functions g and f

$$\frac{\partial \overline{\mathbf{g}}}{\partial \mathbf{M}}\bigg|_0 = \left[\frac{\partial g}{\partial M^{(i)}}\bigg|_0 \frac{\partial g}{\partial M^{(i-1)}}\bigg|_0 \quad \ldots\ldots \quad \frac{\partial g}{\partial M}\bigg|_0\right]$$

$$\frac{\partial \overline{\mathbf{g}}}{\partial \mathbf{S}}\bigg|_0 = \left[\frac{\partial g}{\partial S^{(i)}}\bigg|_0 \frac{\partial g}{\partial S^{(i-1)}}\bigg|_0 \quad \ldots\ldots \quad \frac{\partial g}{\partial S}\bigg|_0\right]$$

$$\frac{\partial \overline{\mathbf{g}}}{\partial \mathbf{S}_a}\bigg|_0 = \left[\frac{\partial g}{\partial S_a^{(i)}}\bigg|_0 \frac{\partial g}{\partial S_a^{(i-1)}}\bigg|_0 \quad \ldots\ldots \quad \frac{\partial g}{\partial S_a}\bigg|_0\right]$$

$$-\frac{\partial f}{\partial S^{(i)}}\bigg|_0 = T^i_{Si}; \quad -\frac{\partial g}{\partial M^{(i)}}\bigg|_0 = T^i_{Mi}$$

$$\frac{\dfrac{\partial f}{\partial M^{(i)}}\bigg|_0}{\dfrac{\partial f}{\partial M}\bigg|_0} = \tau^i_{Mi}; \quad \frac{\dfrac{\partial g}{\partial S^{(i)}}\bigg|_0}{\dfrac{\partial g}{\partial S}\bigg|_0} = \tau^i_{Si}; \quad -\frac{\partial f}{\partial M}\bigg|_0 \cdot \frac{\partial g}{\partial S}\bigg|_0 = \varkappa$$

$Z_T(t)$	torque relative value
$\Omega(t)$	angular velocity relative value
$I(t)$	armature current relative value
$U(t)$	terminal voltage relative value
$U_A(t)$	basic voltage relative value
r	relative feedback index
ϱ	relative internal voltage drop in the armature
T_v	motor electrical time constant (sec)
T_m	motor electromechanical time constant (sec)
β	gain factor of the integral controller ($\sec^{-1}$)
$Z_{\xi 1}$	relative load torque value $1/T_{cs} > \xi_0/T_0$
$Z_{\xi 0}$	relative load torque at the stability limit

Introduction

The linear control system dynamics is a well delimited and almost completely elaborated field in the control theory. When the mathematical model of a real system is represented by a linear, constant coefficient differential equation of arbitrary order, the system analysis and synthesis can be performed without any special difficulties. Due to its properties the mathematical model can be reduced to algebraic system of equations by various integral transformations (Laplace, Fourier) and the principle of superposition holds [1].

The practical design by these methods is in fact an application of Bode's theorems for control systems and presently it is the most common method [2].

This advanced state of the linear control theory is probably due to the fact that even before the emergence of control science the Kirchoff equations for lumped parameter electrotechnical and electronic circuits led to linear differential equations so that the theoretical methods developed here were applicable for the analysis and synthesis of control systems. Until quite recently the theory of these fields differed only in terminology [3].

In many cases the phenomena arising in real physical systems cannot be described by linear mathematical models. The mathematical models describing the signal transmission in the systems often lead to nonlinear differential equations. No uniform theoretical test method can be used in these cases.

There are, however, test methods for some types of nonlinearity and system structure, moreover these methods can be generalized for generally developing static characteristics with discontinuities [4]. By these tests stability problems as "limit cycle" problem can be solved in the first place, and they are usually suitable for systems which can be divided to strictly linear and nonlinear parts. Here the linear and nonlinear parts of the influence diagram are connected in series, in parallel or via a feedback, and so the mathematical operation of summation, subtraction is permitted among the parts beside the series connection. If other operations are used (multiplication, raising to a power division of variables, etc.) these methods prove to be tedious or even unusable.

In these cases one can have resort only to the computer analysis by digital or analog computers. The analog computer is highly preferable for this purpose as compared to the digital machine, because it works in parallel mode at a very high speed, and the runs for various parameters give immediate results as characteristic curves in diagram form.

This paper endeavours to give an examination method for a general control system, and taking into account a property of nonlinearities going beyond the nonlinearity of the static curve system. Essentially it is recognized that given types of nonlinear systems can be described by characteristic equation linearizing about the workpoints of the functions governing the system behaviour, and the root-array of these characteristic equations which depends on the workpoint can constitute a "map" for the nonlinear system.

1. The structure of the system under study

The control systems based on the principle of negative feedback can be fundamentally characterized by the influence scheme of Fig. 1.

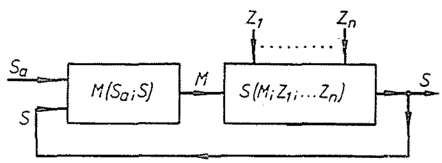

Fig. 1. Influence scheme of the control system

In this scheme the control device (comprising the basic and error forming, sensing, amplifying, actuating and intervening function and their internal feedbacks) can be described by a function in which the two input signals of the controller (S_a and S) appear as independent variables and the output signal M as dependent variable.

This function must be determined by the system designer. He must take into account the interrelationship between the dependent variable S and the $n + 1$ independent variables of the plant, which follows from the technological properties and requirements.

A design is to start in the first place from the functions concerning the equilibrium state. The designer has to examine the control aim, how the signals Z_q influence the S value statically, then after determining the most unfavourable disturbing signal combination pattern he marks out the necessary modification range for M.

With the inclusion of a satisfactory "dynamic margin" to this range of modification the requirement for the control device can numerically be established.

The system dynamics design is "simple", when the relationship among the system variables can be given in terms of linear differential equations. If the controller and plant functions are continuous, and the system functions as a constant value control, so that its characteristics can be linearized at a workpoint, the design method for linear systems can be applied for nonlinear systems as well, at least in the vicinity of the equilibrium workpoint.

In the structure under study the following assumptions are made:

a) The control device and the plant are characterized by nonlinear function defined over a simply connected range T and having continuous partial derivatives there. This restriction aims to have such control system to be studied where both the plant and the control device would be characterized by "smooth" functions. Apart from the relay (discontinuous) systems a good many control systems are featured with this property as most practical linear systems contain inherent nonlinear elements (amplifier saturation, iron saturation, etc.),

b) The functions defined over the range T are of single value. This assumption excludes from the study the hysteresis-type organs, and so the dependent variable assumes a single value for any given combination of the independent variables of the functions describing the system.

According to these conditions the nonlinear system functions

$$\begin{aligned} g^* \left[\overline{\mathbf{M}}(t);\ \overline{\mathbf{S}}(t);\ \overline{\mathbf{S}}_a(t)\right] &= 0 \\ f^* \left[\overline{\mathbf{S}}(t);\ \overline{\mathbf{Z}}_1(t);\ \ldots;\overline{\mathbf{Z}}_\mu(t);\ \overline{\mathbf{M}}(t)\right] &= 0 \end{aligned} \tag{1.1}$$

concerning the control device and the plant are required to be such that within the defined range of the variables, a tangent plane could be drawn to the hypersurfaces (1.1) assuming arbitrary combination of variables; this tangent plane must substitute (1.1) over the range Δ. On this assumption a first order Taylor approximation can be given.

2. Workpoint linearization of the system

Functions (1.1) can be expressed as

$$\begin{aligned} M(t) &= g\left[\overline{\mathbf{M}}(t);\ \overline{\mathbf{S}}(t);\ \overline{\mathbf{S}}_a(t)\right] \\ S(t) &= f\left[\overline{\mathbf{S}}(t);\ \overline{\mathbf{Z}}(t);\ \overline{\mathbf{M}}(t)\right] \end{aligned} \tag{1.2}$$

For the sake of simplicity one disturbing variable ($q = 1$) is assumed in these functions.

Eq. (1.2) can be obtained from (1.1) by arranging according to any factor containing terms $M(t)$ and $S(t)$, then dividing by the coefficient of the factor.*

In steady state the equilibrium workpoint — assuming constant value control — is characterized by the

$$\begin{aligned} M_0 &= g(M_0;\ S_0;\ S_{a0}) \\ S_0 &= f(S_0;\ Z_0;\ M_0) \end{aligned} \tag{1.3}$$

* If $M(t)$ in g^* or $S(t)$ in f^* does not appear, the equation should be arranged according to the least order time derivative of $M(t)$ and $S(t)$. This is the case when the controller (I, PI, PID types) or the plant are of integral type. For the sake of simplicity systems consisting of zero-type (static) control devices and plant will be examined next.

data, as the zero value of the time derivative ($\lambda \equiv 1$) is regarded to be the condition of the equilibrium. If at this workpoint — with the system being in rest — the variables Z and S_a which are independent from the viewpoint of the system, vary relative to their workpoint values, this fact necessarily involves the variation of the values of M and S, too. The question arises, how this dynamic operation evolves with time. This can only be answered by solving Eqs (1.2) for $Z(t)$ and $S_a(t)$. As mentioned before, this is a hard task because of the nonlinearity of functions (1.2) and often the analog modeling has to have recourse to. It is somewhat simpler to evaluate the system dynamics, if the variation of Z and S_a is little at the workpoint. In this case the movement over the nonlinear hypersurface can be approximated by a movement over a tangent plane.

This assumption permits no definite statement as to the system response to large variations, yet it is worthwhile to carry out the investigation if nothing but informatively.

Apply the Taylor approximation to equations (1.2) at the workpoints M_0; S_0; Z_0; S_{a0}, i.e.:

$$S(t) \cong S_0 + \overline{\frac{\partial \mathbf{f}}{\partial \mathbf{S}}}\bigg|_0 \cdot \overline{\mathbf{s}(t)} + \overline{\frac{\partial \mathbf{f}}{\partial \mathbf{Z}}}\bigg|_0 \overline{\mathbf{z}(t)} + \overline{\frac{\partial \mathbf{f}}{\partial \mathbf{M}}}\bigg|_0 \cdot \overline{\mathbf{m}}(t)$$

$$M(t) \cong M_0 + \overline{\frac{\partial \mathbf{g}}{\partial \mathbf{M}}}\bigg|_0 \cdot \overline{m}(t) + \overline{\frac{\partial \mathbf{g}}{\partial \mathbf{S}}}\bigg|_0 \overline{\mathbf{s}}(t) + \overline{\frac{\partial \mathbf{g}}{\partial \mathbf{S}_a}}\bigg|_0 \cdot \overline{\mathbf{s}}_a(t).$$

On this basis the system's "equation of motion" is

$$\begin{aligned} s(t)\left[1 - \overline{\frac{\partial \mathbf{f}}{\partial \mathbf{S}}}\bigg|_0 \cdot \overline{\lambda}_m\right] &= \overline{\frac{\partial \mathbf{f}}{\partial \mathbf{Z}}}\bigg|_0 \cdot \overline{\lambda}_l \cdot z(t) + \overline{\frac{\partial \mathbf{f}}{\partial \mathbf{M}}}\bigg|_0 \cdot \overline{\lambda}_n \cdot m(t) \\ m(t)\left[1 - \overline{\frac{\partial \mathbf{g}}{\partial \mathbf{M}}}\bigg|_0 \cdot \overline{\lambda}_j\right] &= \overline{\frac{\partial \mathbf{g}}{\partial \mathbf{S}}}\bigg|_0 \cdot \overline{\lambda}_k \cdot s(t) + \overline{\frac{\partial \mathbf{g}}{\partial \mathbf{S}_a}}\bigg|_0 \cdot \overline{\lambda}_\varepsilon \cdot s_a(t). \end{aligned} \tag{1.4}$$

Eq. (1.4) is a system of linear, constant coefficient differential equations describing the control system at the workpoint. This system of equations is to be solved for given time functions $Z(t)$ and $S_a(t)$, in order to get the system response functions $s(t)$ and $m(t)$ for $z(t)$ and $s_a(t)$.*

* The "free motion" of the control system in case of excitations $z(t) \equiv 0$, $s_a(t) \equiv 0$ is characterized by the expression

$$s(t)\left[1 - \overline{\frac{\partial \mathbf{g}}{\partial \mathbf{S}}}\bigg|_0 \cdot \overline{\lambda}_m\right] \cdot m(t)\left[1 - \overline{\frac{\partial \mathbf{g}}{\partial \mathbf{M}}}\bigg|_0 \cdot \overline{\lambda}_j\right] = \overline{\frac{\partial \mathbf{f}}{\partial \mathbf{M}}}\bigg|_0 \overline{\lambda}_n \cdot m(t) \cdot \overline{\frac{\partial \mathbf{g}}{\partial \mathbf{S}}}\bigg|_0 \cdot \overline{\lambda}_k \cdot s(t)$$

deriving from the product of Eq. (1.4). This equality must hold at every moment, i.e.

$$\left[1 - \overline{\frac{\partial \mathbf{f}}{\partial \mathbf{S}}}\bigg|_0 \cdot \overline{\lambda}_m\right] \cdot \left[1 - \overline{\frac{\partial \mathbf{g}}{\partial \mathbf{M}}}\bigg|_0 \cdot \lambda_j\right] = \overline{\frac{\partial \mathbf{f}}{\partial \mathbf{M}}}\bigg|_0 \cdot \overline{\lambda}_n \cdot \overline{\frac{\partial \mathbf{g}}{\partial \mathbf{S}}}\bigg|_0 \cdot \overline{\lambda}_k$$

must hold. This is the characteristic equation of the system.

From among the functions describing the system the characteristic equation is the most representative as to the dynamics. In our case this equation can be written as

$$\left[1-\frac{\overline{\partial \mathbf{f}}}{\partial \mathbf{S}}\bigg|_0 \cdot \bar{\boldsymbol{\lambda}}_m\right] \cdot \left[1-\frac{\overline{\partial \mathbf{g}}}{\partial \mathbf{M}}\bigg|_0 \cdot \bar{\boldsymbol{\lambda}}_j\right] - \left(\frac{\overline{\partial \mathbf{f}}}{\partial \mathbf{M}}\bigg|_0 \cdot \bar{\boldsymbol{\lambda}}_n\right) \cdot \left(\frac{\overline{\partial \mathbf{g}}}{\partial \mathbf{S}}\bigg|_0 \cdot \bar{\boldsymbol{\lambda}}_k\right) = 0$$

or

$$\left[1-\sum_0^m \frac{\partial f}{\partial S^{(i)}}\bigg|_0 \cdot \lambda^i\right] \cdot \left[1-\sum_0^j \frac{\partial g}{\partial M^{(i)}}\bigg|_0 \cdot \lambda^i\right] - \\ -\frac{\partial f}{\partial M}\bigg|_0 \cdot \frac{\partial g}{\partial S}\bigg|_0 \cdot \sum_0^n \frac{\left.\frac{\partial f}{\partial M^{(i)}}\right|_0}{\left.\frac{\partial f}{\partial M}\right|_0} \lambda^i \cdot \sum_0^k \frac{\left.\frac{\partial g}{\partial S^{(i)}}\right|_0}{\left.\frac{\partial g}{\partial S}\right|_0} \lambda^i = 0. \tag{1.5}$$

On the basis of this characteristic equation the problems of stability and damping conditions of the system can be analyzed.

3. Study of workpoint stability on the basis of the characteristic equation of the linearized system

Eq. (1.5) is of great importance for the analysis of control systems. Even the analysis of linear systems has led to a so-called characteristic equation, whose roots have determined the performance of the system. These roots appear in the exponent of time function of the force-free system. Their signs, real and imaginary parts determine the dynamic operation.

Eq. (1.5) was obtained as the characteristic equation of a nonlinear system, but because of the linearization this equation is valid for small variations neighbouring the workpoint.

Consider the coefficients of λ in Eq. (1.5)

As $\dim[\lambda^i] = \sec^{-i}$, $\dim[f] = \dim[S]$ $\dim[g] = \dim[M]$ and

$$\begin{aligned} \dim\,[T^i_{Si}] &= \sec^i \\ \dim\,[T^i_{Mi}] &= \sec^i \\ \dim\,[\tau^i_{Si}] &= \sec^i \\ \dim\,[\tau^i_{Mi}] &= \sec^i \\ \dim\,[\varkappa] &= 1 \end{aligned} \tag{1.6}$$

coefficients are numbers with the dimension $\sec^i$, denote them by T^i_{Si}, T^i_{Mi}, τ^i_{Si}, τ^i_{Mi}, $\varkappa$, respectively. Now Eq. (1.5) can be written as

$$\left(1+\sum_{0}^{n} T_{Si}^{i}\cdot\lambda^{i}\right)\left(1+\sum_{0}^{j} T_{Mi}^{i}\lambda^{i}\right)+\varkappa\sum_{0}^{m}\tau_{Mi}^{i}\lambda^{i}\cdot\sum_{0}^{k}\tau_{Si}^{i}\lambda^{i}=0. \tag{1.7}$$

The values of these coefficients are uniquely determined by the workpoint data S_0, M_0, Z_0, S_{a0} as well as the structure of the g and f functions. Beside the variables these functions include various system parameters, which are constant, dimensioned numbers, usually as coefficients or exponents. Denote these parameters by $a, b, c, \ldots$ and $\alpha, \beta, \gamma, \ldots$ in functions f and g, respectively. By this reason the coefficients T_s, τ_s, T_M, τ_M and $\varkappa$ are the functions of this system parameters and of the workpoint data, i.e.

$$\begin{aligned}
T_{Si}^{i} &= f_{Si}(a;\ b;\ c;\ \ldots\ldots\ldots\ldots\ldots\ S_0;\ M_0;\ Z_0;\ S_{a0})\\
\tau_{Si}^{i} &= f_{ki}(a;\ b;\ c;\ \ldots\ldots\ldots\ldots\ldots\ S_0;\ M_0;\ Z_0;\ S_{a0})\\
T_{Mi}^{i} &= f_{ji}(\alpha;\ \beta;\ \gamma;\ \ldots\ldots\ldots\ldots\ldots\ S_0;\ M_0;\ Z_0;\ S_{a0})\\
\tau_{Mi}^{i} &= f_{mi}(\alpha;\ \beta;\ \gamma;\ \ldots\ldots\ldots\ldots\ldots\ S_0;\ M_0;\ Z_0;\ S_{a0})\\
\varkappa &= f(a;\ b;\ c;\ \ldots\ \alpha;\ \beta;\ \gamma;\ \ldots\ S_0;\ M_0;\ Z_0;\ S_{a0})
\end{aligned} \tag{1.8}$$

In linear control systems, as the functions (1.2) are given in terms of linear differential equation, the coefficients T, τ and $\varkappa$ are independent of the workpoint data S_0, M_0, Z_0 and S_{a0}.

Therefore the array of the characteristic equation's zeros for these systems can be plotted by the known root-locus methods where some $a, b, c, \ldots$ or $\alpha, \beta, \gamma \ldots$ type system parameters are taken as variables. Its influence on the damping and the stability for the variation of the parameter over some region is shown on the root-locus curve.

4. Nonlinear system root-locus curve

The coefficients of Eq. (1.8) suggest the idea of expressing the physical parameters of the system static workpoint as functions of the independent variables S_{a0} and Z_0, because in the equilibrium workpoint (1.2) can be given in form of

$$M_0 = g(M_0,\ S_0,\ S_{a0})$$

$$S_0 = f(S_0,\ Z_0,\ M_0)$$

so that

$$M_0 = g_*(S_{a0},\ Z_0)$$

$$S_0 = f_*(S_{a0},\ Z_0).$$

This operation in (1.8) would mean that coefficients T, τ and K are, besides being dependent on the system parameters, functions of only the independent variables S_{a0} and Z_0. The following question can be raised. Assuming constant $a, b, c, \ldots$ and $\alpha, \beta, \gamma, \ldots$ system parameters how the zero array of the characteristic equation of a nonlinear system will vary as a function of the independent system variables, and what kind of zero array will characterize the nonlinear system at the workpoints corresponding to various equilibrium states. This question can be answered by plotting a conventional root-locus curve, which curve, provided a constant basic value (constant value control), gives the root array of linearized characteristic of the nonlinear system over the $Z_{\min} < Z_0 < Z_{\max}$ variation range of the independent variable Z_0. So essentially the root-loci of the characteristic equation (1.7) written as

$$\sum_{0}^{N} A_i(a; b; c; \ldots \alpha; \beta; \gamma; \ldots S_{a0}; Z_0) \cdot \lambda^i = 0 \tag{1.9}$$

are plotted on the complex plane, while for $S_{a0} =$ const Z_0, or for $Z_0 =$ const S_{a0} varies over an interval prescribed by the system operation, and the system parameters $a, b, c, \ldots$ and $\alpha, \beta, \gamma, \ldots$ are of constant value.

The solution of (1.9) for λ does not mean any special difficulties as even relatively small digital computers have corresponding subroutines for solving equation of (1.9) type up to several hundreds of order. Some problem can arise from the repetitive computation, because the coefficients of (1.9) are usually complicated irrational functions of Z_0.

5. Example

A circuit diagram and influence scheme are shown in Figs 2 and 3. The relationship between the physical variables of a winding-up machine drive is illustrated by the influence scheme. The aim of control in this case is to ensure a static functionality $Z_T \cdot \Omega =$ const between the load torque Z_T and the angular velocity. This problem could be reduced to control the system to hold $(1 - \varrho)I^2\Omega$ as constant.

This was done by measuring the signal $(1 - \varrho)I^2\Omega$, comparing its value with the required value S_a, and feeding the difference signal into an integrator. The integrator output alters U_A as long as the $S_{a0} - (1 - \varrho)I_0^2\Omega_0 = 0$ is established at its input. In this equilibrium state a zero signal must appear at the input of the integrator with Ω output, that is

$$S_{a0} = (1 - \varrho)\, Z_{T0}\Omega_0$$

or

$$Z_{T0}\Omega_0 = \text{const} \tag{F1}$$

i.e. the system meets the requirements as to the static properties. The other functionalities in the influence scheme result from the fact that in order to relieve the controller and to permit manual control the drive was supplied by a magnetic amplifier. Its output voltage — as a consequence of the negative current feedback — linearly decreases when the current increases. Thus $Z_T \cdot \Omega =$ const can be reached without control by a properly set feedback.

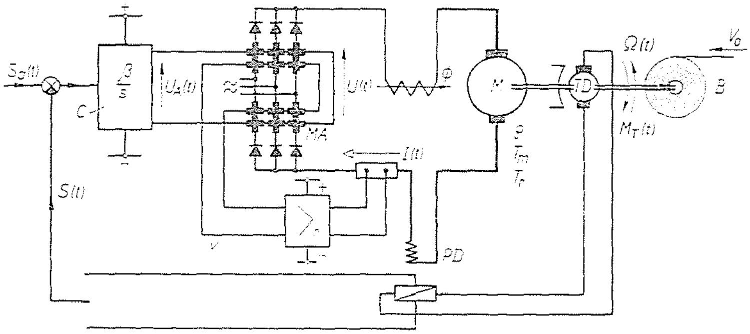

Fig. 2. Drive control of a winding-up machine. C: Control amplifier; MA: Magnetic power amplifier; M: Externally excited DC motor; TD: Tachometer generator; PD: Power detector; V: Feedback circuit; B: Wound up bale

The nonlinearities shown in the influence scheme are due to the operational properties of the series motor. The other effects due to other possible nonlinearities (as saturation and hysteresis of the magnetic amplifier, saturation of other amplifiers nonlinearities in the excitation and armature circuits of the motor, etc.) are disregarded in this example.

The normalized basic equations of the system are (5):

$$
\begin{aligned}
U(t) - \varrho\, I(t) - \varrho\, T_v \frac{dI(t)}{dt} - (1-\varrho) I(t) \Omega(t) &= 0 \\
I^2(t) - Z_T(t) - \frac{1-\varrho}{\varrho}\, T_m \frac{d\Omega(t)}{dt} &= 0 \\
U_A(t) - \varrho r I(t) - U(t) - T \frac{dU(t)}{dt} &= 0 \qquad \text{(F2)} \\
S(t) - (1-\varrho)\, I^2(t) \cdot \Omega(t) &= 0 \\
U_A(t) - \beta \int [S_a(t) - S(t)]\, dt &= 0
\end{aligned}
$$

In this nonlinear system of equations r, ϱ; T_v, T_m, β and T can be regarded as constant system parameters. The dependent variables are $U(t)$, $S(t)$, $I(t)$, $\Omega(t)$ and $U_A(t)$, while $S_a(t)$, $Z_T(t)$ are independent variables. The system operation condition can be described by $S_a(t) = S_{a0} =$ const, while the load torque excurses the interval $1 < Z_{T0} < 4$.

From the linearization of the system the characteristic equation can be gained at the workpoint $(Z_{T0}, S_{a0}, \Omega_0, I_0, S_0, U_{A0}, U_0)$ as

$$T_v T_m T\lambda^4 + \left[T_v T_m + \left(1 + \frac{1-\varrho}{\varrho}\Omega_0\right) T T_m\right]\lambda^3 + \left[2\,TI_0^2 + \left(1 + r + \frac{1-\varrho}{\varrho}\Omega_0\right) T_m\right]\lambda^2 + \left(2I_0^2 + 2\,\beta T_m \frac{1-\varrho}{\varrho}\Omega_0 I_0\right)\lambda + 2\,\beta I_0^3 = 0. \tag{F3}$$

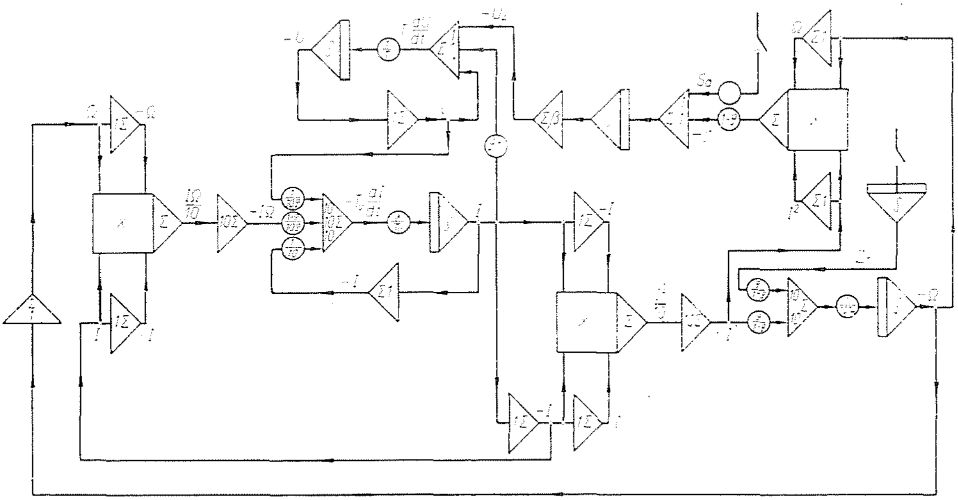

Fig. 3. Power control influence scheme of the winding-up machine

As from (Fl)

$$I_0^2 = Z_{T0}$$

$$(1-\varrho) I_0^2 \Omega_0 = S_{a0}$$

we get

$$I_0(S_{a0}; Z_{T0}) = \sqrt{Z_{T0}}$$

$$\Omega_0(S_{a0}; Z_{T0}) = \frac{S_{a0}}{(1-\varrho) Z_{T0}}$$

therefore (F3) can be written as

$$A_4(S_{a0}; Z_{T0})\lambda^4 + A_3(S_{a0}; Z_{T0})\,\lambda^3 + A_2(S_{a0}; Z_{T0})\,\lambda^2 + A_1(S_{a0}; Z_{T0})\lambda + A_0(S_{a0}; Z_{T0}) = 0. \tag{F4}$$

The coefficients in (F4) are

$$
\begin{aligned}
A_4(S_{a0};\, Z_{T0}) &= a \\
A_3(S_{a0};\, Z_{T0}) &= b + c\left(1 + \alpha \frac{S_{a0}}{Z_{T0}}\right) \\
A_2(S_{a0};\, Z_{T0}) &= eZ_{T0} + f\left(1 + d\frac{S_{a0}}{Z_{T0}}\right) \\
A_1(S_{a0};\, Z_{T0}) &= gZ_{T0} + h\frac{S_{a0}}{\sqrt{Z_{T0}}} \\
A_0(S_{a0};\, Z_{T0}) &= i\sqrt[3]{Z_{T0}^2}
\end{aligned}
\tag{F5}
$$

They can be interpreted as functions of the torque and the required value S_a ($a, b, c, d, e, f, g, h, i$ are constants).

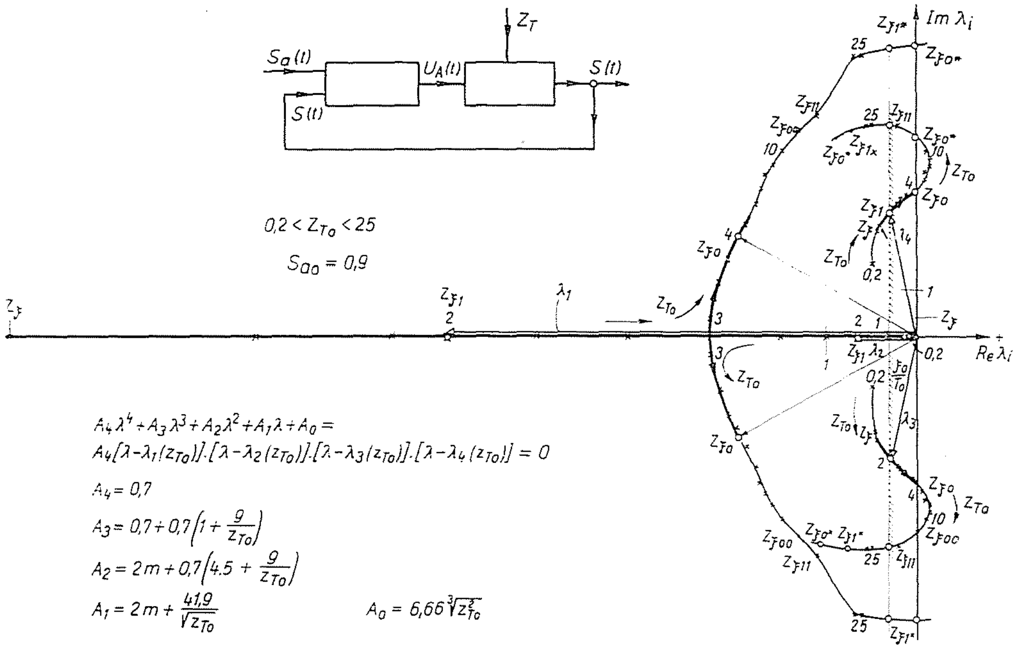

Fig. 4. Root array of the characteristic equation in the case of a torque disturbing signal varying in the range $0.2 < Z_{T_0} < 25$ with a constant basic signal S_{a_0}

For the parameters $[r, \varrho, T_\nu, T_m, T, \beta] = [3.5, 0.1, 1.0, 0.7, 1.0, 3.33]$, $S_{a0} = 0.9$ constant, the root array of the characteristic equation (F5) for the range $0.2 < Z_{T0} < 25$ was computed as a function of Z_{T0}. Computation was carried out on the digital computer ODRA 1013 at the Department of Process Control, Polytechnical University of Budapest (Fig. 4). From the root-locus curve, as it can be seen, for $S_{a0} = 0.9$ and increasing disturbing signal Z_{T0}, the system runs through various workpoints which results in different damping conditions (of course for small input variations). If the torque is in the ranges $Z_\xi < Z_{T0} < Z_{\xi 1}$ and $Z_{\xi 11} < Z_{T0} < Z_{\xi 1*}$ the damping of the system response is $1/T_{cs} > \xi_0/T_0$, if $Z_{\xi 1} < Z_{T0} < Z_{\xi 0}$ and $Z_{\xi 1*} < Z_{T0} < Z_{\xi 0*}$ the damping is $\xi_0/T_0 > 1/T_{cs} > 0$, and if $Z_{\xi 0} \leqq Z_{T0} \leqq Z_{\xi 0*}$ or $Z_{T0} > Z_{\xi 0*}$ the response is not damped and an oscillation with increasing amplitude will arise.

The basic requirement in the design of this system was to form a parameter system $[r, \varrho, T_v, T_m, T; \beta]$ that the $1 < Z_{T0} < 4$ operational torque range lies on that part of the root-locus curve, where $1/T_{cs} > \xi_0/T_0$ holds, as in this case neither stability nor damping problems will arise.

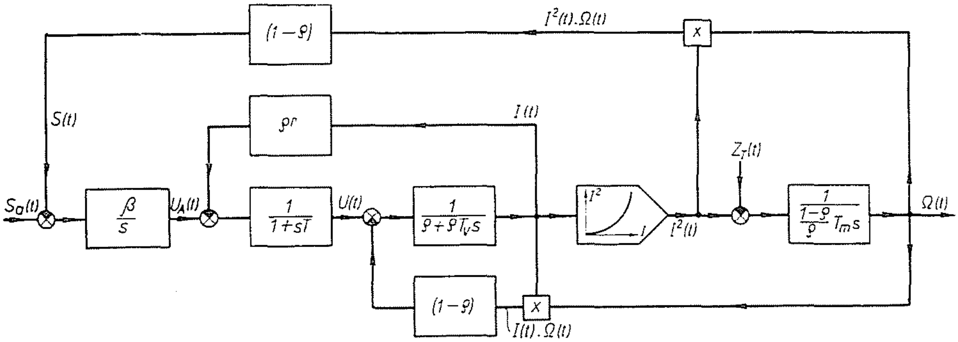

Fig. 5. Scheme of an analog computer program for power control

To check for the damping conditions of the system the equations (F2) were also programmed on an eighty-amplifier analog computer of the Process Control Department. Fig. 5 shows the block diagram. $S_{a0} = 0.9$ and $Z_{T0} = 1$ were set as workpoint values in the static system. The load torque Z_T was varied linearly with a slight slope, so that the variation due to the system's own dynamics took place during the variation of 1% in Z_T. The static curves $\Omega_0(Z_{T0})$ and $S_0(Z_{T0})$ are plotted in Fig. 6a. Figs 6b and 6c show the time functions $\Omega(t)$, when a $Z_{T0} = 1(t)$ step was fed into the system at the workpoints 1, 2, 3 and 4 (Fig. 6b) and when starting from the workpoint 1 the system was excited by $1(t)$, $2.1(t)$, $3.1(t)$, $4.1(t)$ and $7.1(t)$ torque steps. Comparing these time functions with the root-locus curve of Fig. 4 identical results were obtained for the damping conditions.

6. Conclusions

The dynamics of a linear control system with the open-loop transfer function $W(s) = G(s)/H(s)$ can be characterized by the characteristic equation

$$G(s) + H(s) = 0$$

$$K \sum_0^m \tau_i^i s^i + \sum_0^m T_i^i s^i = 0$$

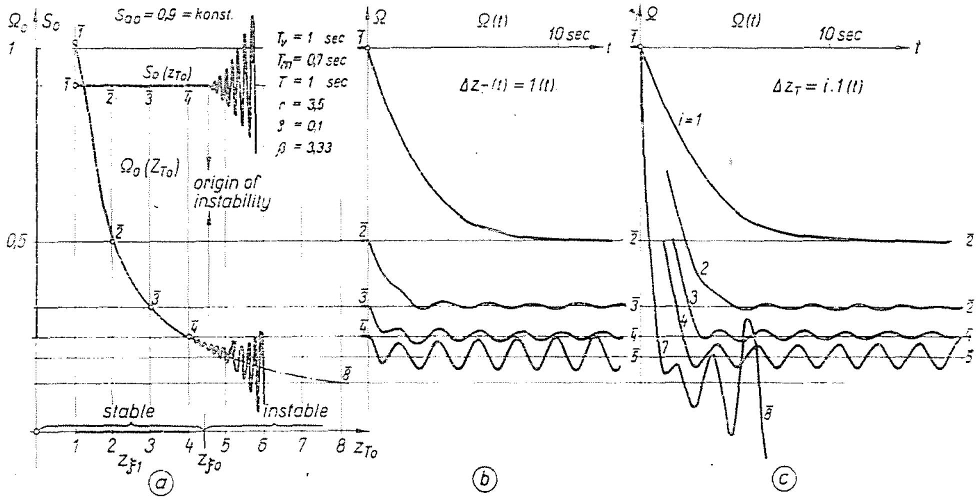

Fig. 6. Dependence of the controlled variable (S) and an internal signal (Ω) of a nonlinear control system on the static disturbing signal *(a)* and its angular velocity response to "small" *(b)* and "large" *(c)* variations of the disturbing signal in separate workpoints

When this equation is written in the form of

$$\sum_{0}^{N} A_i \, s^i = 0$$

the coefficient A_i is a function of the parameters of time constant type and of the loop gain

$$A_i = A_i(T_1, T_2, \ldots, T_n, \tau_1, \tau_2, \ldots, \tau_m, K) .$$

The root-locus curve of the system plotted versus a system parameter (generally the loop gain) shows the stability and damping conditions of the closed-loop system.

The characteristic equation of a nonlinear system with given mathematical structure is valid in the neighbourhood of the workpoint and is significant for the dynamic properties appearing in this region only. Therefore, the characteristic equation written in the form:

$$\sum_{0}^{N} A_i \cdot \lambda^i = 0$$

has different coefficients A_i at various workpoints, i.e. A_i depends beside on the system parameters on independent variables S_{a0}, Z_0, which determine the workpoint values.

$$A_i = A_i(a, b, c, \ldots, \alpha, \beta, \gamma, \ldots, S_{a0}, Z_0).$$

Summary

Assuming constant system parameters a nonlinear system can be characterized by a root-locus curve, which shows the influence of the variation of the disturbing signal in the case of varying or constant basic value on the root-locus plot of the system. The characteristic equation and root-locus curve give the workpoint damping ratio at various workpoints of the system.

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