STABILITY TEST OF LINEAR CONTROL SYSTEMS WITH DEAD TIME BY DIGITAL COMPUTER

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In a previous paper [1] we investigated the possibilities of determining the stability region variations of a linear control system with dead time and second order lag as shown in Fig. 1. The investigations were made in function of the dead time and the system time constants. We have derived that the transcendental equation determining ω_{cr}, the angular frequency belonging to the critical loop gain is, in the case of a PID compensation, as follows:

$$\begin{aligned}(T^2 T_i \omega_{cr}^3 - \omega_{cr} T_i + 2\zeta T\omega_{cr} - 2\zeta T\omega_{cr}^3 T_d T_i) \operatorname{tg} \omega\tau = \\ = 2\zeta T T_i \omega_{cr}^2 - T^2 \omega_{cr}^2 + 1 + \omega^4 T_d T_i T^2 - \omega_{cr}^2 T_d T_i .\end{aligned} \tag{1}$$

Control systems of the types P, I, PI, PD may be calculated as special cases of the above control system. The critical loop gain K_{cr} with arbitrarily accurate approximation values of ω_{cr} may be determined by a simple algebraic equation either by PONTRYAGIN's method or by the use of the NYQUIST stability criterion.

This paper presents graphs showing the stability region variations in function of the dead time and the system time constants for control systems of types P and I with values of K_{cr} obtained with the help of a digital computer.

1. Proportional control

The transcendental equation determining the angular frequency belonging to K_{cr} with the assumption of $T = 1$ is

$$2\zeta\omega_{cr} + (1 - \omega^2) \operatorname{tg} \omega_{cr} \tau = 0 . \tag{2}$$

Fig. 2 shows the values of K_{cr} *vs.* τ/T with ζ as parameter in a diagram of log-log scale for sake of clearness, demonstrating that

a) with increasing ζ the stability region increases,

b) for $\tau/T \to 0$, $K_{cr} \to \infty$,

c) with increasing τ/T K_{cr} sharply diminishes, for $\tau/T \to \infty$, $K_{cr} \to 1$.

The results are up to expectations. With increasing ζ the tendency for fluctuation diminishes. For very low values of τ/T the control may be regarded to be a pure second order lag system, which is structurally stable with any loop gain. On the other hand, for high values of τ/T the control may be substituted by a pure dead time system, for which the stability limit is $K_{cr} = 1$.

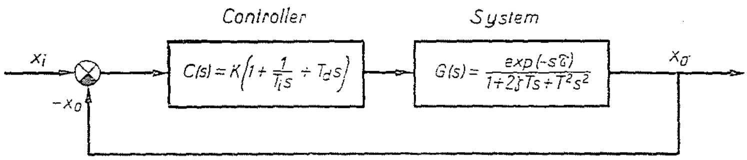


Fig. 1

2. Integral control

With the assumption of $T_i = 1$ the transcendental equation determining ω_{cr} is:

$$1 - T^2\omega_{cr}^2 - 2\zeta T\omega_{cr}\tan\omega_{cr}\tau = 0. \tag{3}$$

Figs 3—6 show the values of K_{cr} for $\tau/T = 0.001,\ 0.1,\ 1,\ 5,\ 10,\ 100$ vs. τ/T_i

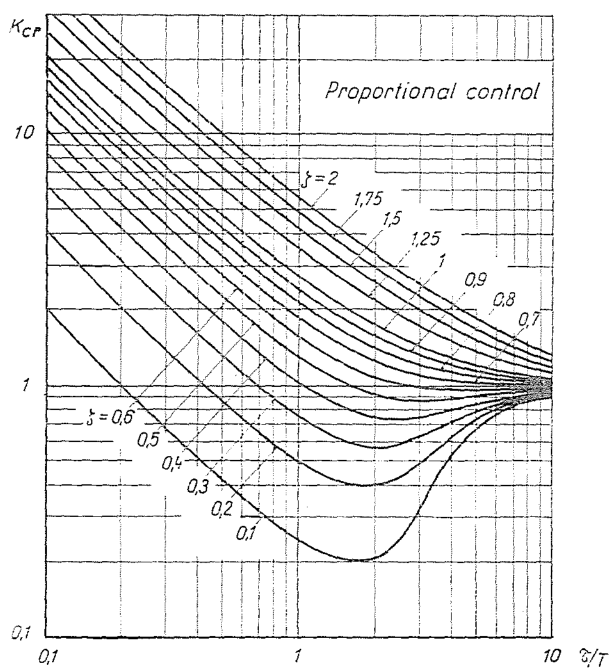


Fig. 2

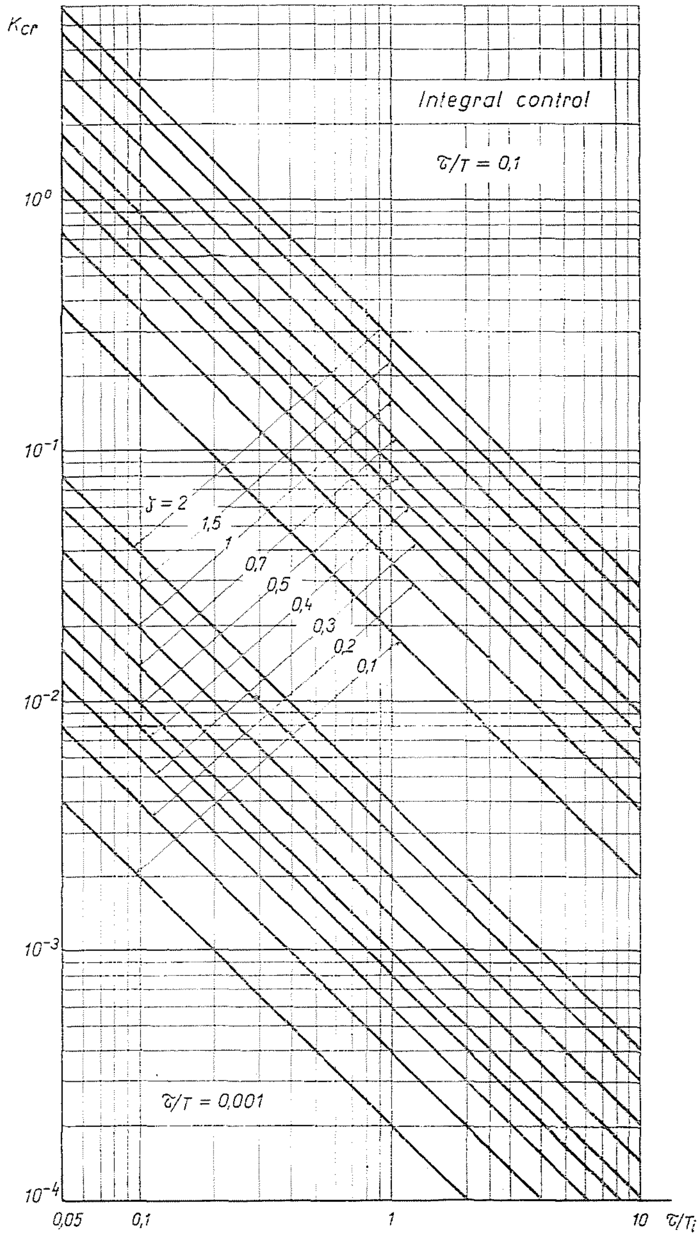

Kcr
Integral control
τ/T = 0,1
10^0
10^-1
10^-2
10^-3
10^-4
ζ = 2
1,5
1
0,7
0,5
0,4
0,3
0,2
0,1
τ/T = 0,001
0,05
0,1
1
10
τ/Ti

Fig. 3

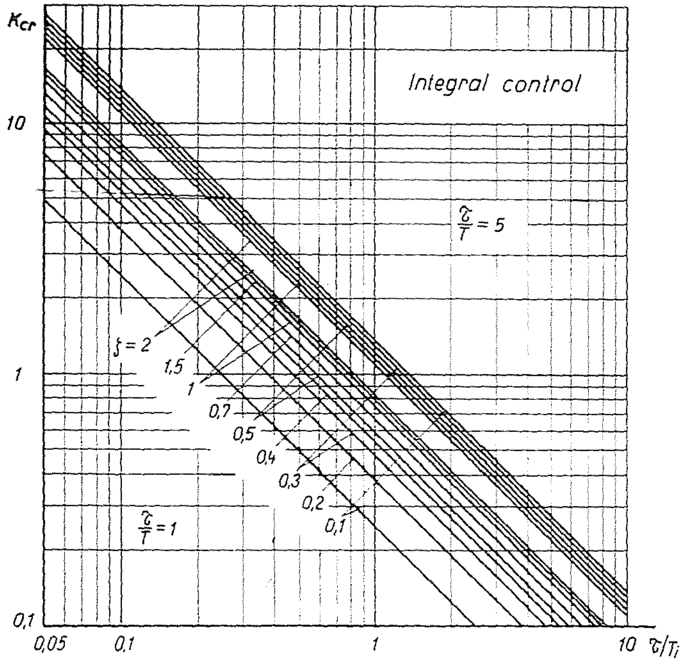

Integral control
K_{cr}
10
1
0,1
0,05
0,1
1
10
τ/T_i
$\frac{\tau}{T} = 5$
$\frac{\tau}{T} = 1$
$\zeta = 2$
1,5
1
0,7
0,5
0,4
0,3
0,2
0,1

Fig. 4

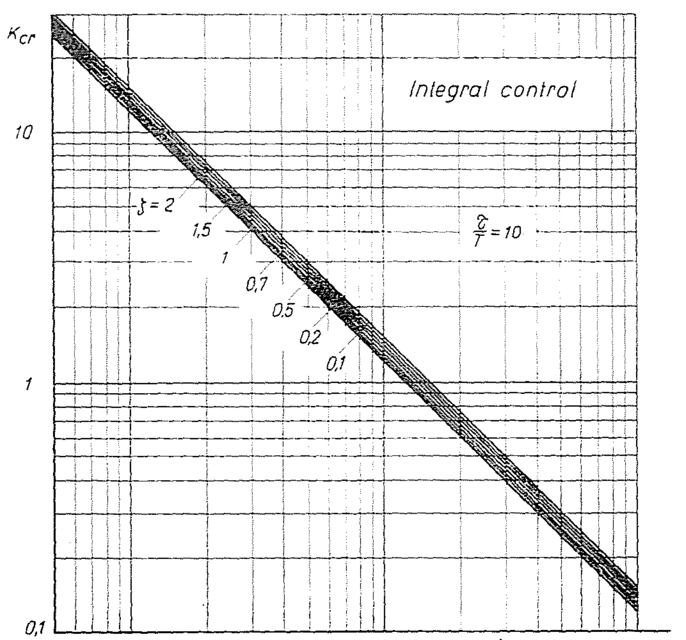
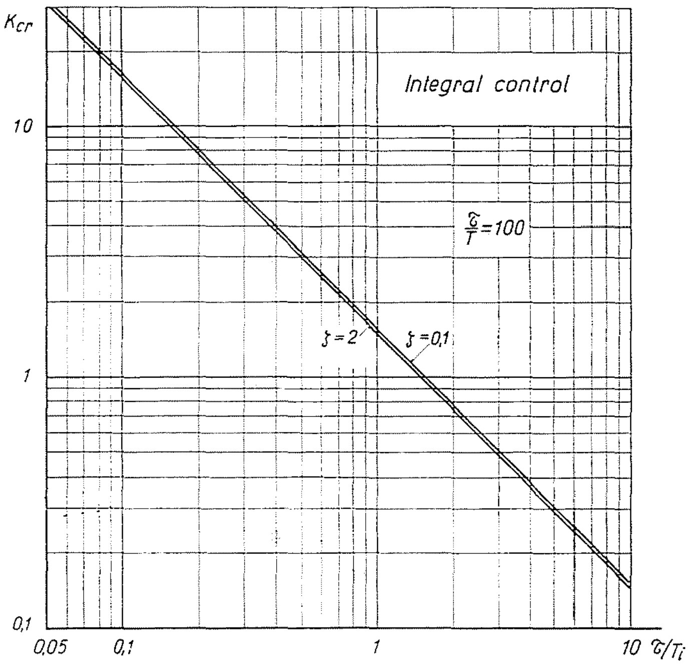
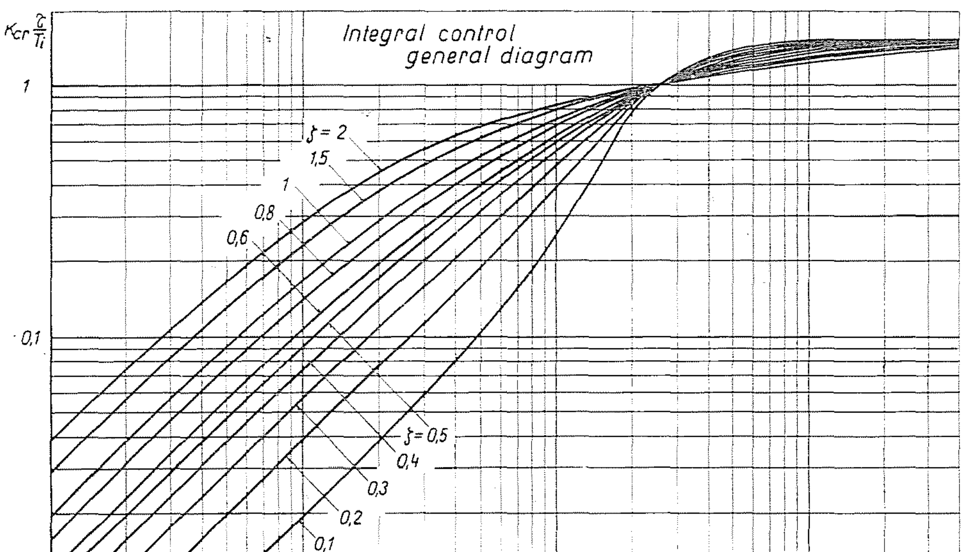

Integral control
K_{cr}
10
1
0,1
0,05
0,1
1
10
τ/T_i
$\frac{\tau}{T} = 10$
$\zeta = 2$
1,5
1
0,7
0,5
0,2
0,1

Fig. 5



K_{cr}
Integral control
$\frac{\tau}{T}=100$
$\zeta=2$
$\zeta=0{,}1$
10
1
0,1
0,05
0,1
1
10 τ/T_i

Fig. 6



$K_{cr}\frac{\tau}{T_i}$
Integral control
general diagram
$\zeta=2$
1,5
1
0,8
0,6
$\zeta=0{,}5$
0,4
0,3
0,2
0,1
1
0,1
0,01
0,01
0,1
1
10
τ/T

Fig. 7

with ζ as parameter, in log-log scale for sake of clearness. The figures demonstrate that

a) with increasing τ/T the stability increases,

b) below a given value of $1 < \tau/T < 5$ the stability region increases with increasing ζ, over this value it diminishes.

As the function $K_{cr} = K_{cr}\,(\tau/T_i)$ plotted in a log-log scale is linear for any ζ, a general graph may be prepared, giving the values of K_{cr} for any arbitrary parameter value.

The stability limit from the resultant transfer function of the open loop system is:

$$Y(s) = K_{cr}\,\frac{e^{-s\tau}}{sT_i\,(1 + 2\,\zeta\,Ts + T^2\,s^2)} = -\,1\,.$$

After transformation we have:

$$K_{cr}\,\frac{\tau}{T_i} = -\,s\tau\,\frac{1 + 2\,\zeta\,T/\tau(s\tau) + (T/\tau)^2\,(s\tau)^2}{e^{-\,s\tau}} = A\,(\zeta\,;\,T/\tau)\,. \tag{4}$$

Consequently, plotting a value of K_{cr} belonging to some ratio τ/T_i versus τ/T with ζ as parameter, K_{cr} belonging to any arbitrary ratio τ/T_i can be evaluated. Supplementary data for the more accurate plotting of the graph may also be determined, the simplest case being $\tau = T_i = 1$.

Fig. 7 shows the set of curves

$$K_{cr}(\tau = 1\,;\; T_i = 1) = K_{cr}(\zeta;\,\tau/T) \tag{5}$$

in a log-log scale again. Table 1 gives the corresponding numerical values with mantissas rounded off to an accuracy of three decimals and with characteristics provided with signs. So if the critical loop gain read off the table is (569 $-$1), this corresponds to $K_{cr} = 0.0569$.

From this general graph the following general laws for linear integral control systems with second order lag and dead time can be drawn.

a) The critical loop gain depends but slighthy on the value of ζ in the vicinity of $\tau/T \sim 2.6$. For increasing ζ, the stability range increases, when $\tau/T < 2.6$ and diminishes, when $\tau/T > 2.6$,

b) when $\tau/T \to 0$, then $K_{cr}\tau/T_i \to 2\,\zeta\,\tau/T$,

c) when $\tau/T \to \infty$, then $K_{cr}\tau/T_i \to \pi/2$.

The correctness of the last two statements is obvious.

For $\tau/T < 100$ the control system may be substituted by a second order lag, integral control system. From the transfer function of this latter the stability limit is:

$$Y(s) = K\,\frac{1}{sT_i\,(1 + 2\,\zeta\,Ts + T^2\,s^2)} = -\,1\,.$$

With the substitution $s = j\omega$ we obtain, from the phase angle condition $\varphi(\omega) = -\pi$, after a simple transposition: $\omega_{cr} = 1/T$. Replacing this into the equation $Y(j\omega) = -1$, we have:

$$K_{cr}\,\tau/T_i = -j\omega\,\tau(1 - T^2\omega^2 + j\,2\,\zeta\,T\omega)\big|_{\omega_{cr}=1/T} = 2\,\zeta\,\tau/T\,.$$

On the other hand, in the case of $\tau/T > 100$ the second order lag character may be neglected in comparison to the dead time. Therefore the preliminary condition of the stability is the validity of the relationship

$$Y(s) = K\,\frac{e^{-s\tau}}{sT_i} = -1\,.$$

This condition is equivalent to the following equation:

$$K_{cr}\,\tau/T_i = -\,j\omega\tau\,(\cos\omega\tau + j\sin\omega\tau)\,.$$

This is valid only in the case, when $\omega\tau = \pi/2$, in which case we have, in fact:

$$K_{cr}\,\tau/T_i = \pi/2\,.$$

According to (6), K_{cr} of an arbitrary integral control system with second order lag, and dead time may be determined by multiplying the value $K_{cr1} = K_{cr}$ $(T_i = 1,\ \tau = 1)$ belonging to given values of ζ, τ/T as read off Fig. 7 by an arbitrary value T_i/τ. For instance let be $\tau/T = 1$, $\zeta = 0.5$. On the basis of Fig. 7 the corresponding value is $K_{cr} = 0.586$. Hence, if e.g. $\tau = 2$, then $K_{cr} = \frac{0.586}{2} = 0.293$, as seen also in Fig. 4.

Conclusion

As a conclusion it can be stated that both the proportional and the integral control calculations are practically suitable. Based on approximate measurements using an analogue computer (Fig. 8 of Ref. [9], Part D, Chapter 10, pp. 10—12), gives the advantageous compensations to be used with a second order lag and dead time system, in function of the dead time. Accordingly, for low dead time values a proportional compensation is preferable, whereas for high dead time values the integral type control proves to be best. This conclusion follows unambiguously also from the stability region

Table 1
Integral control
Values of $K_{cr1} = K_{cr}(\tau = 1;\ T_i = 1)$

τ/T \ ζ	0.1		0.2		0.3		0.4		0.5		0.6		0.7	
0.01	200	−2	398	−2	596	−2	793	−2	990	−2	119	−1	138	−1
0.015	299	−2	596	−2	892	−2	119	−1	148	−1	177	−1	206	−1
0.02	398	−2	794	−2	119	−1	158	−1	196	−1	234	−1	272	−1
0.03	597	−2	119	−1	177	−1	234	−1	291	−1	348	−1	403	−1
0.05	991	−2	196	−1	292	−1	385	−1	477	−1	567	−1	655	−1
0.06	119	−1	235	−1	348	−1	459	−1	567	−1	673	−1	776	−1
0.08	158	−1	311	−1	459	−1	603	−1	743	−1	878	−1	101	+0
0.10	197	−1	386	−1	569	−1	744	−1	913	−1	108	+0	123	+0
0.20	392	−1	754	−1	109	+0	140	+0	169	+0	196	+0	222	+0
0.30	590	−1	111	+0	158	+0	200	+0	238	+0	272	+0	303	+0
0.50	101	+0	183	+0	251	+0	307	+0	356	+0	398	+0	434	+0
0.80	180	+0	298	+0	384	+0	450	+0	502	+0	545	+0	581	+0
1.00	248	+0	382	+0	472	+0	537	+0	586	+0	626	+0	658	+0
1.25	355	+0	495	+0	579	+0	636	+0	679	+0	713	+0	739	+0
1.50	487	+0	610	+0	680	+0	726	+0	761	+0	787	+0	808	+0
1.75	628	+0	720	+0	772	+0	807	+0	832	+0	851	+0	866	+0
2.00	761	+0	821	+0	855	+0	878	+0	894	+0	906	+0	916	+0
2.25	878	+0	909	+0	928	+0	940	+0	948	+0	955	+0	960	+0
2.50	974	+0	985	+0	990	+0	994	+0	996	+0	997	+0	998	+0
2.75	105	+1	105	+1	104	+1	104	+1	104	+1	104	+1	103	+1
3.00	112	+1	110	+1	109	+1	108	+1	107	+1	107	+1	106	+1
3.50	122	+1	119	+1	117	+1	115	+1	114	+1	112	+1	111	+1
4.00	129	+1	125	+1	122	+1	120	+1	119	+1	117	+1	116	+1
5.00	137	+1	134	+1	130	+1	130	+1	126	+1	124	+1	122	+1
6.00	142	+1	139	+1	136	+1	133	+1	131	+1	129	+1	127	+1
8.00	148	+1	145	+1	142	+1	139	+1	137	+1	135	+1	133	+1
10.00	150	+1	148	+1	145	+1	143	+1	141	+1	139	+1	138	+1
15.00	153	+1	151	+1	150	+1	148	+1	146	+1	145	+1	144	+1
20.00	155	+1	153	+1	152	+1	150	+1	149	+1	148	+1	147	+1
50.00	156	+1	156	+1	155	+1	155	+1	154	+1	153	+1	153	+1
100.00	157	+1	156	+1	156	+1	156	+1	156	+1	155	+1	155	+1

Table 1 (continued)

ζ / τ/T	0.8		0.9		1		1.25		1.5		1.75		2.00	
0.01	157	−1	177	−1	196	−1	244	−1	291	−1	338	−1	385	−1
0.015	234	−1	263	−1	291	−1	361	−1	431	−1	499	−1	566	−1
0.02	310	−1	348	−1	385	−1	476	−1	566	−1	654	−1	741	−1
0.03	458	−1	512	−1	566	−1	698	−1	826	−1	951	−1	107	+0
0.05	742	−1	827	−1	910	−1	111	+0	131	+0	149	+0	167	+0
0.06	877	−1	976	−1	107	+0	131	+0	153	+0	174	+0	194	+0
0.08	114	+0	126	+0	138	+0	167	+0	194	+0	219	+0	243	+0
0.10	138	+0	153	+0	167	+0	201	+0	232	+0	260	+0	287	+0
0.20	246	+0	268	+0	289	+0	337	+0	379	+0	415	+0	448	+0
0.30	332	+0	356	+0	383	+0	436	+0	481	+0	520	+0	553	+0
0.50	467	+0	495	+0	521	+0	575	+0	618	+0	653	+0	682	+0
0.80	611	+0	637	+0	660	+0	706	+0	741	+0	768	+0	790	+0
1.00	685	+0	708	+0	728	+0	767	+0	796	+0	819	+0	837	+0
1.25	761	+0	780	+0	796	+0	826	+0	849	+0	866	+0	880	+0
1.50	824	+0	838	+0	857	+0	874	+0	891	+0	903	+0	914	+0
1.75	878	+0	888	+0	896	+0	913	+0	925	+0	934	+0	941	+0
2.00	924	+0	930	+0	936	+0	946	+0	954	+0	959	+0	964	+0
2.25	964	+0	967	+0	970	+0	975	+0	979	+0	982	+0	984	+0
2.50	999	+0	100	+1	100	+1	100	+1	100	+1	100	+1	100	+1
2.75	103	+1	103	+1	103	+1	102	+1	102	+1	102	+1	102	+1
3.00	106	+1	105	+1	105	+1	104	+1	104	+1	104	+1	103	+1
3.50	111	+1	110	+1	109	+1	108	+1	107	+1	106	+1	106	+1
4.00	115	+1	114	+1	113	+1	111	+1	110	+1	109	+1	108	+1
5.00	121	+1	120	+1	118	+1	116	+1	114	+1	113	+1	112	+1
6.00	125	+1	124	+1	123	+1	120	+1	120	+1	116	+1	115	+1
8.00	132	+1	130	+1	129	+1	126	+1	124	+1	122	+1	120	+1
10.00	136	+1	135	+1	133	+1	130	+1	128	+1	126	+1	124	+1
15.00	142	+1	141	+1	140	+1	137	+1	135	+1	133	+1	131	+1
20.00	146	+1	145	+1	144	+1	141	+1	139	+1	137	+1	135	+1
50.00	152	+1	152	+1	151	+1	150	+1	149	+1	147	+1	146	+1
100.00	155	+1	154	+1	154	+1	153	+1	153	+1	152	+1	151	+1

graphs in our Figs 2 and 7 showing proportional and integral types of compensation respectively.

With regard to the frequent use of PI and PID type compensations in control technics, the investigation of these will be dealt with in a coming paper.

Summary

The present paper investigates the stability region variations of linear control system with second order lag and dead time, compensated by P and I type components with the help of data evaluated by a digital computer. The obtained critical loop gain values are plotted in log-log diagrams for the sake of clearness, concerning dead time values of $0 < \tau < 10$ in the case of P type controls and $0 < \tau < \infty$ in the case of I type controls with the system time constants as parameters.

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