THE IMPEDANCE OF RECTANGULAR CONDUCTORS IN GROOVES OF ELECTRIC MACHINES

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(Received August 12, 1968)

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1. Introduction

The discussion of the skin effect in rectangular conductors in the grooves of electric machines can be found in the technical literature. Machine dimensioning has been based on the results of EMDE [1] and ROGOWSKI [2] as a rule which can be criticized from various aspects as has earlier been shown by FODOR [3]. The theory for solving the problem was amended by FODOR, without meeting, however, the boundary conditions. For the case of a single conductor layer STEIDINGER [4] established a solution satisfying the boundary conditions. Since his results can be used for practical calculations only with difficulties, they have not become common knowledge.

In the present paper the problem is discussed on the basis of the solution by STEIDINGER. It has been attempted to present the deduction in a well arranged form and to make results easily manageable during practical calculations. On the basis of the presented results the numerical error of the usual solution can also be estimated.

2. Evaluation of known solutions

In the following the conditions of the arrangement shown in Fig. 1 are examined. In the rectangular section slot of width $2a$ cut in the ferromagnetic material a conductor of similarly rectangular section, of width $2b$ and height h, of specific conductivity σ is arranged, conducting the sinusoidally changing current I. The lines of magnetic field produced by the current are closed through the ferromagnetic material. The permeability of this material is assumed to be infinitely high. An isolating layer of width $a-b$ is built in between the conductor and the ferromagnetic material on both sides. In our calculations the system of coordinates shown in the figure is used. In the following the theories of EMDE, FODOR, and STEIDINGER are briefly surveyed.

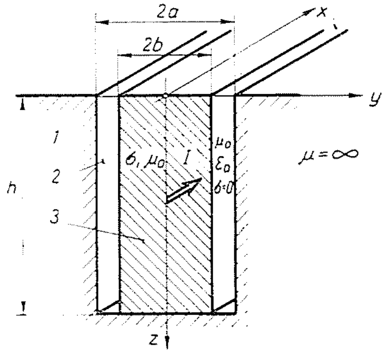

Fig. 1

2.1. *The solution by Emde*

In the solution by EMDE the electric field has only components in the directioe x, while the magnetic field only in the direction y. The field is a function exclusively of the coordinate z, i.e. the derivatives with respect to x and y are equal to zero.

We obtain from the integral relationships of electrodynamics valid for the quasi-stationary case [5] the equations

$$\frac{d^2 H}{dz^2} = \frac{b}{a} j\omega\mu_0 \sigma H ,$$
$$\frac{d^2 E}{dz^2} = \frac{b}{a} j\omega\mu_0 \sigma E , \tag{1}$$

forming the basis of our further calculations. Against these the following objections may be raised:

From the differential form of Maxwell's equations the following relationships can be deduced under the above enumerated conditions:

$$\frac{d^2 H}{dz^2} = j\omega\mu\sigma H$$
$$\frac{d^2 E}{dz^2} = j\omega\mu\sigma E . \tag{2}$$

These equations, contrary to Eq. (1), include no geometrical dimensions Thus the two groups of equations are in contradiction for $a \neq b$.

The solution given by EMDE, as FODOR has shown, can be criticized also on the basis of energy flow. Let us namely examine energy flow in the isolator (Fig. 2). Since E is in the direction x, while H in the direction y, thus the Poynting vector $\mathbf{S}$ is in the direction z, that is, energy can only flow in the direction z in the gap. The value of $\mathbf{S}$ depends on z, i.e. energies of different magnitudes are flowing through the gap sections pertaining to the different z places. For example in the case of a single conductor layer, in the gap section at the upper face of the conductor, at $z = 0$ in our system of coordinates, the

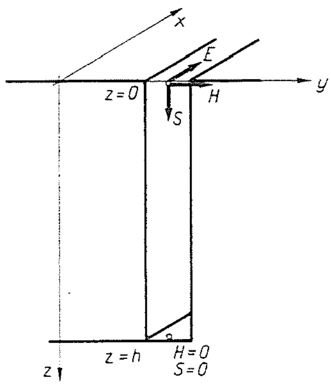

Fig. 2

absolute value of $\mathbf{S}$ is at its maximum, while at $z = h$, at the lower face of the conductor, it is zero. There is, however, no energy loss in the gap, thus the result does not correspond to the principle of the conservation of energy.

In the solution by EMDE the boundary conditions of the electromagnetic field are not satisfied. According to these conditions the field is independent of the coordinates x and y. This should be satisfied also in the gap. In the ferromagnetic material, in consequence of the assumption that $\mu \approx \infty$, there is no field ($E = 0$, $H = 0$). On the surface, however, E must be continuous. This is not the case, since on the gap side $E \neq 0$, on the ferromagnetic material side, in turn, $E = 0$. In spite of the enumerated deficiencies, the results of EMDE are being applied in practice.

2.2. *The method of Fodor*

In view of the errors of the EMDE method, FODOR has abandoned the assumption that the field does not change in the direction of the y axis. (This means that only in the direction of the x axis is there no change.) Thus he obtained, in place of (1), equations in no contradiction to Maxwell's equations. His solution is in accord also with the principle of the conservation of energy. Not all the boundary conditions for the field, however, are satisfied thereby.

The expressions obtained for alternating current resistances are suitable for practical calculations.

The calculation, similarly as the method of EMDE, can be employed also in the case of several conductor layers.

2.3. *The method of Steidinger*

STEIDINGER made his calculations for the case of a single conductor layer. His solution satisfies all boundary conditions. His final results are not suitable for the direct practical calculation of the conductor impedance. In the following the essential steps in his work will be used.

3. Equations for the electromagnetic field of the examined arrangement

In our arrangement there are three media with different material characteristics (Fig. 1). In particular:

1. Ferromagnetic medium,
   permeability: $\mu \approx \infty$
2. Conductor,
   permeability: $\mu = \mu_0$
   specific conductivity: σ
3. Air,
   permeability: $\mu = \mu_0$
   permittivity: $\varepsilon = \varepsilon_0$
   specific conductivity: $\sigma = 0$.

We assume in our calculations that the field is independent of the coordinate $x \left(\frac{\partial}{\partial x} = 0 \right)$.

Under the enumerated conditions the solution of Maxwell's equations can be found for the case of both modes TE and TM. The two deductions produce the same result. Therefore only the calculations of mode TE are described here. This field is deduced of the electric Hertz vector.

The expression of the electric Hertz vector of the conductor, taking into consideration the symmetry and antisymmetry conditions, is the following:

$$\Pi(y, z) = -A \sinh gy \, [e^{-\gamma z} + Ce^{\gamma z}] \tag{3}$$

where A, C, g, and γ are constants to be defined later. Among these g denotes the propagation constant in direction y, while γ that in direction z, for which the relationship

$$g^2 + \gamma^2 = j\omega\mu_0\sigma \tag{4}$$

is valid.

The field components can be determined from the Hertz vector on the basis of the following equations:

$$\begin{aligned} E_z &= 0 & H_z &= -g^2 \Pi \\ E_x &= -j\omega\mu \frac{\partial \Pi}{\partial y} & H_x &= 0 \\ E_y &= 0 & H_y &= \frac{\partial^2 \Pi}{\partial y \, \partial z} . \end{aligned} \tag{5}$$

Upon substituting (3) into these we find that

$$\begin{aligned} E_x &= j\omega\mu_0 Ag \cosh gy \, [e^{-\gamma z} + Ce^{\gamma z}] \\ H_z &= Ag^2 \sinh gy \, [e^{-\gamma z} + Ce^{\gamma z}] \\ H_y &= Ag \gamma \cosh gy \, [e^{-\gamma z} - Ce^{\gamma z}] \\ E_z &= 0 ; \; E_y = 0 ; \; H_x = 0 . \end{aligned} \tag{6}$$

In the left side air gap, at the boundary of the ferromagnetic material, the Hertz vector pertaining to the solution satisfying the boundary condition is:

$$\Pi_0 = -A_0 \sinh g_0(a + y) \, [e^{-\gamma z} + Ce^{\gamma z}] , \tag{7}$$

where for g_0 we obtain, similarly as in (4),

$$g_0^2 + \gamma^2 = -\omega^2 \mu_0 \varepsilon_0 . \tag{8}$$

The field components can be calculated from (7) on the basis of (5):

$$\begin{aligned} E_x &= j\omega\mu_0 A_0 g_0 \cosh g_0(a + y) \, [e^{-\gamma z} + Ce^{\gamma z}] \\ H_z &= A_0 g_0^2 \sinh g_0(a + y) \, [e^{-\gamma z} + Ce^{\gamma z}] \\ H_y &= A_0 g_0 \gamma \cosh g_0(a + y) \, [e^{-\gamma z} - Ce^{\gamma z}] \\ E_z &= 0 ; \; E_y = 0 ; \; H_x = 0 . \end{aligned} \tag{9}$$

4. The relationships for the propagation coefficients

In the equations of Chapter 3, the propagation coefficients g, g_0, γ are unknown values. These can be determined from the following boundary conditions of the field:

a) At $y = \mp a$, $H_z = 0$;
b) at $z = h$, $H_y = 0$;
c) at $y = \mp b$, E_x is continuous;
d) at $y = \mp b$, H_z is continuous.

The expression for Π_0 given under (7) was written so as to meet condition *a)*.

From condition *b)* the constant C can be determined. This is discussed in Chapter 5.

By force of condition *c)*, on the basis of (6) and (9) we obtain:

$$j\omega\mu_0 Ag \cosh gb\,[e^{-\gamma z} + Ce^{\gamma z}] = j\omega\mu_0 A_0 g_0 \cosh g_0(a-b)\,[e^{-\gamma z} + Ce^{\gamma z}] \quad (10)$$

From this we have

$$Ag \cosh gb = A_0\, g_0 \cosh g_0(a-b)\,. \quad (11)$$

According to condition *d)* in turn

$$-Ag^2 \sinh gb\,[e^{-\gamma z} + Ce^{\gamma z}] = A_0 g_0^2 \sinh g_0(a-b)\,[e^{-\gamma z} + Ce^{\gamma z}]\,, \quad (12)$$

that is

$$-Ag^2 \sinh gb = A_0\, g^2 \sinh\, g_0(a-b)\,. \quad (13)$$

Let us form the quotient from equation (13) by (11):

$$-g \tanh gb = g_0 \tanh g_0(a-b)\,. \quad (14)$$

This relationship will be used in our later calculations. The second important equation is obtained as the difference of (4) and (8).

$$g^2 - g_0^2 = \omega^2 \mu_0 \varepsilon_0 + j\omega\mu_0 \sigma\,. \quad (15)$$

In practical cases $\sigma \gg \omega\, \varepsilon_0$ as a rule. Thus

$$g^2 - g_0^2 = j\omega\mu_0 \sigma = p^2 \quad (16)$$

where the symbol p^2 was introduced in the sense as given in the equation. (14) and (16) contain two unknown values (g, g_0). They are difficult to determine since (14) is a transcendent equation.

5. The determination of the propagation coefficients in the case of a single conductor layer

In the following some approximative solutions of (14) and (16) will be examined.

At first let us examine the case where no air gap exists between the conductor and the ferromagnetic material, i.e. for $a = b$. In this case

$$g \tanh gb = 0 \tag{17}$$

and thus

$$g = 0\,. \tag{18}$$

We obtain from (4) for this case that

$$\gamma^2 = j\omega\mu_0\sigma = p^2\,. \tag{19}$$

The solution obtained in this way is identical with the solution for the infinite conductive half space obtained if the skin effect is taken into consideration.

As the next case let us assume that gb and $g_0(a-b)$ are so small that approximations

$$\tanh gb = gb \tag{20}$$

and

$$\tanh g_0(a-b) = g_0(a-b) \tag{21}$$

can be employed. In this case we may write in place of (14) that

$$-g^2 b = g_0^2 (a-b)\,. \tag{22}$$

From (16) and (22) g^2 and g_0^2 can be determined:

$$\begin{aligned} g^2 &= \frac{a-b}{a} p^2 \\ g_0^2 &= -\frac{b}{a} p^2\,. \end{aligned} \tag{23}$$

By using these values, γ^2 can also be calculated on the basis of (4):

$$\gamma^2 = p^2 - g^2 = \frac{b}{a} p^2 = -g_0^2\,. \tag{24}$$

We see in (24) that by this approximation the dependence on z of the field is identical with the function in the calculation of EMDE.

A more precise solution can be obtained by taking into consideration a further term in the series of hyperbolic functions:

$$\tanh gb = gb - \frac{(gb)^3}{3} \tag{25}$$

and

$$\tanh g_0(a-b) = g_0(a-b) - \frac{g_0^3 (a-b)^3}{3}\,. \tag{26}$$

By using (16), (25), and (26), we obtain from (14) a quadratic equation for g_0^2, with the solution:

$$g_0^2 = \frac{3a - 2p^2 b^3 - \sqrt{(3a-2p^2 b^3)^2 - 4p^2 b(p^2 b^2 - 3)[b^3 + (a-b)^3]}}{2[b^3 + (a-b)^3]}\,. \tag{27}$$

The solution of our problem involves the negative value of the square root since then $g_0^2 = 0$ for $a = b$. In the knowledge of g_0^2 we can determine g^2 on the basis of (16), and γ^2 from (8).

In our numerical calculations formula (27) was used.

6. The determination of constants A and C

In the following the values of the Hertz vector constants C and A are determined.

According to the boundary condition *b)* in Chapter 4, we obtain from (6):

$$A\, g\gamma \cosh gy\, [e^{-\gamma h} - Ce^{\gamma h}] = 0. \tag{28}$$

From this

$$C = e^{-2\gamma h}. \tag{29}$$

Substituting this into (6) we obtain for the field components in the conductor:

$$\begin{aligned} E_x &= j\omega\mu_0\, 2\, A\, g \cosh gy\, e^{-\gamma h} \cosh \gamma(h-z) \\ H_z &= 2\, A\, g^2 \sinh gy\, e^{-\gamma h} \cosh \gamma(h-z) \\ H_y &= 2\, A\, g\gamma \cosh gy\, e^{-\gamma h} \sinh \gamma(h-z) \end{aligned} \tag{30}$$

Constant A is proportional to the value of current I exciting the phenomenon. The coefficient of proportionality can be determined from the first Maxwell equation written for the curve surrounding the conductor cross section (Fig. 3).

$$\oint \bar{H}\, d\bar{l} = \int_{l_1} \bar{H}\, d\bar{l} + \int_{l_2} \bar{H}\, d\bar{l} + \int_{l_3} \bar{H}\, d\bar{l} + \int_{l_4} \bar{H}\, d\bar{l} = I\,, \tag{31}$$

where the closed curve was decomposed into 4 lengths as shown in Fig. 3.

Along length l_1, $z = 0$, thus on the basis of (30)

$$\int_{l_1} \bar{H}\, d\bar{l} = \int_{-b}^{b} 2\, A\, g\gamma \cosh gy\, e^{-\gamma h} \sinh \gamma h\, dy = 4\, A\, \gamma e^{-\gamma h} \sinh \gamma h \sinh gb. \tag{32}$$

Along length l_2, $y = b$, thus

$$\int_{l_2} \bar{H}\, d\bar{l} = \int_0^h 2\, A\, g^2 \sinh gb\, e^{-\gamma h} \cosh \gamma (h - z)\, dz = \\ = 2\, \frac{g^2}{\gamma}\, A\, e^{-\gamma h} \sinh gb \sinh \gamma h\,. \tag{33}$$

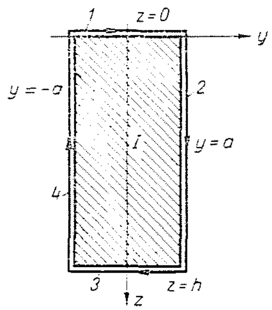

Fig. 3

Along length l_3, $z = h$, thus

$$\int_{l_3} \bar{H}\, d\bar{l} = \int_b^{-b} H_y\, dy = 0\,. \tag{34}$$

The value of the integral along l_4 is identical with that along l_2:

$$\int_{l_4} \bar{H}\, d\bar{l} = \int_{l_2} \bar{H}\, d\bar{l}\,. \tag{35}$$

Substituting these values into (31) we may write

$$\oint \bar{H}\, d\bar{l} = \int_{l_1} \bar{H}\, d\bar{l} + 2 \int_{l_2} \bar{H}\, d\bar{l} = 4\, A\, \frac{p^2}{\gamma}\, e^{-\gamma h} \sinh \gamma h \sinh gb = I\,. \tag{36}$$

From this we have

$$A = \frac{I}{4\, \dfrac{p^2}{\gamma}\, e^{-\gamma h} \sinh \gamma h \sinh gb}\,. \tag{37}$$

In the knowledge of A, g, and g_0, A_0 can be calculated on the basis of (11).

In the preceding, constants g, g_0, γ, A, A_0, and C have been determined. Thus the field as given in our arrangement may be regarded as known.

7. The determination of the impedance of the conductor

For the calculation of the impedance we determine the complex power flowing through the surface of the conductor of length l in the direction x. This can be obtained by forming the integral of the complex Poynting vector on the surface of the conductor.

On the upper plane of the conductor $z = 0$. The component of the Poynting vector perpendicular to the surface is, on the basis of (30):

$$S_1 = \frac{1}{2} E_x H_y^* = \tag{38}$$

$$= \frac{1}{2} j\omega\mu_0 \, 2\, A\, g \cosh gy\, e^{-\gamma h} \cosh \gamma h \; (2\, A\, g\, \gamma \cosh gy\, e^{-\gamma h} \sinh \gamma h)^*$$

(* designates the conjugate).

For performing the calculation, let us decompose γ and g into real and imaginary parts:

$$\gamma = \alpha + j\beta \tag{39}$$

$$g = v + jk$$

Substituting these into (38) we find:

$$S_1 = \frac{1}{2} j\omega\mu_0 |Ag|^2 \gamma^* e^{-2\alpha h} (\cosh 2vy + \\ + \cos 2ky)(\sinh 2\alpha h - j \sin 2\beta h)\,. \tag{40}$$

The complex power flowing into the conductor across the upper plane can be calculated simply:

$$P_1 + jQ_1 = l \int_{-b}^{b} S_1\, dy = \frac{l}{2} j\omega\mu_0 |Ag|^2 \gamma^* e^{-2\alpha h} (\sinh 2\,\alpha h - \\ - j \sin 2\,\beta h)\left(\frac{1}{v} \sinh 2\, vb + \frac{1}{k} \sin 2\, kb\right). \tag{41}$$

The powers flowing into the conductor across the two side planes are equal, thus it is sufficient to calculate one of them. At place $y = b$, the component of the complex Poynting vector perpendicular to the surface is found to be, on the basis of (30):

$$S_2 = \frac{1}{2} E_x H_z^* = \\ = \frac{1}{2} j\omega\mu_0 |Ag|^2 g^* e^{-2\alpha h}(\sinh 2\,vb - j \sin 2\,kb)\,[\cosh 2\,\alpha(h-z) + \\ + \cos 2\,\beta(h-z)]\,. \tag{42}$$

From this the power flowing into the conductor across the side plane with the coordinate $y = a$ is:

$$P_2 + jQ_2 = l\int_0^h S_2\,dz = \frac{l}{2} j\omega\mu_0 |Ag|^2 g^* e^{-2\alpha h}(\sinh 2\,vb - \\ - j\sin 2\,kb)\left(\frac{1}{\alpha}\sinh 2\,\alpha h + \frac{1}{\beta}\sin 2\,\beta h\right). \tag{43}$$

At the lower plane ($z = h$) the component perpendicular to the surface of the Poynting vector is zero.

The expression for the complex power flowing into the conductor is:

$$P + jQ = P_1 + jQ_1 + 2\,(P_2 + jQ_2) = I_{\mathrm{eff}}^2(R + jX)\,. \tag{44}$$

The impedance of the conductor can be calculated therefrom by using (37), (42), and (43). The result of the lengthy calculation is:

$$R = \frac{l}{4\omega\mu_0\sigma^2}\,\frac{(\alpha^2+\beta^2)(v^2+k^2)}{(\cosh 2\,\alpha h - \cos 2\,\beta h)(\cosh 2\,vb - \cos 2\,kb)}\left\{\left(\frac{\sinh 2\,vb}{v} + \right.\right. \\ \left. + \frac{\sin 2\,kb}{k}\right)(\beta\sinh 2\,\alpha h + \alpha\sin 2\,\beta h) + \\ \left. + \left(\frac{\sinh 2\,\alpha h}{\alpha} + \frac{\sin 2\,\beta h}{\beta}\right)(k\sinh 2\,vb + v\sin 2\,kb)\right\} \tag{45}$$

$$X = \frac{l}{4\,\omega\mu_0\sigma^2}\,\frac{(\alpha^2+\beta^2)\,(v^2+k^2)}{(\cosh 2\,\alpha h - \cos 2\,\beta h)(\cosh 2\,vb - \cos 2\,kb)}\left\{\left(\frac{\sinh 2\,vb}{v} + \right.\right. \\ \left. + \frac{\sin 2\,kb}{k}\right)(\alpha\sinh 2\,\alpha h - \beta\sin 2\,\beta h) + \\ \left. + \left(\frac{\sinh 2\,\alpha h}{\alpha} + \frac{\sinh 2\,\beta h}{\beta}\right)(v\sinh 2\,vb - k\sin 2\,kb)\right\}. \tag{46}$$

8. Numerical example. Evaluation of results

The obtained formulae have been applied to calculate the alternating current impedance of a rectangular ($b = 9$ mm; $h = 18$ mm) copper conductor arranged in a groove, as compared to the direct current resistance R_0 for a frequency of 50 c/s, for cases of different groove dimensions, i.e. different dimensions of the isolating layer. In our example the isolating material was air. These results are summarized in Table 1. The R/R_0 values as calculated by the methods of Emde and of Fodor are given in columns I and II, respectively of Table 2 the latter values are given as published in the paper of Fodor [3]).

Table 1

a/mm	g/m^{-1}	g_0/m^{-1}	γ/m^{-1}	$(R+jX)/R_0$
9.0	0	—106.0 +j106.0	—106.0 —j106.0	1.792+j1.792
9.9	31.03+j32.76	—100.8 +j101.4	—101.4 —j100.8	1.689+j1.624
10.8	41.01+j44.89	— 95.97+j 97.95	— 97.95—j 95.97	1.606+j1.424
11.7	46.97+j53.83	— 91.50+j 95.21	— 95.21—j 91.50	1.535+j1.264
12.6	51.25+j60.62	— 87.32+j 93.14	— 93.14—j 87.32	1.476+j1.141
13.5	54.49+j66.26	— 83.37+j 91.51	— 91.51—j 83.37	1.427+j1.045

Table 2

a mm	I	II
9.0	1.792	1.835
9.9	1.693	1.678
10.8	1.610	1.543
11.7	1.539	1.441
12.6	1.480	1.351
13.5	1.430	1.266

It can be seen that the values calculated by the formula of Emde and those given in Table 1 differ by less than 1%. This implies that the results of Emde can be well used in practice, in spite of the theoretical deficiencies, and that the application of the more exact but essentially more complicated formulae (45) and (46) is not justified.

Summary

The impedance of rectangular conductors in grooves of electric machines is usually calculated in practice on the basis of the method by EMDE—ROGOWSKI. The pertaining theory can be criticized, as has been shown by FODOR. In the present paper a more exact theory is described for the case of a single conductor layer, by using the results of STEIDINGER. On the basis of this theory the impedance can be calculated. In our numerical calculations the error of the EMDE results was found to be within 1%.

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Edit HOLLÓS
Budapest XI., Egri József u. 18—20, Hungary