THEORETICAL FOUNDATIONS OF THE REDUCTION OF TRIPHASE NETWORKS TO COMPONENTS

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1. Introduction

One of the customary calculation methods for triphase linear networks is based on its reduction to components. Thus, a usual method is the application of symmetrical components and of the components α, β, 0. The idea of the reduction to symmetrical components was originated by C. L. FORTESCUE [1]. Calculation methods that can be used in practice can be found in the book by WAGNER—EVANS [2]. The formulation of the equations in the matrix form was first applied by HOFFMAN—KOCH—PRYCE [3] and RISLEY—BURLINGTON [4]. The works of BACH [5], SZENDY [6] have laid the foundations for the reduction to symmetrical components as a reduction with respect to the eigenvectors of the impedance matrix characterizing the network. The idea of the α, β, 0 vectors originated by E. CLARKE [7]. Other authors [8, 9] have used other reductions too.

The aim of the present paper is the clarification of the unified theoretical foundations of the various methods of reduction. Generally reduction is made with respect to the eigenvalues of the transfer matrix G characterizing the network. In the case of symmetrical reciprocal networks the two eigenvalues of G are identic, thus, various reductions satisfying the given conditions are possible. These are, among others, the reduction to symmetrical components and to the components α, β, 0. In the case of symmetrical or reciprocal networks the reduction with respect to the eigenvectors leads to the reduction to the symmetrical components, as has also been proved by BACH [5].

2. General correlations

2.1. Transfer matrix

At an arbitrary place in a triphase network the three voltages (U_{1a}, U_{1b}, U_{1c}), resp. the three intensities (I_{1a}, I_{1b}, I_{1c}) pertaining to the three phases are called primary voltages, resp. primary intensities, while the three voltages (U_{2a}, U_{2b}, U_{2c}) or intensities (I_{2a}, I_{2b}, I_{2c}) arising at another place are denomi-

nated as secondary quantities. In the case of a linear network a linear correlation can be written between the primary and secondary quantities. E.g. if the primary quantities are voltages and the secondary quantities intensities, then this linear correlation has the following form:

$$\begin{aligned} I_{2a} &= Y_{11}U_{1a} + Y_{12}U_{1b} + Y_{13}U_{1c} \\ I_{2b} &= Y_{21}U_{1a} + Y_{22}U_{1b} + Y_{23}U_{1c} \\ I_{2c} &= Y_{31}U_{1a} + Y_{32}U_{1b} + Y_{33}U_{1c} \end{aligned} \tag{2.1}$$

The relationships given under (2.1) can be more compactly formulated in the matrix form:

$$\mathbf{I}_2 = \boldsymbol{Y}\mathbf{U}_1, \tag{2.2}$$

where $\mathbf{U}_1$ is the column vector of the primary voltage, and $\mathbf{I}_2$ that of the secondary intensity:

$$\mathbf{U}_1 = \begin{bmatrix} U_{1a} \\ U_{1b} \\ U_{1c} \end{bmatrix} \text{ and } \qquad \mathbf{I}_2 = \begin{bmatrix} I_{2a} \\ I_{2b} \\ I_{2c} \end{bmatrix}, \tag{2.3}$$

and $\boldsymbol{Y}$ is the quadratic admittance matrix:

$$\boldsymbol{Y} = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{21} & Y_{22} & Y_{23} \\ Y_{31} & Y_{32} & Y_{33} \end{bmatrix}. \tag{2.4}$$

The vectors of the primary intensity and of the secondary voltage, respectively, can be defined similarly as in (2.3).

$$\mathbf{I}_1 = \begin{bmatrix} I_{1a} \\ I_{1b} \\ I_{1c} \end{bmatrix} \text{ and } \qquad \mathbf{U}_2 = \begin{bmatrix} U_{2a} \\ U_{2b} \\ U_{2c} \end{bmatrix}. \tag{2.5}$$

Between the various primary and secondary quantities relationships similar to (2.2) can be written, such as

$$\mathbf{U}_2 = \boldsymbol{Z}\mathbf{I}_1 \quad \mathbf{U}_2 = \boldsymbol{G}_u\mathbf{U}_1 \quad \mathbf{I}_2 = \boldsymbol{G}_i\mathbf{I}_1, \tag{2.6}$$

where $\boldsymbol{Z}$, $\boldsymbol{G}_u$, and $\boldsymbol{G}_i$ designate the impedance, the voltage transfer and the intensity transfer matrix, respectively. In the following the transfer matrix $\boldsymbol{G}$ will be equally interpreted as the admittance matrix $\boldsymbol{Y}$, the impedance matrix $\boldsymbol{Z}$, the voltage transfer matrix $\boldsymbol{G}_u$, or the intensity transfer matrix $\boldsymbol{G}_i$.

2.2. *Reduction with respect to the eigenvalue*

The calculation of the network becomes more simple if in $\boldsymbol{G}$ the elements outside the main diagonal are equal to zero, then $\boldsymbol{G}$ is a diagonal matrix.

$$<\boldsymbol{G}> = \begin{bmatrix} G_{11} & 0 & 0 \\ 0 & G_{22} & 0 \\ 0 & 0 & G_{33} \end{bmatrix} = < G_{11}\, G_{22}\, G_{33} > \tag{2.7}$$

If (2.7) is satisfied, (2.1) assumes the following form:

$$\begin{aligned} I_{2a} &= G_{11}\, U_{1a} \\ I_{2b} &= G_{22}\, U_{1b} \\ I_{2c} &= G_{33}\, U_{1c} \end{aligned} \tag{2.8}$$

(2.8) means that (2.1) and (2.6), respectively, are decomposed to three independent equations. Thus, the calculation is naturally more simple. Equations (2.1), (2.2), and (2.6) can be subjected to such a transformation that they assume a form similar to that of equation (2.8). This task means the decomposition of the matrix $\boldsymbol{G}$ with respect to the eigenvectors. The equation for the determination of the eigenvalues λ_i pertaining to the eigenvectors is:

$$\det\left[\boldsymbol{G} - \lambda_i \boldsymbol{E}\right] = 0, \tag{2.9}$$

where $\boldsymbol{E}$ is the unit matrix, having elements in the main diagonal with a value of 1, while the others are zero. By writing (2.9) in the ordinates, we obtain the following equation:

$$\begin{vmatrix} G_{11} - \lambda_i & G_{12} & G_{23} \\ G_{21} & G_{22} - \lambda_i & G_{23} \\ G_{31} & G_{32} & G_{33} - \lambda_i \end{vmatrix} = 0\,. \tag{2.10}$$

(2.10) is an equation of the third degree with respect to λ_i, generally having three different roots. In the knowledge of the eigenvalues the pertaining matrix Lagrange polynomial $\boldsymbol{L}_k$ can be determined.

$$\boldsymbol{L}_k(\boldsymbol{G}) = \prod_{\substack{i=1 \\ i \neq k}}^{3} \frac{\boldsymbol{G} - \lambda_i \boldsymbol{E}}{\lambda_k - \lambda_i} \qquad k = 1, 2, 3. \tag{2.11}$$

The decomposition of the column vectors $\mathbf{U}$, or $\mathbf{I}$ can be made by utilizing the matrix Lagrange polynomials.

$$\mathbf{U} = \sum_{k=1}^{3} \boldsymbol{L}_k \mathbf{U} \qquad \mathbf{I} = \sum_{k=1}^{3} \boldsymbol{L}_k \mathbf{I}. \tag{2.12}$$

In (2.12) the individual terms in the sums give the components in the direction of the eigenvectors, i.e. with the aid of the Lagrange polynomials the coordinate system is transformed in the directions of the eigenvectors.

2.3. *Examination of the system of perpendicular eigenvectors*

In the followings our attention will be concentrated on that case when the eigenvectors are perpendicular to each other. A transformed system of this kind is called an orthogonal system. The reduction of $\mathbf{U}$, or $\mathbf{I}$ in the new system can also be written with the aid of the unit column vector $\mathbf{T}_k$ of the new coordinate system.

$$\mathbf{U} = \sum_{k=1}^{3} \mathbf{T}_k (\overline{\mathbf{T}}_k^* \mathbf{U}) = \left(\sum_{k=1}^{3} \mathbf{T}_k \overline{\mathbf{T}}_k^*\right) \mathbf{U}, \tag{2.13}$$

where * indicates the transpose, the dash the conjugate, i.e. $\overline{\mathbf{T}}_k^*$ is the row vector. $(\overline{\mathbf{T}}_k^* \mathbf{U})$ is the scalar product of $\overline{\mathbf{T}}_k$ and of $\mathbf{U}$, that is the projection of $\mathbf{U}$ in the direction of $\mathbf{T}_k$.

$\mathbf{T}_k \overline{\mathbf{T}}_k^*$ is the dyadic product of $\mathbf{T}_k$ with itself (dyad) and thus this is a quadratic matrix.

By comparing (2.12) and (2.13), it is evident that

$$\boldsymbol{L}_k = \mathbf{T}_k \overline{\mathbf{T}}_k^*. \tag{2.14}$$

According to (2.14) $\boldsymbol{L}_k$ is a dyad, the column vector of which is just the k-th eigenvector, while its row vector can be obtained by the transposing and conjugating of $\mathbf{T}_k$. Thus, we obtain $\mathbf{T}_k$ from $\boldsymbol{L}_k$ by dividing the elements in the first column by the square root of the first element in the first row.

In the new system of coordinates the individual components of $\mathbf{U}$ are equal to the coefficients of $\mathbf{T}_k$ in (2.13). Let $\mathbf{U}_T$ designate the column vector formed by these:

$$\mathbf{U}_T = \begin{bmatrix} \overline{\mathbf{T}}_1^* \mathbf{U} \\ \overline{\mathbf{T}}_2^* \mathbf{U} \\ \overline{\mathbf{T}}_3^* \mathbf{U} \end{bmatrix} = \begin{bmatrix} \overline{\mathbf{T}}_1^* \\ \overline{\mathbf{T}}_2^* \\ \overline{\mathbf{T}}_3^* \end{bmatrix} \mathbf{U} = \boldsymbol{T}\mathbf{U} = \begin{bmatrix} U_{T1} \\ U_{T2} \\ U_{T3} \end{bmatrix}, \tag{2.15}$$

where the $\boldsymbol{T}$ transformation matrix is

$$\boldsymbol{T} = \begin{bmatrix} \overline{\mathbf{T}}_1^* \\ \overline{\mathbf{T}}_2^* \\ \overline{\mathbf{T}}_3^* \end{bmatrix} \tag{2.16}$$

Similarly as in (2.15), the transformed form of $\mathbf{I}$ can also be written:

$$\mathbf{I}_T = \boldsymbol{T}\mathbf{I}. \tag{2.17}$$

From equations (2.15) and (2.17):

$$\mathbf{U} = \boldsymbol{T}^{-1}\mathbf{U}_T \qquad \mathbf{I} = \boldsymbol{T}^{-1}\mathbf{I}_T. \tag{2.18}$$

(2.18) can also be written in terms of the unit vectors $\mathbf{T_k}$:

$$\mathbf{U} = \mathbf{T}_1 U_{T1} + \mathbf{T}_2 U_{T2} + \mathbf{T}_3 U_{T3} = [\mathbf{T}_1\,\mathbf{T}_2\,\mathbf{T}_3]\begin{bmatrix} U_{T1} \\ U_{T2} \\ U_{T3} \end{bmatrix} = [\mathbf{T}_1\,\mathbf{T}_2\,\mathbf{T}_3]\,\mathbf{U}_T\,. \tag{2.19}$$

By comparing (2.18) and (2.19) we find that

$$\boldsymbol{T}^{-1} = [\mathbf{T}_1\,\mathbf{T}_2\,\mathbf{T}_3] = \bar{\boldsymbol{T}}^*, \tag{2.20}$$

that is

$$\boldsymbol{T}\,\bar{\boldsymbol{T}}^* = \boldsymbol{E}. \tag{2.21}$$

(2.21) is the condition that the eigenvectors are to be perpendicular to each other. The orthogonality supplies a condition also for the Lagrange polynomials. Namely, in the case of perpendicularity

$$(\bar{\boldsymbol{L}}_k\,\bar{\mathbf{U}})^*(\boldsymbol{L}_i\,\mathbf{U}) = \bar{\mathbf{U}}^*\bar{\boldsymbol{L}}_k^*\boldsymbol{L}_i\,\mathbf{U} = 0. \tag{2.22}$$

Equation (2.22) can be fulfilled independently of $\mathbf{U}$ only if

$$\bar{\boldsymbol{L}}_k^*\,\boldsymbol{L}_i = 0. \tag{2.23}$$

Since $\mathbf{U} = \boldsymbol{Z}\mathbf{I}$, by utilizing (2.15), (2.17), and (2.18) the relation between $\mathbf{U}_T$ and $\mathbf{I}_T$ can be expressed:

$$\mathbf{U}_T = \boldsymbol{T}\,\mathbf{U} = \boldsymbol{T}\boldsymbol{Z}\,\mathbf{I} = \boldsymbol{T}\boldsymbol{Z}\boldsymbol{T}^{-1}\,\mathbf{I}_T = \boldsymbol{Z}_T\,\mathbf{I}_T, \tag{2.24}$$

where

$$\boldsymbol{Z}_T = \boldsymbol{T}\,\boldsymbol{Z}\,\boldsymbol{T}^{-1}\,. \tag{2.25}$$

Similarly as in (2.25) the transformed transfer matrix $\boldsymbol{G}_T$ can also be defined.

$$\boldsymbol{G}_T = \boldsymbol{T}\,\boldsymbol{G}\,\boldsymbol{T}^{-1} \tag{2.26}$$

$\boldsymbol{G}_T$ and naturally also $\boldsymbol{Z}_T$ and $\boldsymbol{Y}_T$ are diagonal matrixes in the main diagonals of which just the eigenvalues can be found. A single transformed primary component is the function of only a single transformed secondary component.

2.3.1. *Power conditions of the orthogonal transformed system*

For calculating the power at a prescribed place the connected **U** and **I** values arising at that place should be taken into consideration, namely in such a way, that the voltages U_a, U_b, U_c should give physically real powers with the intensities I_a, I_b, I_c, respectively. Then the expression for the complex power is found to be:

$$P + jQ = \bar{I}_a^* U_a + \bar{I}_b^* U_b + \bar{I}_c^* U_c = \bar{\mathbf{I}}^* \mathbf{U}, \tag{2.27}$$

where P is the effective and Q the reactive power. On the basis of (2.18) we have

$$\bar{\mathbf{I}}^* = \overline{(\boldsymbol{T}^{-1} \mathbf{I}_T^*)} = \bar{\mathbf{I}}_T^* \, \bar{\boldsymbol{T}}^{*-1}, \tag{2.28}$$

From equations (2.18), (2.21), (2.27) and (2.28) the complex power can also be expressed with the aid of the transformed voltages and intensities.

$$\begin{aligned} P + j\,Q &= \mathbf{I}_T \, \bar{\boldsymbol{T}}^{*-1} \boldsymbol{T}^{-1} \mathbf{U}_\mathrm{T} = \bar{\mathbf{I}}_T^* (\bar{\boldsymbol{T}}^* \boldsymbol{T})^{-1} \mathbf{U}_\mathrm{T} = \\ &= \bar{\mathbf{I}}_T^* \boldsymbol{E} \, \mathbf{U}_T = \bar{\mathbf{I}}_T^* \, \mathbf{U}_T = \bar{I}_{T1}^* U_{T1} + \bar{I}_{T2}^* U_{T2} + \bar{I}_{T3}^* U_{T3} . \end{aligned} \tag{2.29}$$

Formula (2.29) means, that after the transformation the individual voltage components produce a power only with the respective intensity components, since transformed components not having identical indices are orthogonal to each other, thus, no power can originate from them.

3. Symmetrical reciprocal triphase networks

In the case of symmetrical reciprocal networks the matrices $G(Z, Y)$ are symmetrical and all the elements in the main diagonal (G_o) are equal to each other and the elements outside the main diagonal (G_k) are also identical.

$$G = \begin{bmatrix} G_o & G_k & G_k \\ G_k & G_o & G_k \\ G_k & G_k & G_o \end{bmatrix}. \tag{3.1}$$

According to (3.1), formula (2.10) for the eigenvalues has now the following form:

$$\begin{vmatrix} G_o - \lambda & G_k & G_k \\ G_k & G_o - \lambda & G_k \\ G_k & G_k & G_o - \lambda \end{vmatrix} = (G_o + 2G_k - \lambda)(G_o - G_k \lambda)^2 = 0. \tag{3.2}$$

The solutions of (3.2) are

$$\lambda_0 = G_o + 2G_k$$
$$\lambda_{12} = G_o - G_k, \tag{3.3}$$

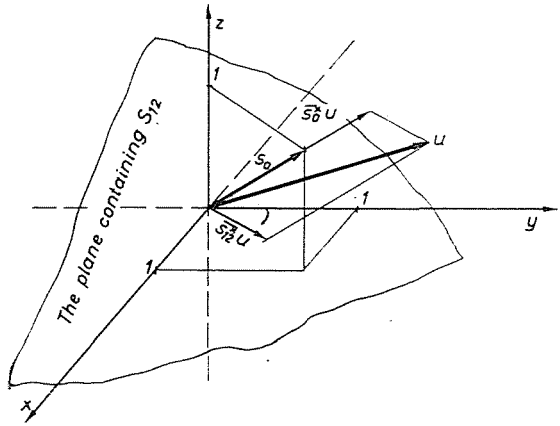

Fig. 1

where λ_{12} is a double root. In the knowledge of the eigenvalues, on the basis of (2.11) the respective Lagrange polynomials can be determined. Thus, the Lagrange polynomial pertaining to λ_0 is:

$$\boldsymbol{L}_0 = \frac{\boldsymbol{G} - \lambda_{12}\boldsymbol{E}}{\lambda_0 - \lambda_{12}} = \frac{1}{G_o + 2G_k - G_o + G_k} \times$$
$$\times \begin{bmatrix} G_o - G_o + G_k & G_k & G_k \\ G_k & G_o - G_o + G_k & G_k \\ G_k & G_k & G_o - G_o + G_k \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}. \tag{3.4}$$

In symmetrical networks the eigenvectors are denoted by S. The eigenvector S_0 pertaining to λ_0 can be obtained by dividing the elements in the first column of $\boldsymbol{L}_0$ by the square root of the first element in the first row:

$$\mathrm{S}_0 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \tag{3.5}$$

S_0 is a unit vector forming the same angle with all the three datum lines.

λ_{12} is a double root of the characteristic equation. Accordingly $\boldsymbol{L}_{12}\mathbf{U}$ is a vector in the plane perpendicular to S_0. That is, the reduction with respect to the eigenvectors means a reduction to a component in the direction S_0 and to a component perpendicular to the former (Fig. 1). $\boldsymbol{L}_0$ similarly to $\boldsymbol{L}_{12}$, it can be calculated on the basis of (2.11) and (3.3), that

$$\boldsymbol{L}_{12}=\frac{\boldsymbol{G}-\lambda_0\boldsymbol{E}}{\lambda_{12}-\lambda_0}=\frac{1}{3}\begin{bmatrix}2&-1&-1\\-1&2&-1\\-1&-1&2\end{bmatrix}. \tag{3.6}$$

It can be seen in formulae (3.4) and (3.6) that the values G_0 and G_k characterizing the network are not figuring either in $\boldsymbol{L}_0$ or in $\boldsymbol{L}_{12}$. This means that in a symmetrical reciprocal triphase network the matrixes characteristic for the reduction with respect to the eigenvalues are independent of the actual network and of the place in the network where the primary and secondary quantities are chosen. The reduction is independent of the actual network, accordingly the transforms of the Kirchhoff equations written for the three phases disintegrate to the equations with respect to the eigenvalues. In turn it follows from this that the correlations which can be deduced from the Kirchhoff equations are valid for each component separately. It is evident from this that the advantages of the reduction with respect to the eigenvalues are especially effective in the case of symmetrical networks.

The component $\boldsymbol{L}_{12}\mathbf{U}$ is perpendicular to $\mathbf{S}_0$ independently of $\mathbf{U}$, i.e. equation 2.23 is fulfilled.

$$\overline{\boldsymbol{L}}_{12}^{*}\boldsymbol{L}_0=0. \tag{3.7}$$

The correctness of this relationship can easily be proved by susbtituting (3.5) and (3.6). The component of $\mathbf{U}$ perpendicular to $\mathbf{S}_0$, i.e. $\boldsymbol{L}_{12}\mathbf{U}$ can be reduced to two components. The directions of these should be identical with those of the unit vectors $\mathbf{S}_1$ and $\mathbf{S}_2$. According to the reduction, $\boldsymbol{L}_{12}$ can be written similarly to (2.13), as the sum of two matrixes, $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$.

$$\boldsymbol{L}_{12}=\boldsymbol{L}_1+\boldsymbol{L}_2=\mathbf{S}_1\overline{\mathbf{S}}_1^{*}+\mathbf{S}_2\overline{\mathbf{S}}_2^{*}. \tag{3.8}$$

Let the ordinates of $\mathbf{S}_1$ be x, y, z, that is

$$\mathbf{S}_1=\begin{bmatrix}x\\y\\z\end{bmatrix}. \tag{3.9}$$

$\mathbf{S}_1$ is a unit vector, that is

$$\overline{\mathbf{S}}_1^{*}\,\mathbf{S}_1=\bar{x}x+\bar{y}y+\bar{z}z=1. \tag{3.10}$$

$\mathbf{S}_1$ is perpendicular to $\mathbf{S}_0$, i.e. by utilizing (3.5) and (3.9),

$$\overline{\mathbf{S}}_0^{*}\,\mathbf{S}_1=\frac{1}{\sqrt{3}}[1\ 1\ 1]\begin{bmatrix}x\\y\\z\end{bmatrix}=\frac{1}{3}(x+y+z)=0. \tag{3.11}$$

The ordinates of vector $\mathbf{S}_1$ should be chosen in such a way, that they should satisfy equations (3.10) and (3.11). Under these conditions, however, they can be chosen arbitrarily. $\boldsymbol{L}_1$ can be determined from $\mathbf{S}_1$ on the basis of (3.8):

$$\boldsymbol{L}_1 = \mathbf{S}_1 \bar{\mathbf{S}}_1^* = \begin{bmatrix} x\bar{x} & x\bar{y} & x\bar{z} \\ y\bar{x} & y\bar{y} & y\bar{z} \\ z\bar{x} & z\bar{y} & z\bar{z} \end{bmatrix}. \tag{3.12}$$

By choosing $\mathbf{S}_1$, we obtain $\boldsymbol{L}_2$ from equations (3.6), (3.8), and (3.12).

$$\boldsymbol{L}_2 = \boldsymbol{L}_{12} - \boldsymbol{L}_1 = \begin{bmatrix} \frac{2}{3} - x\bar{x} & -\left(\frac{1}{3} + x\bar{y}\right) & -\left(\frac{1}{3} + x\bar{z}\right) \\ -\left(\frac{1}{3} + y\bar{x}\right) & \frac{2}{3} - y\bar{y} & -\left(\frac{1}{3} + y\bar{z}\right) \\ -\left(\frac{1}{3} + z\bar{x}\right) & -\left(\frac{1}{3} + z\bar{y}\right) & \frac{2}{3} - z\bar{z} \end{bmatrix} = \mathbf{S}_2 \bar{\mathbf{S}}_2^*. \tag{3.13}$$

From (3.13) $\mathbf{S}_2$ can be calculated in the already described way:

$$\mathbf{S}_2 = \frac{1}{\sqrt{\frac{2}{3} - x\bar{x}}} \begin{bmatrix} \frac{2}{3} - x\bar{x} \\ -\left(\frac{1}{3} + y\bar{x}\right) \\ -\left(\frac{1}{3} + z\bar{x}\right) \end{bmatrix} = \frac{1}{\sqrt{\frac{2}{3} - z\bar{z}}} \begin{bmatrix} -\left(\frac{1}{3} + x\bar{z}\right) \\ -\left(\frac{1}{3} + y\bar{z}\right) \\ \frac{2}{3} - z\bar{z} \end{bmatrix}. \tag{3.14}$$

The second form of (3.14) could be more expediently used in such cases when $2/3 - x\bar{x} = 0$.

According to the choice of $\mathbf{S}_1$ several reductions are possible. From these two are utilized in practice: Reduction with respect to the symmetrical components and to the components α, β, 0. In the followings these two reductions are examined.

3.1. Symmetrical components

The basic idea of the reduction to symmetrical components is that the three ordinates of the component in the direction $\mathbf{S}_1$ should form a symmetrical triphase system, that is,

$$\mathbf{S}_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ a^2 \\ a \end{bmatrix} = \mathbf{S}^{(+)}, \tag{3.15}$$

where

$$a = e^{j\frac{2}{3}\pi} . \tag{3.16}$$

The following correlations for a are known:

$$\begin{aligned} a^2 &= \bar{a} = a^{-1} \\ a^3 &= 1 \\ 1 + a + a^2 &= 0. \end{aligned} \tag{3.17}$$

By selecting $\mathbf{S}_1$ according to (3.15) the equations (3.10) and (3.11) are evidently satisfied.

The expression for $\boldsymbol{L}_1 = \boldsymbol{L}^{(+)}$ on the basis of (3.12) and (3.15) is

$$\boldsymbol{L}_1^{(+)} = \mathbf{S}_1^{(+)}\,\overline{\mathbf{S}_1^{(+)}}^{*} = \frac{1}{3}\begin{bmatrix}1\\ a^2\\ a\end{bmatrix}[1\ a\ a^2] = \frac{1}{3}\begin{bmatrix}1 & a & a^2\\ a^2 & 1 & a\\ a & a^2 & 1\end{bmatrix}. \tag{3.18}$$

$\boldsymbol{L}_2 = \boldsymbol{L}^{(-)}$ can now be determined already by using (3.8) and (3.18).

$$\boldsymbol{L}^{(-)} = \boldsymbol{L}_{12} - \boldsymbol{L}^{(-)} = \frac{1}{3}\begin{bmatrix}1 & a^2 a & \\ a & 1 & a^2\\ a^2 a & 1 & \end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\ a\\ a^2\end{bmatrix}\frac{1}{\sqrt{3}}[1\ a^2\ a] = \mathbf{S}_2\,\overline{\mathbf{S}}_2^{*}. \tag{3.19}$$

$\mathbf{S}_2 = \mathbf{S}^{(-)}$ can be calculated either on the basis of (3.14) or of (3.19).

$$\mathbf{S}^{(-)} = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\ a\\ a^2\end{bmatrix}. \tag{3.20}$$

$\mathbf{S}^{(-)}$ is perpendicular to $\mathbf{S}^{(+)}$, namely

$$\overline{\mathbf{S}}^{(-)*}\,\mathbf{S}^{(+)} = \frac{1}{3}[1\ a^2\ a]\begin{bmatrix}1\\ a^2\\ a\end{bmatrix} = \frac{1}{3}(1 + a^4 + a^2) \equiv 0. \tag{3.21}$$

In the knowledge of $\mathbf{S}_0$, $\mathbf{S}^{(+)}$, $\mathbf{S}^{(-)}$ that is on the basis of (3.5), (3.15) and (3.20), the transformation matrix $\boldsymbol{T} = \boldsymbol{S}$ of the reduction to symmetrical components according to (2.16) can be written:

$$\boldsymbol{S} = \begin{bmatrix}\overline{\mathbf{S}}^{0*}\\ \overline{\mathbf{S}}^{(+)*}\\ \overline{\mathbf{S}}^{(-)*}\end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix}1 & 1 & 1\\ 1 & a & a^2\\ 1 & a^2 & a\end{bmatrix}, \tag{3.22}$$

and

$$\boldsymbol{S}^{-1} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} = \overline{\boldsymbol{S}}^* . \tag{3.23}$$

It is conceivable on the basis of (3.22) and (3.23) that equation (2.21) is also fulfilled for $\boldsymbol{S}$.

The reduction of the voltage and intensity to symmetrical components can be performed on the ground of (2.15) and (2.17):

$$\mathbf{U}_S = \boldsymbol{S}\,\mathbf{U} \text{ and } \mathbf{I}_S = \boldsymbol{S}\,\mathbf{I}. \tag{3.24}$$

We may further write that

$$\mathbf{U} = \boldsymbol{S}^{-1}\mathbf{U}_S \text{ and } \mathbf{I} = \boldsymbol{S}^{-1}\mathbf{I}_S. \tag{3.25}$$

It should be mentioned that in the relevant literature on writing $\boldsymbol{S}$, frequently 1/3 figures in place of the factor $1/\sqrt{3}$. In this case in the expression of $\boldsymbol{S}^{-1}$ in place of $1/\sqrt{3}$ we should write 1. This mode of writing is identical in principle with the equations also applied by us, but in this case $\mathbf{S}_0$, $\mathbf{S}_1$, $\mathbf{S}_2$ are not unit vectors.

On reducing to symmetrical components, the component in the direction $\mathbf{S}_1$ is called the positive order component, the one in the direction $\mathbf{S}_2$ the negative order component, and the one in the direction $\mathbf{S}_0$ the zero order component. As we have seen in the case of symmetrical networks, the Kirchhoff-equations and all the relations to be deduced therefrom can be written separately for the individual components and on calculating the power only the products of currents and voltages of the same order are to be taken into consideration.

We determine hereafter, by taking (2.24), (3.11), (3.22), and (3.23) into consideration, the transformed transfer matrix as defined by (2.24):

$$G_S = \boldsymbol{S}\,\boldsymbol{G}\,\boldsymbol{S}^{-1} = < G_0 - 2G_k \;\; G_0 - G_k \;\; G_0 - G_k > . \tag{3.26}$$

In the main diagonal just the eigenvalues are figuring. If $\boldsymbol{G} = \boldsymbol{Z}$, then on the basis of (3.26) and (2.6),

$$\begin{aligned} U_2^0 &= (Z_0 + 2Z_k)\, I_1^0 \\ U_2^+ &= (Z_0 - Z_k)\, I_1^{(+)} \\ U_2^- &= (Z_0 - Z_k)\, I_1^{(-)} , \end{aligned} \tag{3.27}$$

where the zero, positive, and negative order components were designated by upper indices 0, +, and — respectively. According to (3.27) the zero order impedance is $Z_0 + 2Z_k$, while the positive and negative impedance $Z_0 - Z_k$.

3.2. The α, β, 0 components

The basic point of the reduction to components α, β, 0 is that the reduction should form a system in which the ordinates of the individual components are in phase or in anti-phase with each other. Then a reduction of $\boldsymbol{L}_{12}$ to such dyad sums is to be found, with which all the ordinates are real values. This is not connected with any special difficulty, since $\boldsymbol{L}_{12}$ is symmetrical and its elements are real, accordingly we may obtain S_1 also in such a way that the first column of $\boldsymbol{L}_{12}$ is divided by the square root of the first element. S_1 is now designated by S_α, S_2 by S_β, $\boldsymbol{L}_1$ by $\boldsymbol{L}_\alpha$, and $\boldsymbol{L}_2$ by $\boldsymbol{L}_\beta$. Thus

$$\mathrm{S}_\alpha = \frac{1}{\sqrt{3}}\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}. \tag{3.28}$$

Further on the ground of (3.12),

$$\boldsymbol{L}_\alpha = \mathrm{S}_\alpha \mathrm{S}_\alpha^* = \frac{1}{3}\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}[2 \; -1 \; -1] = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 1/2 & 1/2 \\ -1 & 1/2 & 1/2 \end{bmatrix}. \tag{3.29}$$

The conjugate designation has been omitted since all values are real. $\boldsymbol{L}_\beta$ can be determined on the basis of (3.13).

$$\boldsymbol{L}_\beta = \boldsymbol{L}_{12} - \boldsymbol{L}_\alpha = \frac{1}{3}\begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{3}{2} & -\frac{3}{2} \\ 0 & -\frac{3}{2} & \frac{3}{2} \end{bmatrix}. \tag{3.30}$$

S_β can be calculated from (3.14). Since now $2/3 - x\bar{x} = 0$, the second expression for S_β is used:

$$\mathrm{S}_\beta = \frac{1}{\sqrt{3}}\begin{bmatrix} 0 \\ \sqrt{\frac{3}{2}} \\ \sqrt{\frac{3}{2}} \end{bmatrix}. \tag{3.31}$$

S_α and S_β are perpendicular, since

$$\mathrm{S}_\alpha^* \mathrm{S}_\beta \equiv 0. \tag{3.32}$$

The components in the direction S_α, S_β, and S_0 are called the alpha, beta and zero components, respectively.

With the knowledge of S_0, S_α, and S_β, i.e. on the basis of (3.5), (3.28), and (3.31), the transformation matrix $\boldsymbol{T} = \boldsymbol{S}_R$ corresponding to (2.16) can be written.

$$\boldsymbol{S}_R = \begin{bmatrix} \overline{S}_0^* \\ \overline{S}_\alpha^* \\ \overline{S}_\beta^* \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ \sqrt{2} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \sqrt{\frac{3}{2}} & -\sqrt{\frac{3}{2}} \end{bmatrix} \tag{3.33}$$

and thus

$$\boldsymbol{S}_R^{-1} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & \sqrt{2} & 6 \\ 1 & -\frac{1}{\sqrt{2}} & \sqrt{\frac{3}{2}} \\ 1 & -\frac{1}{\sqrt{2}} & -\sqrt{\frac{3}{2}} \end{bmatrix} \tag{3.34}$$

Let us also determine on the basis of (2.24), (3.1), (3.33), and (3.34) the transformed transfer matrix $\boldsymbol{G}_S$:

$$\boldsymbol{G}_S = \boldsymbol{S}_R \boldsymbol{G} \boldsymbol{S}_R^{-1} = < G_o - 2G_k \;\; G_o - G_k \;\; G_o - G_k > \tag{3.35}$$

In the main diagonal, now again the eigenvalues are standing.

4. Symmetrical, non-reciprocal networks

Triphase electric machines form a symmetrical triphase non-reciprocal network. The transfer matrix of such networks is cyclic, i.e. of the following form:

$$\boldsymbol{G} = \begin{bmatrix} G_0 & G_1 & G_2 \\ G_2 & G_0 & G_1 \\ G_1 & G_2 & G_0 \end{bmatrix}. \tag{4.1}$$

Accordingly the characteristic equation (2.11) is

$$\begin{bmatrix} G_0 - \lambda & G_1 & G_2 \\ G_2 & G_0 - \lambda & G_1 \\ G_1 & G_2 & G_0 - \lambda \end{bmatrix} = 0. \tag{4.2}$$

On adding the second and third row to the first one, then by deducing the first column from the second and the third, we obtain

$$(G_0 + G_1 + G_2 - \lambda)\begin{vmatrix} G_0 - G_2 - \lambda & G_1 - G_2 \\ G_2 - G_1 & G_0 - G_1 - \lambda \end{vmatrix} = 0. \tag{4.3}$$

On deducing the a-fold of the second row from the first row of the determinant, then by adding the a-fold of the first column to the second, we obtain the root factor form of the characteristic equation:

$$(G_0 + G_1 + G_2 - \lambda)(G_0 + a^2 G_1 + aG_2 - \lambda)(G_0 + aG_1 + a^2 G_2 - \lambda) = 0. \tag{4.4}$$

From this the eigenvalues are:

$$\begin{aligned} \lambda_0 &= G_0 + G_1 + G_2 \\ \lambda_1 &= G_0 + a^2 G_1 + aG_2 \\ \lambda_2 &= G_0 + a\,G_1 + a^2 G_2. \end{aligned} \tag{4.5}$$

In the knowledge of the eigenvalues, on the ground of (2.11) we write the Lagrange polynomials.

$$\boldsymbol{L}_0 = \frac{\boldsymbol{G} - \lambda_1 \boldsymbol{E}}{\lambda_0 - \lambda_1}\,\frac{\boldsymbol{G} - \lambda_2 \boldsymbol{E}}{\lambda_0 - \lambda_2} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\frac{1}{\sqrt{3}}[1\ 1\ 1] = \mathbf{S}_0\overline{\mathbf{S}}_0^{*}\,.$$

$$\boldsymbol{L}_1 = \frac{\boldsymbol{G} - \lambda_0 \boldsymbol{E}}{\lambda_1 - \lambda_0}\,\frac{\boldsymbol{G} - \lambda_2 \boldsymbol{E}}{\lambda_1 - \lambda_2} = \frac{1}{3}\begin{bmatrix} 1 & a^2 & a \\ a & 1 & a^2 \\ a^2 & a & 1 \end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ a \\ a^2 \end{bmatrix}\frac{1}{\sqrt{3}}[1\ a^2\ a] = \mathbf{S}_1\overline{\mathbf{S}}_1^{*}\,.$$

$$\boldsymbol{L}_2 = \frac{\boldsymbol{G} - \lambda_0 \boldsymbol{E}}{\lambda_2 - \lambda_0}\,\frac{\boldsymbol{G} - \lambda_1 \boldsymbol{E}}{\lambda_2 - \lambda_1} = \frac{1}{3}\begin{bmatrix} 1 & a & a^2 \\ a^2 & 1 & a \\ a & a^2 & 1 \end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ a^2 \\ a \end{bmatrix}\frac{1}{\sqrt{3}}[1\ a\ a^2] = \mathbf{S}_2\overline{\mathbf{S}}_2^{*}. \tag{4.6}$$

It is obvious from (4.6) that the eigenvectors of the transfer matrix of symmetrical non-reciprocal networks are the unit vectors of the symmetrical components, i.e. the expressions written under (3.5), (3.15), and (3.20). The transformation matrix $\mathbf{S}$ is given by equation (3.22), its reciprocal value by (3.23).

It is thus evident that in the case of symmetrical non-reciprocal networks the reduction with respect to the eigenvalues of the transfer matrix is identical with the reduction with that of the symmetrical components. Thus, the symmetrical components are promoted from among the other com-

ponent systems. Naturally the reduction is even now independent of the actual network, thus the Kirchhoff equations can be written for the individual components separately. Let us also write the transformed transfer matrix on the basis of (2.24), (3.22), (3.23), and (4.1):

$$\boldsymbol{G}_S = \boldsymbol{S}\boldsymbol{G}\boldsymbol{S}^{-1} = \begin{bmatrix} G_0 + G_1 + G_2 & 0 & 0 \\ 0 & G_0 + a^2 G_1 + aG_2 & 0 \\ 0 & 0 & G_0 + aG_1 + a^2 G_2 \end{bmatrix}. \quad (4.7)$$

Since the values figuring in the main diagonal represent the correlation between the components of various orders, therefore $(G_0 + G_1 + G_2)$ is called the transfer function of the zero order, $(G_0 + a^2 G_1 + aG_2)$ the transfer function of the positive order, and $(G_0 + aG_1 + a^2 G_2)$ the transfer function of the negative order. These are just equal to their eigenvalues.

Summary

One of the usual calculation methods of triphase linear networks is based on the reduction to components of various systems. The aim of the present paper is to clarify the unified theoretical foundations of the various reductions.

It is proved that reduction is made with respect to the eigenvalues of the transfer matrix $\boldsymbol{G}$ characterizing the network. In this case, namely, to a component of a given order of a primary quantity (intensity, voltage), a secondary quantity of the same order (intensity or voltage) belongs. In the case of symmetrical reciprocal networks the two eigenvalues of $\boldsymbol{G}$ are identical, therefore several reductions satisfying the given conditions are possible. There are, among others the symmetrical components and the $\alpha, \beta, 0$ components. The reduction of symmetrical non-reciprocal networks with respect to the eigenvalues leads to the symmetrical components.

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