SOME APPLICATIONS OF THE SPECIAL PURPOSE ANALOGUE MACHINE POLCOMP

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1. Introduction

For the designing practice of the linear control technique numerous investigating methods are at disposal which enable the determination of the expected behaviour of a control loop still in the phase of design. We think here first of all of the procedures worked out in the field of Laplace transformation and frequency method which give certain aid to examine the transient response of the system.

There is essentially a double reason why the investigating and designing methods fully described in the literature are not widely used in Hungary.

a) The investigating methods, especially in the case of control systems described by higher order differential equations require considerable worktime. The worktime requirement is still further increasing if we take into consideration that even solving one single problem requires the examination of several variations.

b) In many cases unsufficient and/or inaccurate data are at disposal for the construction of the mathematical model of the process to be controlled which results in endeavours for building up a system rather for experimental, testing works.

For reducing the worktime — requiring in many respects routine work only — as mentioned in a), beyond the application of universal analogue and digital computers numerous special purpose machines and modelling systems have been developed all over the world. The advantage of the special purpose machines is that their investment cost is only a fraction of that of the universal machines, their handling is more simple and they are suitable for investigation of the variations of one problem within a short time. Their disadvantage is that they are considerably less accurate than the digital computers.

Experimental model *Polcomp* designed and built in the Automation Research Institute of the Hungarian Academy of Sciences has technical data similar to those of Esiac [6] and enables the quick application of some designing methods. The picture of the *Polcomp* is shown in Fig. 1.1.

Owing to limited space we refer to the corresponding literature [8] as far as details of application are concerned. It deals in detail with the theoretical questions of the model [1] and [2] while [3] it contains the technical description and details of circuit realization.

On *Polcomp* the model of the frequency and the transfer function of the examined control circuit or transfer block can be built up. The equipment is suitable for examining systems the dynamical characteristics of which can be described or approached in the frequency domain by polinomial equations of a complex variable.

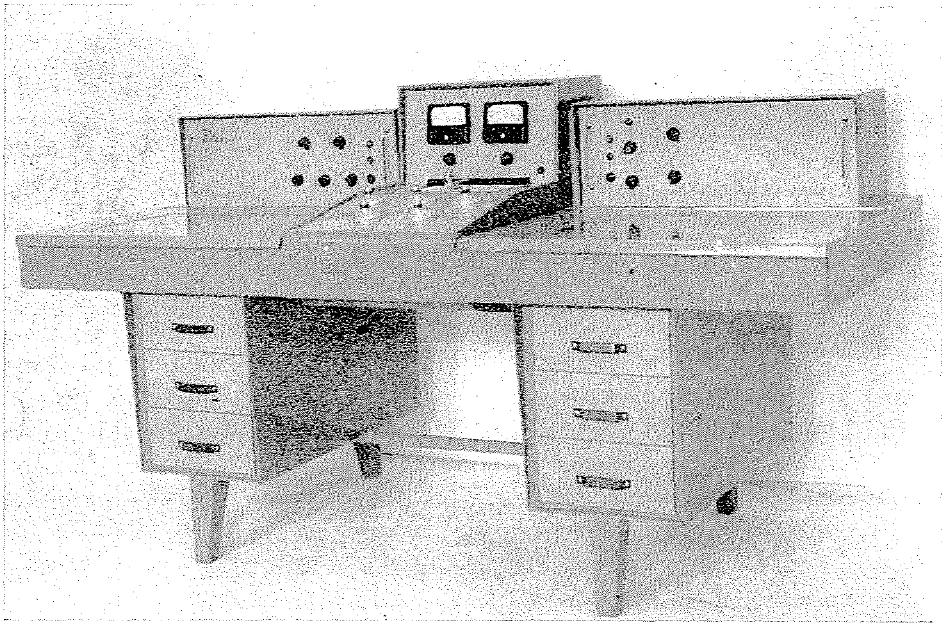

Fig. 1.1

On *Polcomp* the transfer functions can be programmed according to either of the following two factored forms:

$$Y(s) = K_h s^i \prod_{n=1}^{N} (s - s_n)^{c_n} \qquad (1.1)$$

$$Y(s) = K\ s^i \prod_{n=1}^{N} \left(1 - \frac{s}{s_n}\right)^{c_n} . \qquad (1.2)$$

The transfer functions (1.1) and (1.2) only differ in the real multiplier recalculation which can be carried out according to:

$$K = K_h \prod_{n=1}^{N} (- s_n)^{c_n} . \qquad (1.3)$$

(K and K_h real coefficients are not distinguised on the equipment as they are necessary only for the comparison of the achieved results in case of using forms (1.1) and (1.2). Real coefficient on *Polcomp* is marked with K.)

From among the quantities figuring in equations (1.1) and (1.2) the complex quantity characterized by magnitude and angle of Y, s and s_n:

$$\begin{aligned} Y &= |Y|\, e^{j\sphericalangle r} \\ s &= |s|\, e^{js\sphericalangle l} \\ s_n &= |s_n|\, e^{j\sphericalangle s_n} \end{aligned} \tag{1.4}$$

while K, i and c_n can assume real values. In the modeled transfer function s_n ($n = 1, 2, \ldots, N$) means the zeros, if the sign of c_n exponent is positive, and the poles if sign of c_n is negative. Magnitude of c_n exponent determines the order of the pole and zero, respectively.

2. Taking the root locus plot

Useful characteristic of *Polcomp* is that by means of it tracing of root locus plots, which otherwise requires considerable calculating and designing work, can be carried out within a short time. In such cases $Y(s)$ means the open loop transfer function, the obtained root locus plots give the poles of the closed loop system (roots of the characteristic equation) on the complex plane in the course of the change of the open loop coefficient. The root locus plot is situated at the points of the complex plane where

$$Y(s) = -1 \quad \text{i.e.} \quad |Y(s)| = 1 \quad \text{and} \quad \sphericalangle Y(s) = 180^\circ \tag{2.1}$$

is valid.

By means of the equipment, after programming $Y(s)$, by varying $|s|$ and $\sphericalangle s$ the points (curves) of the complex plane satisfying condition (2.1) can easily be found. In connection with detailed description of the root locus plot system we refer to the literature [4].

The points of the root locus plot can be achieved by solving the phase equation, while by solving the amplitude equation the value of the open loop coefficient K belonging to the given root can be obtained.

2.1. Example

Transfer function of control system according to Fig. 2.1 should be

$$Y(s) = \frac{K}{s(1 + 2.5s)(1 + 0.5s)(1 + 0.067s)} \,. \tag{2.2}$$

The root locus plots of the closed loop system during the changing of open loop gain $0 < K < \infty$ are to be determined.

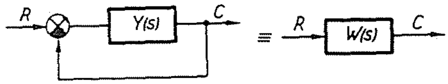

Fig. 2.1

It is suggested to transcribe and program $Y(s)$ transfer function according to equation (1.2):

$$Y(s) = Ks^{-1}\left(1 - \frac{s}{s_1}\right)^{-1}\left(1 - \frac{s}{s_2}\right)^{-1}\left(1 - \frac{s}{s_3}\right)^{-1} \tag{2.3}$$

where

$$s_1 = -\frac{1}{2.5} = 0{,}4\underline{/180^\circ}$$

$$s_2 = -\frac{1}{0.5} = 2\underline{/180^\circ}$$

$$s_3 = -\frac{1}{0.067} \cong 15\underline{/180^\circ}.$$

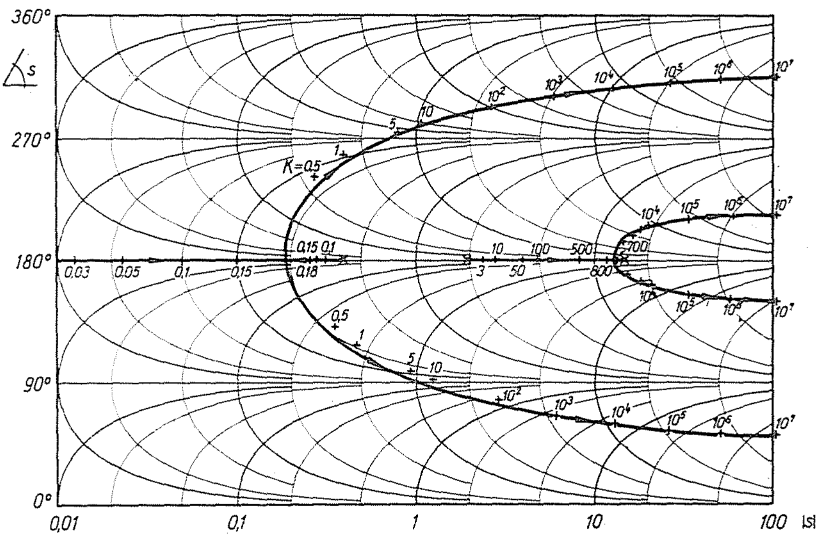

Fig. 2.2

The resulting root locus plot is shown in Fig. 2.2. It can be seen on the figure that control is becoming unstable at $K_{crit} \sim 4$ open loop gain. At value $K > K_{crit}$ the characteristic equation of the closed loop system has two roots with positive real components.

3. Inverse root loci

At the synthese of control circuits it occurs that the transfer characteristics of the open loop are to be determined knowing the required transfer function of the closed loop system. If $W(s)$ is the transfer function of the closed loop to be realized and the feedback is rigid the $Y(s)$ transfer function of the open loop (configuration of poles and zeros) can be determined definitely. Zeros of $Y(s)$ transfer functions are given by zeros of $W(s)$ transfer function, while poles of $Y(s)$ are identical with the roots of equation

$$1 - W(s) = 0. \tag{3.1}$$

For determining these, the factored forms $W(s)$ are to be programmed on *Polcomp* and when $|Y| = 1$ and $\sphericalangle Y = 0°$ is valid, then by changing $|s|$ and $\sphericalangle s$ roots of equation (3.1) can be found, i. e. inverse root loci on the complex plane.

4. Bode gain-plot

When taking the BODE gain-plot, a solution is to be found for the magnitude of the examined transfer function while the complex variable ($s = j\omega$) is running through the given part of the positive imaginary axis [5]. The angle of the complex variable must be fixed on angle value of $\sphericalangle s = 90°$ after programming the examined $Y(s)$ function on the amplitude unit of *Polcomp*. The amplitude unit will be balanced by the changing of $|Y|$ for the preset values of angular frequency $\omega = |s|$. By adjusting the registering device for fixing the values $|Y|$ and $|s|$ which belong together, the BODE plot can be drawn up in the accustomed form.

4.1. Example

Let us determine the BODE gain-plot of the control system shown in Fig. 2.1 for the open loop system $A(\omega)$ and for the closed loop system $M(\omega)$ when the open loop gain $K = 1$ [1/sec].

The transfer function of the *open loop* (according to 2.3) is programmed on *Polcomp* at value $K = 1$. The obtained curve is shown in the angular frequency range 0.01 [1/sec] $< \omega <$ 100 [1/sec] in Fig. 4.1.

Transfer function of the *closed loop:*

$$W(s) = \frac{Y(s)}{1 + Y(s)} = \frac{1}{\left(1 - \frac{s}{s_\alpha}\right)\left(1 - \frac{s}{s_\beta}\right)\left(1 - \frac{s}{s_\gamma}\right)\left(1 - \frac{s}{s_\delta}\right)} \tag{4.1}$$

where $s_\alpha \ldots s_\delta$ are the poles of the closed loop system which can be read from the root locus plot shown in Fig. 2.2, for open loop gain value of $K = 1$.

$$\begin{aligned} s_\alpha &= -2.2 \\ s_\beta &= -15 \\ s_{\gamma,\delta} &= 0.5\underline{/180^\circ \pm 78^\circ}. \end{aligned} \tag{4.2}$$

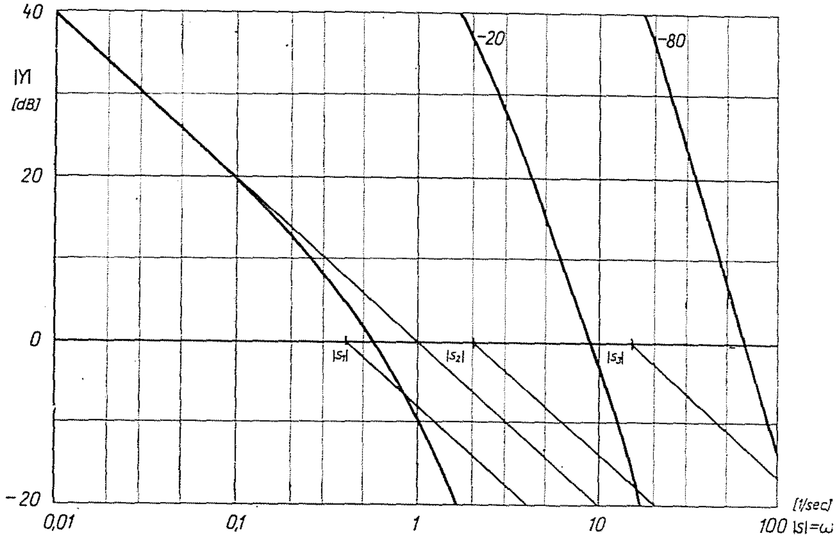

Fig. 4.1

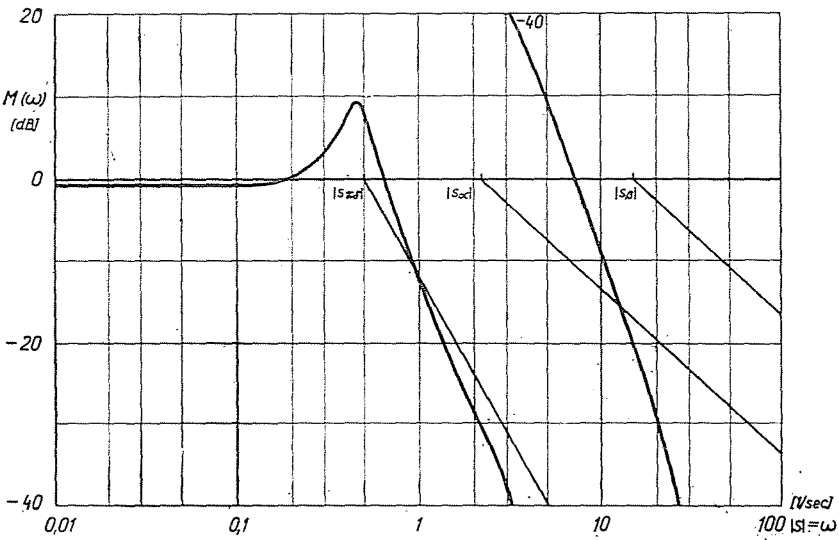

Fig. 4.2

The frequency curve $M(\omega)$ of the system with the transfer function $W(s)$ programmed according to equation (4.2) is shown in Fig. 4.2 in the angular frequency range 0.01 [1/sec] $< \omega <$ 40 [1/sec].

5. Bode phase-plot

When taking the BODE phase-plot a solution is to be found for the angle of the examined transfer function while the complex variable is running through the given part of the positive imaginary axis [5].

After programming in phase unit the examined transfer function of the angle of the complex variable must be adjusted to value $\sphericalangle s = 90°$. At $\omega = |s|$ values of the angular frequency the phase unit must be compensated by changing $\sphericalangle Y$. Adjusting the registering device for plotting values of $\sphericalangle Y$ and $|s|$ belonging together, the BODE diagram can be obtained in its usual form.

5.1. Example

Let us determine the $\varphi(\omega)$ and the $\alpha(\omega)$ phase shift of the open loop system and that of the closed loop system, respectively, in the angular frequency range 0.01 [1/sec] $< \omega <$ 100 [1/sec] at the open loop gain $K = 1$. The diagrams are shown in Fig. 5.1.

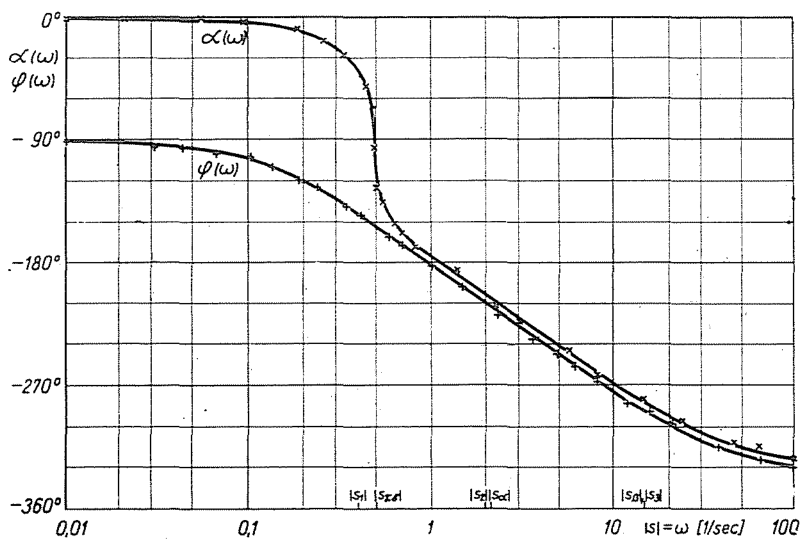

Fig. 5.1

6. Determination of the transient response

By means of *Polcomp* the transient response of the control system or block defined by its transfer function can be determined. The equipment performs the inverse Laplace transformation by applying the expansion theorem [5], defines the magnitude, damping and angular frequency of oscillation of the single components of the transient response. On the basis of these the transient process can easily be evaluated even without plotting the curve (See details under [8]).

6.1. Example

Let us determine the transient response of the system shown in Fig. 2.1 when a unit step reference input is applied. Let the open loop transfer function of the system given by Equ. (2.2) at the open loop gain $K = 1$ [1/sec]. In this case the $W(s)$ transfer function of the closed loop system is given by Equs. (4.1) and (4.2). The Laplace transform of the transient response is to be programmed in the form:

$$\frac{W(s)}{s} = \frac{1}{s\left(1+\dfrac{s}{2.2}\right)\left(1+\dfrac{s}{15}\right)\left(1-\dfrac{s}{0.5/\underline{102^\circ}}\right)\left(1-\dfrac{s}{0.5/\underline{258^\circ}}\right)} =$$

$$= \frac{8{,}25}{s(s+2.2)(s+15)(s-0.5/\underline{102^\circ})(s-0.5/\underline{258^\circ})} \qquad (6.1)$$

Measured and theoretical results are contained in Table I.

Table I

k pole index	s_k insulated pole	R_k residue belonging to pole s_k	
		measured	theoretical
1	$2.2/\underline{180^\circ}$	$0.05/\underline{180^\circ}$	$0.065/\underline{180^\circ}$
2	$15/\underline{180^\circ}$	$2.2\cdot 10^{-4}/\underline{0^\circ}$	$1.94\cdot 10^{-4}/\underline{0^\circ}$
3	$0.5/\underline{102^\circ}$	$0.55/\underline{160^\circ}$	$0.53/\underline{152.9^\circ}$
4	$0.5/\underline{258^\circ}$	$0.55/\underline{201^\circ}$	$0.53/\underline{207.1^\circ}$
5	0 ($i = 0$, $s \to 0$)	$0.93/\underline{0^\circ}$	$1/\underline{0^\circ}$

Transient response of the system:

$$v(t) = 1 - 0.05e^{-2{,}2t} + 2.2\cdot 10^{-4}\,e^{-15t} + 1.1e^{-0{,}104t}\,\cdot$$

$$\cdot\cos\left(0.49t + \frac{160^\circ}{360^\circ}\,2\pi\right).$$

Leaving out of consideration the quickly damping, and small-weight components the approximate form of the transient response is the following:

$$v(t) \simeq 1 + 1.1e^{-0{\cdot}104t} \cos (0.49t + 0.89\pi).$$

7. Factoring polynomials

On *Polcomp* the transfer functions can be programmed in factored forms. It often occurs that the transfer function to be evaluated is given in the form of ratio of polynomials. The polynomials can be factored by means of the equipment. The polynomial to be factored has the form:

$$\sum_{k=0}^{K} a_k s^k = a_0 + a_1 s + a_2 s^2 + \ldots + a_{K-1} s^{K-1} + a_K s^K \tag{7.1}$$

where s is the complex variable and a_k the real coefficient of k^{th} term of K-order polynome.

Polynome (7.1) can be factored in two ways, which are shown in equations (1.1) and (1.2):

$$\sum_{k=0}^{K} a_k s^k = a_K \prod_{k=1}^{K} (s - s_k) \tag{7.2}$$

$$\sum_{k=0}^{K} a_k s^k = a_0 \prod_{k=1}^{K} \left(1 - \frac{s}{s_k}\right) \tag{7.3}$$

where s_k is the k^{th} root of the polynome. In both cases the factoring requires repeated application of the root locus method so that only the root configuration belonging to only one single real parameter value must be determined in each case.

8. Examination of systems containing a dead-time element

Control systems containing an element of transfer function e^{-Ts}(T is the dead-time) in the open loop system, can be analyzed approximately on *Polcomp*.

For approaching the transfer function of the dead-time element several methods are described [7]. Out of these methods those given in [8] are applied most advantageously for our purpose.

Programming on the equipment the rational fractional function in factored form approaching the exponential transfer function, the all examination described in Points 2—6 can be carried out for the tested system.

9. Examination of sampled data systems

Analysis and synthesis of sampled data systems can be effected on the *Polcomp* by modelling and testing of the z-transform of the discrete transfer function of the system.

Discrete transfer function $Y^*(s)$ of the open loop contains exponential expressions of form e^{sT} (i.e. it has poles, resp. zeros of infinite number) and so even when not considering the farther poles programming of $Y^*(s)$ might be problematic.

On the other hand, the z-transform of discrete transfer function of either the open loop, or the closed loop system can be given in form of factored multipliers or polynomes in many cases, and can be programmed immediately.

In possession of the z-transform of discrete transfer function $Y(z)$ the following tasks can be solved:

a) Root locus plot of the system in z-plane can be determined, according to method described under Point 2, considering that on the z-plane the area within the circle of unit radius complies with the left-side half-plane of s-plane.

b) The NYQUIST-diagram of the open loop of sampled data system can be determined which enables the application of the NYQUIST-criterion. Considering that the frequency response according to ω_0 is periodical, it is sufficient to plot in the region $0 \leq \omega \leq \omega_0$ the NYQUIST-diagram. On the z-plane counter-clockwise turning of the unit-radius circle complies with the ranging over of the positive imaginary axis between 0 and ω_0 on s-plane, according to connection:

$$\sphericalangle z = 360° \frac{\omega}{\omega_0} . \tag{9.1}$$

The points of the NYQUIST-diagram are given by the values belonging together of $|Y(z)|$ and $\sphericalangle Y(z)$ which are taken by function $Y(z)$ during parameter change $\sphericalangle z$ according to the angular frequency (9.1) at the condition $|z| = 1$.

c) The NYQUIST-diagram of the closed loop of sampled data system can be taken. From root locus plot taken according to point a) the poles of the closed loop system on z-plane can be read, with the help of which the z-transform of discrete transfer function of the closed loop system $W(z)$ can be given in factored form. $W(z)$ can be programmed immediately on *Polcomp* and the frequency response of the closed loop system can be achieved according to the methods described in point b).

9.1. Example

Let us determine the root locus plot of a simple, closed loop sampled data control system and the NYQUIST-diagram of its open loop on the basis of block diagram shown in Fig. 9.1.

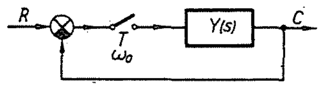

Fig. 9.1

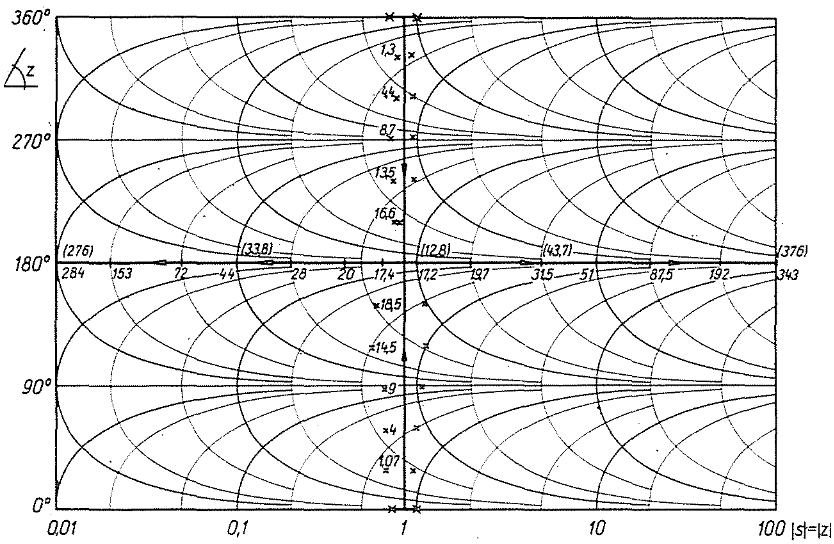

Fig. 9.2

Should the transfer function of the unsampled open loop be

$$Y(s) = \frac{K}{s(1+s)} \tag{9.2}$$

and the sampling angular frequency and the period time:

$$\omega_0 = 20[1/\text{sec}]\,, \quad T = \frac{2\pi}{\omega_0} = 0.314\,[\text{sec}]. \tag{9.3}$$

Discrete transfer function of the open loop by using (9.2) and (9.3) is the following:

$$Y^*(s) = \frac{Ke^{sT}(1-e^{-T})}{(e^{sT}-e^{-T})(e^{sT}-1)}\,.$$

The z-transform of discrete transfer function of the open loop:

$$Y(z) = \frac{Kz(1-e^{-T})}{(z-e^{-T})(z-1)} = \frac{0.27\,Kz}{(z-0.73)(z-1)} . \tag{9.4}$$

For the determination of the root locus plot the model of transfer function $Y(z)$ must be built up on *Polcomp* (in consequence of the formal replacement of $s = z$ the equipment now represents the z-plane instead of the s-plane)

Table II

ω rad/sec	$\sphericalangle z$ degree	$Y(j\omega)$ measured	$Y(j\omega)$ theoretical
0	0	—	$\infty/\underline{270°}$
1.7	30	$4.5/\underline{217°}$	$3.78\ /\underline{210°}$
3.4	60	$1.22\ /\underline{200°}$	$1.11\ /\underline{194.5°}$
5.0	90	$0.573/\underline{193°}$	$0.57\ /\underline{189°}$
6.7	120	$0.37\ /\underline{191°}$	$0.388/\underline{185°}$
8.4	150	$0.29\ /\underline{187°}$	$0.314/\underline{182°}$
10.0	180	$0.28\ /\underline{180°}$	$0.29\ /\underline{180°}$
11.7	210	$0.3\ /\underline{177°}$	$0.314/\underline{178°}$
13.4	240	$0.37\ /\underline{174°}$	$0.388/\underline{175°}$
15.0	270	$0.565/\underline{170°}$	$0.57\ /\underline{171°}$
16.7	300	$1.12\ /\underline{165°}$	$1.11\ /\underline{165.5°}$
18.4	330	$3.82\ /\underline{154°}$	$3.78\ /\underline{150°}$
20.0	360	—	$\infty/\underline{90°}$

and under condition that $Y(z) = 1/\underline{180°}$ the root locus plot is to be taken in the z-plane in the manner described in Point 2. The taken root locus plot is shown in Fig. 9.2. A continuous line marks the exact situation of the root locus plot on the Figure. The measured values, which were taken with two different methods, are marked with crosses along the branches outside the real axis.

Along the branches situated on the negative real axis the values in brackets mark the exact value of the open loop gain.

The NYQUIST-diagram of the open loop can be determined under condition $|z| = 1$ from transfer function $Y(z)$, by adjusting the angular frequency according to (9.1). The measured and theoretical values for $0.27K = 1$ [1/sec] open loop gain are shown in Table II.

The author expresses his warm thanks to Prof. F. CSÁKI for valuable suggestions and advice.

Summary

This paper deals with the main field of application of the analog special purpose machine *Polcomp* which was designed and manufactured in the Automation Research Institute of the Hungarian Academy of Sciences.

The paper gives a review on the fast application of several designing and analysing methods of the linear control theory by means of *Polcomp* such as the root locus method, *Bode* plots, inverse root loci, determining the components of the step response; factorizing polynomials, modelizing dead-time element and analyzing sampled data systems.

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