STATISTICAL SYNTHESIS METHODS OF SINGLE- AND MULTIVARIABLE CONTROL SYSTEMS*

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1. Introduction

In control engineering no unique theory has been developed, not even concerning the linear systems. As the most important design methods, the following may be mentioned:

1. The "classic" method.
2. The statistical method.
3. The phase-space method.

From among the enumerated methods the first one is based on typical deterministic signals (e.g. unit impulse, unit step, unit velocity step, unit acceleration step). The prescribed value of the so-called steady-state control deviation determines the magnitude of the loop gain (for simplicity's sake we shall consider a simple, single variable, single loop control system). With the loop amplification known, the stability of the system may already be investigated, e.g. by one of the procedures of NYQUIST, BODE, NICHOLS, MIKHAILOV, LEONHARD, ROUTH, HURWITZ, EVANS, etc. [1—6].

In most cases the system proves to be instable. In order to cancel the contradiction between the low value of the steady-state error and the securing of the necessary degree of stability also some signal-forming (compensation) organs, resp. elements are to be included. The investigations are most purposefully carried out in the frequency domain and it is perhaps the simplest to use the logarithm magnitude — logarithm frequency, i.e. the so-called BODE diagrams.

In this case we start out from the transfer function of the open control loop. In many cases by the inclusion of an appropriate compensation element it is possible to reach, that the approximating BODE diagram, even without diminishing the loop gain should intersect the 0 dB axis with a slope of —20 dB/decade (6 dB/octave). If this section of —20 dB/decade is long enough to the right and to the left, then usually the quality requirements are also satisfied. For checking this we should return into the time domain by the inverse Laplace transformation. But this may also be avoided, if we

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rely on the CHESTNUT—MAYER performance charts [7], with the aid of which the most important quality characteristics for a given BODE diagram may be estimated. In the knowledge of the settling time, the maximum overshoot, the oscillation frequency it is possible to check the progress of the control process. If we happened to transgress against some of these prescriptions, then we reiterate the whole procedure with the inclusion of a different compensation element or loop amplification. So the "classic" method might be connected with a certain amount of trial and error as well. This method looks back by now to a past of over three decades.

The second method chooses, for the truer approximation of reality, signals of an irregular random course, varying according to a random process and the control system it may be designed on the basis of a certain statistical characteristic of the signal. Further on we shall deal with this method in detail. Here preliminarily we only wish to mention, that this method has been developed on the basis of the work of WIENER and KOLMOGOROV about two decades ago [8—13].

Finally the third method realizes the design in the so-called phase space. On the coordinate system axes the state variables of the control system, in most cases its phase coordinates (signal, signal velocity, signal acceleration, etc.) figure. The task is to find the choice of a control signal (or signals) in such a way, that the system should convert from one state into another in the most favourable manner. Among the usual processes the variational calculus, the PONTRYAGIN maximum principle and the BELLMANN dynamic programming may be mentioned [14, 15]. The third method is still younger than a decade.

Optimization plays an important role in all three methods. In the case of the first method the question arises thus: By what compensation elements can the foreseen quality characteristics be optimally fulfilled, with as little as possible steady-state error? In the case of the second method an optimum weighting function, resp. an optimum transfer function is to be determined for the whole system or for a part of it (e.g. for the compensation element) in such a way, that the output signal should optimally approximate the desired stochastic signal. Finally in the case of the third method the switch-over time instants are to be determined in such a way that a certain cost functional, e.g. the changing time, or the energy consumption should reach a minimum. The first method may be regarded as already nearly closed, the second one approaches closure, whereas the third one is still greatly developing.

II. The fundamental variant of the statistical design method

For the introduction of the statistical design method we shall start out from the simplest possible variant (Fig. 1).

The stochastic signals are denoted by the lower case letters of the Latin alphabet. It is assumed that the input signal $r(t)$ of the linear, invariant concentrated parameter, single variable control system of weighting function $w(t)$ consists of two components: of the useful command signal component $s(t)$ and of the disturbing signal component (in short: noise) $n(t)$. At the system output the stochastic output signal $c(t)$ arises. This is compared with the ideal output signal $i(t)$, in other words with the desired signal. It is to be noted, that the ideal signal is sometimes considered to be derived from the command

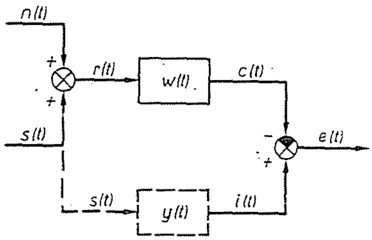


Fig. 1

signal with the aid of a certain transfer member $y(t)$. The weighting function $y(t)$ — as we shall see later — is not one which must be physically realizable unconditionally. The difference between the signals $i(t)$ and $c(t)$ is the error signal $e(t)$.

The statistical design of the control system sets the task of determining the weighting function $w(t)$, or the transfer function $W(s)$. The transfer function — as is well known — is the Laplace transform of the weighting function: $W(s) = \mathscr{L}[w(t)]$.

The starting conditions are:

1. The statistical characteristics, e.g. the autocorrelation function $\varphi_{rr}(\tau)$, or the power-density spectrum $\Phi_{rr}(s)$ are assumed as being known.

2. The ideal output signal must be chosen. E.g. if $i(t) = s(t)$, then the task is the true filtering [in this case $y(t) = \delta(t)$, where $\delta(t)$ is the DIRAC unit impulse function, more correctly distribution]. It rarely happens, that it should be $i(t) = s(t - T_d)$, i.e. $y(t) = \delta(t - T_d)$, i.e. that the task should be the following of the command signal delayed by deadtime T_d. The prediction is very interesting, when $i(t) = s(t + T_s)$, i.e. $y(t) = \delta(t + T_s)$, so in this case the signal sped by time T_s is to be followed with as little error as possible. This task arises, for instance, in the case of anti-aircraft batteries, or of rockets, but similar tasks are met with in every system, where the controlled section contains a deadtime and the task is the true following of the command signal.

3. Also the optimization criterion must be given. Most often the set task is the minimization of the error signal mean-square value, i.e.

$$\overline{e^2(t)} = \overline{[e(t)]^2} = \text{minimum} \tag{1}$$

not as if this were always the best criterion, but doubtlessly this is the criterion to be treated mathematically as the simplest, which is independent from the sign of the error signal.

III. A few relationships of the stochastic signals

It must be mentioned here that the stochastic signals are assumed to be stationary and ergodic, because otherwise the difficulties would greatly increase. We regard the stochastic signal as stationary, when its statistical characteristics do not depend from the beginning of the time count (e.g. the correlation function is only the function of time shift τ and not the two-variable function of time data t_a and $t_b = t_a + \tau$). The ergodic hypothesis assumed that the ensemble average of a great number of signals and the time mean value of a single representative signal coincide (see e.g. [11, 16—20]).

With the assumption of the ergodicity of stationary process e. g. the definition of the cross-correlation function of the actual and the ideal output signals on the basis of the time average formation is:

$$\varphi_{ci}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} c(t)\, i(t+\tau)\, dt \tag{2}$$

while e.g. the autocorrelation function of the error signal is:

$$\varphi_{ee}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e(t)\, e(t+\tau)\, dt\,. \tag{3}$$

The stochastic signals can be characterized not only by the correlation functions, but also by the power-density spectra. The latter are, according to the WIENER—KHINCHIN relationship, the Fourier transforms of the correlation functions

$$\Phi_{ci}(s) = \int_{-\infty}^{\infty} \varphi_{ci}(\tau)\, e^{-s\tau}\, d\tau = \mathcal{F}[\varphi_{ci}(\tau)]\,, \tag{4}$$

or

$$\Phi_{ee}(s) = \int_{-\infty}^{\infty} \varphi_{ee}(\tau)\, e^{-s\tau}\, d\tau = \mathcal{F}[\varphi_{ee}(\tau)]\,. \tag{5}$$

Here and further on too $s = j\omega$.

From the power density spectrum the correlation function may be determined by the inverse Fourier transformation:

$$\varphi_{ci}(\tau) = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \Phi_{ci}(s)\, e^{\tau s}\, ds = \mathcal{F}^{-1}\left[\Phi_{ci}(s)\right], \tag{6}$$

or

$$\varphi_{ee}(\tau) = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \Phi_{ee}(s)\, e^{\tau s}\, ds = \mathcal{F}^{-1}\left[\Phi_{ee}(s)\right]. \tag{7}$$

The significance of the autocorrelation function, resp. of the power-density spectrum is especially evident by the fact, that they may be brought into direct relationship with the mean-square value of the error signal

$$\overline{e^2(t)} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^2(t)\, dt \tag{8}$$

figuring in the design criterion according to the relationships

$$e^2(t) = \varphi_{ee}(0), \tag{9}$$

resp.

$$\overline{e^2(t)} = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \Phi_{ee}(s)\, ds. \tag{10}$$

The first relationship is simpler than the second one, yet — as we shall see later during the calculations — our goal is reached easier, if we start from the second one. Though here we are not going to extend upon the features of the correlation functions and the power-density spectra, as these may be found elsewhere, e.g. [9, 11, 12], yet we shall mention in short the rules of the index inversion and of index changing [11, 19, 20].

According to the rule of index inversion for instance

$$\varphi_{ci}(\tau) = \varphi_{ic}(-\tau), \tag{11}$$

or

$$\Phi_{ci}(s) = \Phi_{ic}(-s). \tag{12}$$

On the other hand if $r(t)$ and $c(t)$ are the input, resp. output signals of a control element or system with $w(t)$ weighting function, then the formal index changing rules are, for instance, as follows:
Changing of the second index:

$$\varphi_{cc}(\tau) = \int_{-\infty}^{\infty} w(\vartheta)\,\varphi_{cr}(\tau - \vartheta)\,d\vartheta \tag{13}$$

resp.

$$\Phi_{cc}(s) = W(s)\,\Phi_{cr}(s)\,. \tag{14}$$

Changing of the first index:

$$\varphi_{cr}(\tau) = \int_{-\infty}^{\infty} w(\zeta)\,\varphi_{rr}(\tau + \zeta)\,d\zeta \tag{15}$$

resp.

$$\Phi_{cr}(s) = W(-s)\,\Phi_{rr}(s)\,. \tag{16}$$

Changing of both indices:

$$\varphi_{cc}(\tau) = \int_{-\infty}^{\infty} w(\zeta) \int_{-\infty}^{\infty} w(\vartheta)\,\varphi_{rr}(\tau - \vartheta + \zeta)\,d\vartheta\,d\zeta\,, \tag{17}$$

resp.

$$\Phi_{cc}(s) = W(-s)\,W(s)\,\Phi_{rr}(s)\,. \tag{18}$$

We will not deal here with the proof of these rules but refer to the literature [9, 11, 12, 18—20].

IV. Variations of the statistical design method

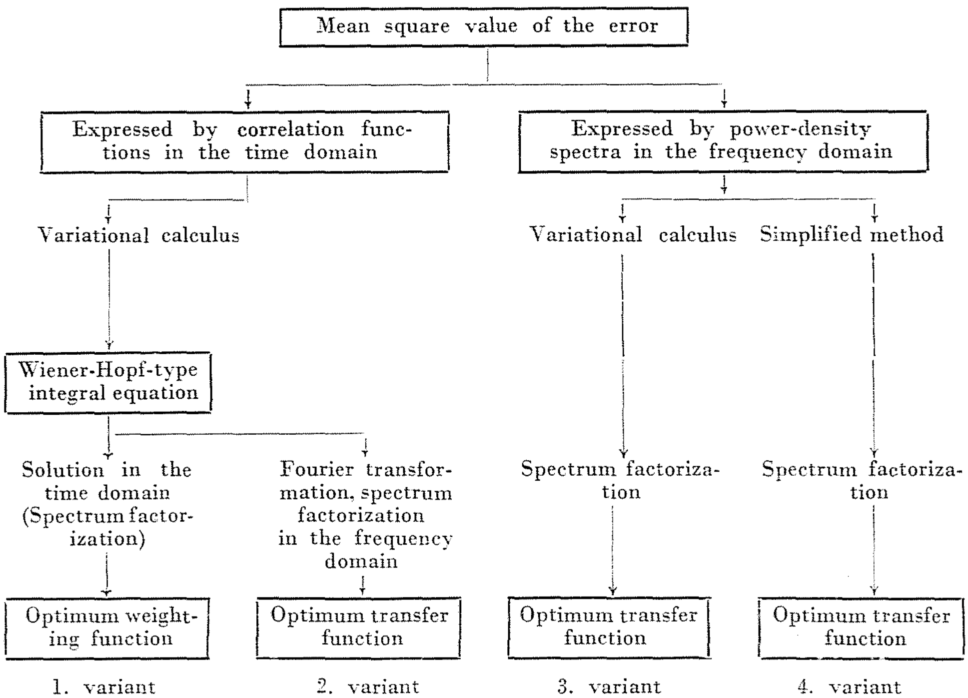
The variants of the statistical design method are summed up in Table I. For a start first the mean square values of the error signal must be written up. In the first, two variants the latter is expressed with the aid of the correlation functions. Then at the first variant we arrive through the variational calculus performed in the time domain to the WIENER—HOPF integral equation, which further on is reduced by a few mathematical manipulations and conversions to an integral equation of the first kind. The solution of the first kind integral equation in the time domain gives the optimum weighting function.

The second procedure solves the first kind integral equation by the Fourier transformation; in this way the optimum transfer function can be determined.

The first two procedures were developed by WIENER [8].

The third method expresses the mean-square value of the error signal by power-density spectra. Then by variational calculus and spectrum factorization in the frequency domain it finally results in the optimum transfer function. The third method comes from TSIEN [21, 22].

Table I



The fourth method also starts from the power density spectra, but instead of the variational calculus it arrives by simple elementary considerations to the optimum transfer function. This method was first applied by BODE and SHANNON [23], but they assumed the existence of an uncorrelated command signal and a noise. The author of this paper made the solution independent of this constraint and, at the same time, proposed a considerably simpler procedure [24—26].

The advantage of the simplified method shows itself especially when going over from the simpler configurations to the more complex ones, e.g. from the totally free configuration to the semi-free or to the constrained semi-free configuration, as it was first proposed by NEWTON [27, 28]. Similarly when going over from single variable systems to multivariable ones.

In the following we shall consider each variant in turn for the sake of comparison. The starting point is the expression of the error:

$$e(t) = i(t) - c(t) \tag{19}$$

resp. the square of the error

$$e^2(t) = i^2(t) - c(t)i(t) - i(t)c(t) + c^2(t), \tag{20}$$

or still better the mean-square value of the error

$$\overline{e^2(t)} = \overline{i^2(t)} - \overline{c(t)\,i(t)} - \overline{i(t)\,c(t)} + \overline{c^2(t)}\,. \tag{21}$$

V. First variant: Variational calculus in the time domain

In the first variant [8, 11] the mean square value of the error is expressed with the aid of the correlation functions:

$$\overline{e^2(t)} = \varphi_{ii}(0) - \varphi_{ci}(0) - \varphi_{ic}(0) + \varphi_{cc}(0)\,. \tag{22}$$

Taking into consideration the rules of index changing and of index inversion:

$$\begin{aligned}\overline{e^2(t)} = \varphi_{ii}(0) - 2\int_{-\infty}^{\infty} w(t_1)\,\varphi_{ri}(t_1)\,dt_1 + \\ + \int_{-\infty}^{\infty} w(t_1)\int_{-\infty}^{\infty} w(t_2)\,\varphi_{rr}(t_1 - t_2)\,dt_2\,dt_1\,.\end{aligned} \tag{23}$$

Our task is to determine a weighting function $w_m(t)$, which minimizes the mean square value of the deviation. For its determination we use the variational calculus. Let us assume, that there exists a minimizing weighting function $w_m(t)$; then if the latter is altered, the error mean-square value must increase. Let us produce the weighting function $w(t)$, as a linear combination of $w_m(t)$ minimizing and of $w_\varepsilon(t)$ arbitrary, but physically realizable weighting functions:

$$w(t) = w_m(t) + \varepsilon\, w_\varepsilon(t), \tag{24}$$

where ε is a variational parameter. According to the above the minimum mean-square error occurs at $\varepsilon = 0$. In other words the derivative of the mean-square error is zero:

$$\left.\frac{d\,\overline{e^2(t)}}{d\varepsilon}\right|_{\varepsilon=0} = 0\,. \tag{25}$$

Starting out from this condition and carrying out the derivation according to ε and the substitution $\varepsilon = 0$, we have:

$$-2 \int_{-\infty}^{\infty} w_{\varepsilon}(t_1) \varphi_{ri}(t_1)\, dt_1 + \int_{-\infty}^{\infty} w_m(t_1) \int_{-\infty}^{\infty} w_{\varepsilon}(t_2) \varphi_{rr}(t_1 - t_2)\, dt_2\, dt_1 +$$

$$+ \int_{-\infty}^{\infty} w_{\varepsilon}(t_1) \int_{-\infty}^{\infty} w_m(t_2) \varphi_{rr}(t_1 - t_2)\, dt_2\, dt_1 = 0. \tag{26}$$

The last two terms evidently coincide, thus:

$$2 \int_{-\infty}^{\infty} w_{\varepsilon}(t_1) \Big[\int_{-\infty}^{\infty} w_m(t_2) \varphi_{rr}(t_1 - t_2)\, dt_2 - \varphi_{ri}(t_1) \Big] dt_1 = 0. \tag{27}$$

As $w_{\varepsilon}(t)$ is physically realizable, thus for negative times it must be identically zero, i.e. it must be a so-called positive-time function. (According to the definition, the weighting function is the output signal arising under the effect of the impulse function, which is acting at the time instant $t = 0$.) But for positive times the left side of the preceding expression can be only zero, when also the arbitrary weighting function $w_{\varepsilon}(t)$ is taken into consideration, if the expression in the square brackets is zero:

$$\int_{-\infty}^{\infty} w_m(t_2) \varphi_{rr}(t_1 - t_2)\, dt_2 - \varphi_{ri}(t_1) = 0, \qquad 0 \leqq t_1. \tag{28}$$

This important relationship is the WIENER—HOPF integral equation. It is this equation which must be satisfied by the minimizing weighting function $w_m(t)$. Unfortunately the solution of the integral equation in the time domain is not an easy task.

Before continuing we shall prove that the calculated extreme value is really minimum. Physically this is clear, because the maximum of the mean-square value is evidently infinite. For its mathematical proof the second derivate must be determined:

$$\frac{d^2 \overline{e^2(t)}}{d\varepsilon^2} = 2 \int_{-\infty}^{\infty} w_{\varepsilon}(t_1) \int_{-\infty}^{\infty} w_{\varepsilon}(t_2) \varphi_{rr}(t_1 - t_2)\, dt_2\, dt_1 .$$

The right side term is the mean square value of an output signal, which arises on the effect of an input signal led through the system of weighting function $w_{\varepsilon}(t)$. As the mean square value is never negative, the second differential quotient is positive, i.e. the extreme value is indeed minimum.

VI. Spectrum factorization in the time domain

If the WIENER—HOPF integral equation were valid not only for $0 \leqq t_1$ time, but for every t_1 time, so we should have to deal with an integral equation of the first kind, which is easier to solve. So let us try to convert the WIENER—HOPF integral equation into an equation of the first kind.

As a first step we shall produce the autocorrelation function φ_{rr} as the convolution of a positive-time function φ_{rr}^{+} and of a negative-time function φ_{rr}^{-}, so

$$\varphi_{rr}(t_1 - t_2) = \int_{-\infty}^{\infty} \varphi_{rr}^{-}(t_3)\, \varphi_{rr}^{+}(t_1 - t_2 - t_3)\, dt_3. \tag{29}$$

It is to be noted, that φ_{rr}^{+} and φ_{rr}^{-} functions may be called the time-domain spectrum factors. The cross-correlation function φ_{ri} may also be expressed similarly:

$$\varphi_{ri}(t_1) = \int_{-\infty}^{\infty} \varphi_{rr}^{-}(t_3)\, \psi(t_1 - t_3)\, dt_3. \tag{30}$$

The auxiliary function ψ figuring here has generally values differing from zero both for positive and for negative times. Introducing these relationships and changing the sequence of the integrations we have:

$$\int_{-\infty}^{\infty} \varphi_{rr}^{-}(t_3) \left[\int_{-\infty}^{\infty} w_m(t_2)\, \varphi_{rr}^{+}(t_1 - t_3 - t_2)\, dt_2 - \psi(t_1 - t_3) \right] dt_3 = 0 \qquad 0 \leqq t_1. \tag{31}$$

But this latter relationship for a negative t_3 time can only be fulfilled, if the expression appearing in square brackets is zero:

$$\int_{-\infty}^{\infty} w_m(t_2)\, \varphi_{rr}^{+}(t_1 - t_3 - t_2)\, dt_2 - \psi(t_1 - t_3) = 0 \tag{32}$$

i.e. when

$$t_3 \leqq 0\,; \qquad 0 \leqq t_1;$$

$$0 \leqq t_1 - t_3.$$

Introducing the substitutions

$$t_1 - t_3 = \tau; \qquad t_2 = \vartheta$$

we obtain:

$$\int_{-\infty}^{\infty} w_m(\vartheta)\, \varphi_{rr}^{+}(\tau - \vartheta)\, d\vartheta - \psi(\tau) = 0 \qquad 0 \leqq \tau. \tag{33}$$

This integral equation is in a form similar to the original WIENER—HOPF integral equation, but now it is only in consequence of the $\psi(\tau)$ function, that the left side is not zero for negative τ time, because the convolution integral is zero for a negative time, as both w_m and φ_{rr}^{+} are positive-time functions, considering, that w_m is a physically realizable weighting function.

Let us separate the auxiliary function $\psi(\tau)$ into $\psi_{-}(\tau)$ negative-time and $\psi_{+}(\tau)$ positive-time function components:

$$\psi(\tau) = \psi_{-}(\tau) + \psi_{+}(\tau)$$

where

$$\psi_{-}(\tau) \equiv 0, \text{ when } 0 \leqq \tau;$$

$$\psi_{+}(\tau) \equiv 0, \text{ when } \tau < 0.$$

As the left side of the above equation differs from zero in the case of a negative shifting time τ only because of the $\psi_{-}(\tau)$ component, if we drop this latter, so the integral equation already gives for the whole τ domain zero result:

$$\int_{-\infty}^{\infty} w_m(\vartheta)\, \varphi_{rr}^{+}(\tau - \vartheta)\, d\vartheta - \psi_{+}(\tau) = 0 \qquad (34)$$

$$-\infty < \tau < \infty.$$

This already is a common integral equation of the first kind, which is easier to solve.

VII. Second variant: Transition to the frequency domain

The second variant solves the integral equation of the first kind by the Fourier transformation.

Relationships (29), (30) and (34) of the preceding point, Fourier-transformed (with the notation $s = j\omega$), give:

$$\Phi_{rr}(s) = \Phi_{rr}^{-}(s)\, \Phi_{rr}^{+}(s)\,, \qquad (35)$$

$$\Phi_{ri}(s) = \Phi_{rr}^{-}(s)\, \Psi(s)\,, \qquad (36)$$

$$W_m(s)\, \Phi_{rr}^{+}(s) - \Psi_{+}(s) = 0 \qquad (37)$$

namely, the convolution integrals go over into the product of the transformed functions. It is a known feature of the Fourier transformation, that all poles of the positive-time (negative-time) function transforms are positioned on the left (right) half of the plane of complex quantities. In addition in the frequency domain the spectrum factorization given by relationship (35) must be performed in such a way, that all zero points of the spectrum factors Φ_{rr}^{+}, resp. Φ_{rr}^{-} should also be on the left (resp. right) half of the plane of complex quantities. The two spectrum factors are conjugates of each other. From relationship (37)

$$W_m(s) = \frac{\Psi_+(s)}{\Phi_{rr}^{+}(s)}. \tag{38}$$

On the other hand according to relationship (36)

$$\Psi(s) = \frac{\Phi_{ri}(s)}{\Phi_{rr}^{-}(s)}. \tag{39}$$

Determining the $\psi(\tau)$ function by the inverse Fourier transformation, then separating the positive time function component $\psi_+(\tau)$ and performing the transformation, we obtain

$$\Psi_+(s) = \mathscr{L}\{\mathscr{F}^{-1}[\Psi(s)]\} \tag{40}$$

as the last two steps can be performed simultaneously with the common Laplace transformation. From relationship (39)

$$\Psi_+(s) = \left[\frac{\Phi_{ri}(s)}{\Phi_{rr}^{-}(s)}\right]_+, \tag{41}$$

by the substitution of which into expression (38), finally the requested, physically realizable optimum transfer function is:

$$W_m(s) = \frac{1}{\Phi_{rr}^{+}(s)}\left[\frac{\Phi_{ri}(s)}{\Phi_{rr}^{-}(s)}\right]_+. \tag{42}$$

VIII. Third variant: Variational calculus in the frequency domain

In the further variants the mean-square value of the error is expressed by power-density spectra according to (10):

$$\overline{e^2(t)} = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \Phi_{ee}(s)\,ds \tag{43}$$

where, taking (21) into consideration

$$\Phi_{ee}(s) = \Phi_{ii}(s) - \Phi_{ci}(s) - \Phi_{ic}(s) + \Phi_{cc}(s). \tag{44}$$

Applying the index changing rule:

$$\begin{aligned}\Phi_{ee}(s) = \Phi_{ii}(s) - W(-s)\,\Phi_{ri}(s) - \Phi_{ir}(s)\,W(s) + \\ + W(-s)\,\Phi_{rr}(s)\,W(s).\end{aligned} \tag{45}$$

Now we perform the variational calculus in the frequency domain [21, 22

Let it once more be

$$W(s) = W_m(s) + \varepsilon W_\varepsilon(s). \tag{46}$$

The derivative of the mean-square error is zero:

$$\left.\frac{d\,\overline{e^2(t)}}{d\varepsilon}\right|_{\varepsilon=0} = 0. \tag{47}$$

From this condition

$$\begin{aligned}\frac{1}{2\pi j}\int_{-j\infty}^{j\infty} W_\varepsilon(-s)\left[\Phi_{rr}(s)\,W_m(s) - \Phi_{ri}(s)\right] ds + \\ + \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \left[W_m(-s)\,\Phi_{rr}(s) - \Phi_{ir}(s)\right] W_\varepsilon(s)\,ds = 0.\end{aligned} \tag{48}$$

By the way, let us observe that the integrand in the second line is the conjugate complex expression of that in the first line. Performing the spectrum factorization given by expression (35) and taking also relationship (36) into consideration, we have:

$$\begin{aligned}\frac{1}{2\pi j}\int_{-j\infty}^{j\infty} W_\varepsilon(-s)\,\Phi_{rr}^{-}(s)\left[\Phi_{rr}^{+}(s)\,W_m(s) - \Psi(s)\right] ds + \\ + \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \left[W_m(-s)\,\Phi_{rr}^{-}(s) - \Psi(-s)\right]\Phi_{rr}^{+}(s)\,W_\varepsilon(s)\,ds = 0.\end{aligned} \tag{49}$$

It may be seen, that

$$\frac{1}{2\pi j}\int_{-j\infty}^{j\infty} W_\varepsilon(-s)\,\Phi_{rr}^{-}(s)\,\Psi_{-}(s)\,ds = 0 \tag{50}$$

and
$$\frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \Psi_-(-s)\,\Phi_{rr}^+(s)\,W_\varepsilon(s)\,ds = 0\,, \tag{51}$$

as in the individual integrals only transforms belonging to negative-time (resp. positive-time) functions appear, i.e. only right-side (resp. left-side) poles are occurring. Closing the integration path along the imaginary axis by a left-hand (right-hand) semicircle of a radius approaching infinity, zero will result according to the residuum theorem, because within the closed loop there is no pole. There is that only to be proved, that the value of the integrals along the semicircles of a radius approaching infinity is zero, but this is satisfied, as can be shown, that the denominators of the terms to be integrated are at least by two degrees higher, than their numerators. Indeed the denominators of W_ε and Ψ_- are at least one degree higher, than their numerators, while the numerators of Φ_{rr}^- and Φ_{rr}^+ are at most of the same degree as their denominators. Deducing the last two terms (50) and (51) from (49), we have:

$$\begin{aligned} &\frac{1}{2\pi j}\int_{-j\infty}^{j\infty} W_\varepsilon(-s)\,\Phi_{rr}^-(s)\left[\Phi_{rr}^+(s)\,W_m(s) - \Psi_+(s)\right] ds\, + \\ &+\frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\left[W_m(-s)\,\Phi_{rr}^-(s) - \Psi_+(-s)\right]\Phi_{rr}^+(s)\,W_\varepsilon(s)\,ds = 0\,. \end{aligned} \tag{52}$$

This relationship is fulfilled then and only then, when the terms in the square brackets are zero. As the latter are the conjugates of each other, it is enough to consider the first one, from which is again

$$W_m(s) = \frac{\Psi_+(s)}{\Phi_{rr}^+(s)}\,, \tag{53}$$

resp.

$$W_m(s) = \frac{1}{\Phi_{rr}^+(s)}\left[\frac{\Phi_{ri}(s)}{\Phi_{rr}^-(s)}\right]_+ . \tag{54}$$

IX. Fourth variant: The simplified method

The first steps of the simplified method [24—26] are the same, as in the third variant. However, now we perform no variational calculus, but we convert expression (45). The key to the conversion is given by the introduction of an auxiliary transfer function

$$G(s) = \frac{\Phi_{ri}(s)}{\Phi_{rr}(s)}\,. \tag{55}$$

As the power-density spectra are known, so $G(s)$ may also regarded as known. If we now substitute the expression

$$\Phi_{ri}(s) = \Phi_{rr}(s)\, G(s) \tag{56}$$

and its conjugate into (45), we obtain the form:

$$\begin{aligned}\Phi_{ce}(s) &= \Phi_{ii}(s) - G(-s)\,\Phi_{rr}(s)\,G(s) + \\ &+ [G(-s) - W(-s)]\,\Phi_{rr}(s)\,[G(s) - W(s)]\,.\end{aligned} \tag{57}$$

It is to be remarked, that only the last term contains the minimizing transfer function $W(s)$. The mean-square error would evidently be minimum, if the last term was missing altogether from the expression of the power-density spectrum. This would occur, if it were

$$W_o(s) = G(s) = \frac{\Phi_{ri}(s)}{\Phi_{rr}(s)}\,. \tag{58}$$

So we obtained an optimum transfer function $W_o(s)$, but the latter is, in general, physically unrealizable. The physically realizable optimum transfer function $W_m(s)$ may be obtained by spectrum factorization.

X. Spectrum factorization in the frequency domain

Relationship (56) with consideration of (58) can be written thus:

$$\Phi_{rr}(s)\, W_o(s) = \Phi_{ri}(s)\,. \tag{59}$$

Now if we put in place of $W_o(s)$ the transfer function

$$W_m(s) = W_o(s) - W_n(s),$$

where $W_n(s)$ is a certain physically unrealizable transfer function, so a new term appears on the right side, but this may contain only right-half-plane poles

$$\Phi_{rr}(s)\, W_m(s) = \Phi_{ri}(s) + F_-(s) \tag{60}$$

where

$$\Phi_{rr}(s)\, W_n(s) = -F_-(s)\,. \tag{61}$$

Introducing by relationship (35) the power-spectrum factors, relationship (60) may be converted into:

$$\Phi_{rr}^{+}(s) W_m(s) = \frac{\Phi_{ri}(s)}{\Phi_{rr}^{-}(s)} + \frac{F_-(s)}{\Phi_{rr}^{-}(s)} \, . \tag{62}$$

By separating the components belonging to the positive-time and the negative-time functions, we have:

$$\Phi_{rr}^{+}(s) W_m(s) = \left[\frac{\Phi_{ri}(s)}{\Phi_{rr}^{-}(s)}\right]_+ + \left[\frac{\Phi_{ri}(s)}{\Phi_{rr}^{-}(s)}\right]_- + \frac{F_-(s)}{\Phi_{rr}^{-}(s)} \, . \tag{63}$$

On the left side only left-half-plane poles appear, therefore on the right side the terms containing the right-half-plane poles must neutralize each other:

$$\left[\frac{\Phi_{ri}(s)}{\Phi_{rr}^{-}(s)}\right]_- + \frac{F_-(s)}{\Phi_{rr}^{-}(s)} = 0 \, , \tag{64}$$

from the latter, if necessary, the function $F_-(s)$, which was unknown up till now, may be determined. On the other hand, from the terms belonging to the positive-time function, we again have:

$$W_m(s) = \frac{1}{\Phi_{rr}^{+}(s)} \left[\frac{\Phi_{ri}(s)}{\Phi_{rr}^{-}(s)}\right]_+ \, . \tag{65}$$

So the simplified method determined first by elementary considerations the physically unrealizable $W_o(s)$ transfer function—formula (58)—, then from this the physically realizable $W_m(s)$ optimum transfer function by spectrum factorization. From the expression of $W_o(s)$ often the expression of $W_m(s)$ can be directly written up.

XI. More complex single variable configurations

The simplified method is the simplest of those introduced above. Instead of the variational calculus only a $G(s)$ auxiliary transfer function had to be adopted in order to arrive at form (57). The advantages of the simplified method are especially evident, when going over from a totally free configuration to a more complex one. But in such cases besides the auxiliary transfer function also the introduction of auxiliary power spectra is necessary [24—26]. For instance, in the case of the semi-free configuration (Fig. 2) the part with weighting function $w_f(t)$ of the control system is given (fixed) in advance and the task is to optimally choose the control, or compensation organ of weighting function $w_c(t)$. With the omission of detailed calculations we only refer to the fact, that in these cases by the introduction of the auxiliary power density spectra

$$\Phi_{fi}(s) W_f(-s) \Phi_{ri}(s), \tag{66}$$

$$\Phi_{ff}(s) = W_f(-s) \Phi_{rr}(s) W_f(s) \tag{67}$$

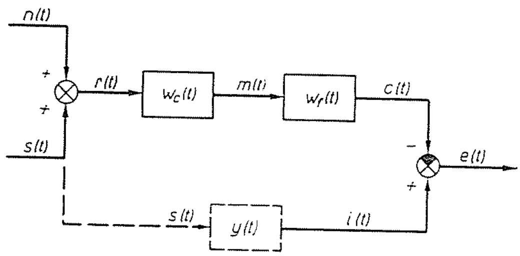


Fig. 2

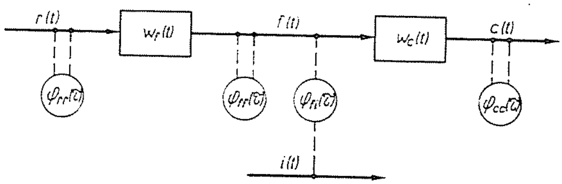


Fig. 3

(which can be interpreted according to Fig. 3) the task may be reduced to the preceding one and so the solution can be directly written, as

$$W_{cm}(s) = \frac{1}{\Phi_{ff}^{+}(s)} \left[\frac{\Phi_{fi}(s)}{\Phi_{ff}^{-}(s)} \right]_{+}, \tag{68}$$

or taking into consideration (66) and (67) as

$$W_{cm}(s) = \frac{\left[\dfrac{W_f(-s)\Phi_{ri}(s)}{[W_f(-s)W_f(s)]^{-}\Phi_{rr}^{-}(s)} \right]_{+}}{[W_f(-s)W_f(s)]^{+}\Phi_{rr}^{+}(s)} \tag{69}$$

for the physically realizable optimum transfer function of the controller.

The constraint in the case of a semi-free configuration (Fig. 4) may be expressed in the form of

$$\overline{b^2(t)} = \varphi_{bb}(0) = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \Phi_{bb}(s)\, ds \leq \sigma^2 \tag{70}$$

where

$$\Phi_{bb}(s) = W_k(-s)\,W_c(-s)\,\Phi_{rr}(s)\,W_c(s)\,W_k(s)\,. \tag{71}$$

The task may now be solved by the Lagrangean conditional extreme value calculus. The function to be minimized is now

$$\overline{x^2(t,\lambda)} = \overline{e^2(t)} + \lambda\overline{b^2(t)}\,, \tag{72}$$

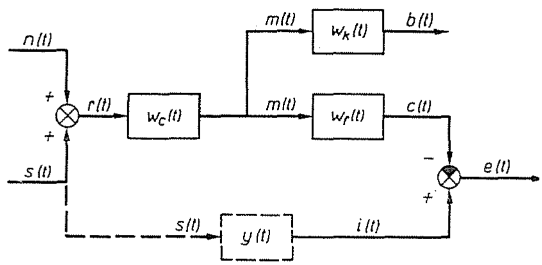


Fig. 4

or

$$\overline{x^2(t,\lambda)} = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}[\Phi_{ee}(s) + \lambda\,\Phi_{bb}(s)]\,ds. \tag{73}$$

Now the introduction of the auxiliary power density spectrum

$$\Phi_{aa}(s,\lambda) = [W_f(-s)\,W_f(s) + \lambda W_k(-s)\,W_k(s)]\,\Phi_{rr}(s) \tag{74}$$

and of the auxiliary transfer function

$$G_a(s,\lambda) = \frac{\Phi_{fi}(s)}{\Phi_{aa}(s,\lambda)} \tag{75}$$

is necessary. This then yields the physically realizable optimum transfer function of the controller in the form of

$$W_{cm}(s) = \frac{1}{\Phi_{aa}^{+}(s,\lambda)}\left[\frac{\Phi_{fi}(s)}{\Phi_{aa}^{-}(s,\lambda)}\right]_{+} \tag{76}$$

or in more detail

$$W_{cm}(s,\lambda) = \frac{\left[\dfrac{W_f(-s)\,\Phi_{ri}(s)}{[W_f(-s)\,W_f(s) + \lambda W_k(-s)\,W_k(s)]^{-}\,\Phi_{rr}^{-}(s)}\right]_{+}}{W_f(-s)\,W_f(s) + \lambda W_k(-s)\,W_k(s)]^{+}\,\Phi_{rr}^{+}(s)}\,. \tag{77}$$

It is to be emphasized, that in this case the Lagrangean indeterminate factor λ figures in the term of the physically realizable optimum transfer function. The factor λ may be eliminated with the aid of the constraining unequality, i.e. relationship (70). In expression (71) of $\Phi_{bb}(s)$ the transfer function $W_c(s)$ may be substituted by the physically realizable optimum transfer function $W_{cm}(s, \lambda)$, and we determine the mean-square value $\overline{b^2(t)}$ of the constrained output signal $b(t)$ on the basis of the Cauchy theorem of residues, then we choose factor λ in such a way, that the constraining unequality (70) should be satisfied. In this way the physically realizable optimum transfer function $W_{cm}(s)$ becomes independent of factor λ.

XII. Multivariable systems

The task of the optimum statistical design may also be extended over multivariable control systems [29—33]. In such cases generally the minimization of the sum of the error signals' mean-square values is required:

$$\sum_{l=1}^{L} \overline{e_l^2(t)} = \text{minimum.} \tag{78}$$

For the treatment of the multivariable systems the matrix calculus offers itself. From the signals we generally form row vectors, from the transfer (or weighting) functions square matrices. The signal vectors and the matrices are distinguished from each other by upper indices, the lower indices serve for the notation of the rows, resp. of the columns.

The physically realizable optimum function matrix [34, 35] of the totally free multivariable configuration (Fig. 5) may be expressed in the form of

$$W_{kl}^{m}(s) = \left[\Phi^{+}_{r_{k''} r_k}(s)\right]^{-1} \left\{\left[\Phi^{-}_{r_{k'} r_{k''}}(s)\right]^{-1} \Phi_{r_{k'} i_l}(s)\right\}_{+} \tag{79}$$
$$(k = k',\ k'' = 1, \ldots K;\ l = 1, \ldots L;\ K = L).$$

Here $\Phi_{r_k i_l}(s)$ is the power-density spectrum matrix belonging to the cross-correlation functions of the input signals and the ideal output signals, while the spectrum factor matrices must be chosen in such a way, that on the one hand

$$\Phi_{r_{k'} r_k}(s) = \Phi^{-}_{r_{k'} r_{k''}}(s)\, \Phi^{+}_{r_{k''} r_k}(s) \tag{80}$$

should be satisfied, where $\Phi_{r_{k'} r_k}(s)$ is the power-density spectrum matrix belonging to the correlation functions formed from the input signals, on the other hand the elements of matrix $\Phi^{-}_{r_{k'} r_{k''}}(s)$ and its inverse may contain only right-half-plane poles, the elements of matrix $\Phi^{+}_{r_{k''} r_k}(s)$ and its inverse may contain only left-half-plane poles.

In the semi-free multivariable configuration (Fig. 6) the optimum transfer function matrix of the controller [36, 37] is:

$$W^{cm}_{kl}(s) = \left[\Phi^{+}_{r_{k''} r_k}(s)\right]^{-1} \left\{\left[\Phi^{-}_{r_k, r_{k''}}(s)\right]^{-1} \Phi_{r_{k'} i_l}(s)\, W^{f}_{l'j'}(-s) \times \right.$$
$$\left. \times \left[(W^{f}_{jl}(s)\, W^{f}_{l'j'}(-s))^{-}\right]^{-1}\right\}_{+} \left[(W^{f}_{jl}(s)\, W^{f}_{l'j'}(-s))^{+}\right]^{-1} \tag{81}$$
$$(k, k', k'' = 1, \ldots K;\ j = 1, \ldots J;\ l = 1, \ldots L;\ K = J = L).$$

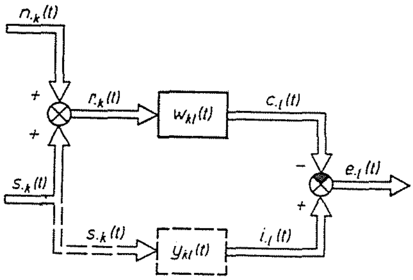


Fig. 5

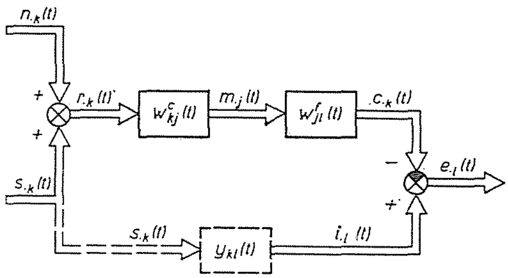


Fig. 6

It is to be noted, that when the matrix $W^{f}_{jl}(s)$ of the fixed part (e.g. the controlled section) is of minimum phase, i.e.

$$(W^{f}_{jl}(s)\, W^{f}_{l'j'}(-s))^{-} = W^{f}_{l'j'}(-s)$$
$$(W^{f}_{jl}(s)\, W^{f}_{l'j'}(-s))^{+} = W^{f}_{jl}(s)$$

then (81) may be simplified:

$$W^{cm}_{kl}(s) = \left[\Phi^{+}_{r_{k''} r_k}(s)\right]^{-1} \left\{\left[\Phi^{-}_{r_k r_{k''}}(s)\right]^{-1} \Phi_{r_{k'} i_l}(s)\right\}_{+} \left[W^{f}_{jl}(s)\right]^{-1}. \tag{82}$$

In such cases

$$W^{cm}_{kl}(s) = W^{m}_{kl}(s) \left[W^{f}_{jl}(s)\right]^{-1}. \tag{83}$$

For constrained semi-free configuration multivariable systems (Fig. 7) it is usual to give the constraint in the form of

$$\sum_{h=1}^{H} \overline{b_h^2(t)} \leqq \sigma^2 \,. \tag{84}$$

The constrained signal vector $b_{.h}(t)$ $(h = 1, \dots H)$ is produced by the weighting function matrix $w_{jh}^{k}(t)$ from the intermediate modified signal vector $m_{.j}(t)$

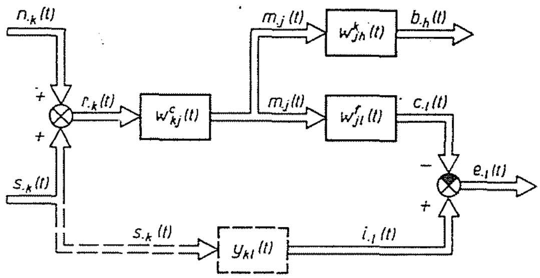


Fig. 7

$(j = 1, \dots J)$. For instance if $w_{jh}^{k}(t)$ is the diagonal unit impulse function matrix, then the $m_{.j}(t)$ signals are directly constrained, if $w_{jh}^{k}(t) = w_{jl}^{f}(t)$, then the $c_{.l}(t)$ output signals are constrained.

It ought to be mentioned, that the constraint may be expressed in the form of

$$\sum_{h=1}^{H} \overline{b_h^2(t)} = \mathrm{tr}\, \varphi_{b_h b_h}(0) = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \mathrm{tr}\, \Phi_{b_{h'} b_h}(s)\, ds \leqq \sigma^2 \tag{85}$$

where tr means the trace of the matrix, i.e. the sum of the main diagonal elements, further

$$\Phi_{b_{h'} b_h}(s) = W_{h'j'}^{k}(-s)\, W_{j'k'}^{c}(-s)\, \Phi_{r_{k'} r_k}(s)\, W_{kj}^{c}(s)\, W_{jh}^{k}(s) \,. \tag{86}$$

For the determination of the optimum controller transfer matrix it is most practical to again use the simplified derivation method. The required optimum transfer matrix [38]:

$$W_{kj}^{cm}(s, \lambda) = \left[\Phi_{a_{k''} a_k}^{+}(s, \lambda)\right]^{-1} \times$$
$$\times \left\{\left[\Phi_{a_{k'} a_{k''}}^{-}(s, \lambda)\right]^{-1} \Phi_{r_{k'} i_l}(s)\, W_{l'j'}^{f}(-s) \left[\left(W_{jl}^{f}(s)\, W_{l'j'}^{f}(-s)\right)^{-}\right]^{-1}\right\}_{+} \times$$
$$\times \left[\left(W_{jl}^{f}(s)\, W_{l'j'}^{f}(-s)\right)^{+}\right]^{-1} \tag{87}$$

contains for the time being also the parameter λ. The auxiliary power density spectrum figuring here can be determined on the basis of the following relationship:

$$\begin{aligned} & W^f_{l'j'}(-s)\,W^c_{j'k'}(-s)\,\Phi_{r_k r_k}(s)\,W^c_{kj}(s)\,W^f_{jl}(s) + \\ & + \lambda\,W^k_{h'j'}(-s)\,W^c_{j'k'}(-s)\,\Phi_{r_k r_k}(s)\,W^c_{kj}(s)\,W^k_{jh}(s) = \\ & = W^f_{l'j'}(-s)\,W^c_{j'k'}(-s)\,\Phi_{a_k a_k}(s,\lambda)\,W^c_{kj}(s)\,W^f_{jl}(s). \end{aligned} \tag{88}$$

The auxiliary matrix $\Phi_{a_k a_k}(s,\lambda)$ does not depend on the transfer matrix $W^c_{kj}(s)$, but only on the transfer matrices $W^f_{jl}(s)$, $W^k_{jh}(s)$, the power-density spectrum matrix $\Phi_{r_k r_k}(s)$ and the parameter λ, if

$$W^k_{jh}(s)\,[W^f_{jl}(s)]^{-1} = G_0(s)\,I_{jj} \tag{89}$$

where $G_0(s)$ is a certain transfer function, while I_{jj} a unit matrix of $j \times j$ dimension.

Parameter λ may be eliminated, if we substitute relationship (87) and its conjugate into expression (86), then we determine with the aid of the constraining unequality (85) the permissible value of λ. Substituting this latter value back into formula (87) the transfer function of the optimum controller $W^{cm}_{kj}(s)$ becomes independent of factor λ.

The minimum sum of the mean-square values of the error signals may be computed with the following formula:

$$\sum_{l=1}^{L} \overline{e_l^2(t)} = \operatorname{tr} \varphi_{e_l e_l}(0)\,\frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \operatorname{tr} \Phi_{e_l e_l}(s)\,ds \tag{90}$$

where

$$\begin{aligned} \Phi_{e_l e_l}(s) = {} & \Phi_{i_l i_l}(s) - \Phi_{i_l r_k}(s)\,W^{cm}_{kj}(s)\,W^f_{jl}(s) - \\ & - W^f_{l'j'}(-s)\,W^{cm}_{j'k'}(-s)\,\Phi_{r_k i_l}(s) + \\ & + W^f_{l'j'}(-s)\,W^{cm}_{j'k'}(-s)\,\Phi_{r_k r_k}(s)\,W^{cm}_{kj}(s)\,W^f_{jl}(s). \end{aligned} \tag{91}$$

Finally we note, that the most general case is the multivariable, semi-free configuration with constraint. So e.g. by the substitution of $\lambda = 0$ the results of the unconstrained semi-free configuration may be regained. On the other hand, going over from the multivariable system to the single variable system we regain the results of point XI.

XIII. Complementary remarks

Recently the pulsed-data (sampling and digital) control systems are gaining ever more significance. The investigations of the continuous control systems may easily be extended and similar results, up to a certain measure, may be determined for the pulsed-data type single and multivariable systems as well, as those described under points XI and XII [39, 40].

XIV. Some conclusions

Finally we refer in short to some further tasks concerning the statistical analysis and design of the control systems.

In connection with linear systems it is worthwhile to deal with the solution of tasks deviating from the criterion $\overline{e^2(t)} = \text{min.}$, resp. $\sum_{l=1}^{L} \overline{e_l^2(t)} = \text{min.}$, though this will complicate the design. In multivariable systems the spectrum factorization of the matrix plays a decisive role. We have to investigate here also the case, where the rank is smaller, than the order of the matrix.

It is similarly interesting that the spectrum factorization of matrices built up of elements containing e^{-sT_d} transcendent factors. All these investigations may be extended to pulsed-data control systems as well. With the exception of the linear systems the consideration of the variable parameter and non-linear systems, further the non-stationary processes is still in the stage of its starting steps; the difficulties of the investigations keep increasing, the more so, as the transformation methods lose their validity.

Finally, on the basis of the theoretical results the applying of correlators for the system analysis and synthesis seems to be of great importance. Here, for instance, practical difficulties arise by the white noise not being perfect even in a finite frequency domain, and the integration time T cannot be increased in an arbitrary measure.

Summary

This paper wishes to give an insight into the range of problems of the statistical synthesis of single- and multivariable control systems. It compares the four variants, which may be used for the design and shows the advantages of the so-called simplified method. It also gives a few results on the range of the single- and multivariable continuous control systems.

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