ROOT LOCI IN THE CASE OF $A(p) + xB(p) + x^2C(p) = 0$

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1. The dependence of the network function upon the values of the circuit elements

As is known, according to BODE's bilinear theorem [1, 6] in a linear invariant lumped-parameter network the dependence of the network functions on any two-terminal circuit element or controlled source x shows the bilinear form

$$F(x, p) = \frac{A(p) + B(p)x}{C(p) + D(p)x} \tag{1}$$

where $F(x, p)$ is the network function (input impedance or transfer function), p is the complex frequency and $A(p)$, $B(p)$, $C(p)$, $D(p)$ are constants determined by the other network elements which are, in general, polynoms of p.

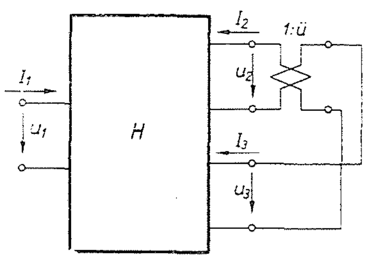

Fig. 1

Let us consider, however, a network containing an ideal transformer and single out the ideal transformer from the network as shown in Fig. 1. Then the following equations can be written:

$$\begin{aligned}
U_1 &= Z_{11} I_1 + Z_{12} I_2 + Z_{13} I_3 \\
U_2 &= Z_{21} I_1 + Z_{22} I_2 + Z_{23} I_3 \\
U_3 &= Z_{31} I_1 + Z_{32} I_2 + Z_{33} I_3 \\
U_3 &= \ddot{u} U_2
\end{aligned}$$

$$I_3 = -\frac{1}{ü} I_2 \tag{2}$$

From these equations the input impedance can be determined by means of the Cramer rule:

$$Z_{\text{in}} = \frac{U_1}{I_1} = \frac{A + üB + ü^2 C}{D + üE + ü^2 F} \tag{3}$$

It can be seen that the input impedance depends biquadratically on the transformation ratio. Similarly it can be shown that similar formula is also valid for the transfer functions. Further it can be demonstrated that besides the ratio of the transformer, biquadratic relation holds value also for the gyrator as well as for the negative impedance converter [5]. The gyrator and the negative impedance converter are fundamental circuit elements of the synthesis of non-reciprocal, respectively, active networks [9]. Therefore, it is important to investigate the dependence of the network functions also on these four-terminal circuit elements. The effect of small variations of the circuit elements can be described by the notion of sensitivity, while the investigation of the effect of large variations brings about the problems of variable-parameter networks. In the following the problems arising from the variation of poles and zeros of the network function will be investigated.

In the case of the bilinear formula this can be made by means of the root locus method well-known in control engineering [2, 4]. The investigation of the biquadratic function makes the extension of root loci for a quadratic case necessary.

2. The root locus method

Root locus is said to be the arrangement of roots of the characteristic function

$$A(p) + xB(p) = 0 \tag{4}$$

in the complex frequency plane if the parameter x is a positive real parameter going from zero to plus infinity or a negative real one going from zero to minus infinity. $A(p)$ and $B(p)$ are polynoms with real coefficients. The coefficient of their highest power is 1 and they are given by their roots. The root locus taken for the negative parameters is said to be the supplement of the root locus obtained for the positive parameters. The roots of $A(p)$ are called poles and those of $B(p)$ zeros.

The root locus points can be determined from the phase condition (5) and the appertaining parameter values from the amplitude condition (6).

$$\sum_{i=1}^{n} \operatorname{arc}(p - p_{pi}) - \sum_{j=1}^{m} \operatorname{arc}(p - p_{0i}) = \begin{cases} 180^\circ \pm k\,360^\circ & \text{if } x > 0 \\ 0 \pm k\,360^\circ & \text{if } x < 0 \end{cases} \tag{5}$$

$$|x| = \frac{\prod_{i=1}^{n} |p - p_{0i}|}{\prod_{j=1}^{m} |p - p_{0j}|} \tag{6}$$

where p_{0i} is the i-th pole and p_{0j} is the j-th zero.

The main rules for constructing root loci will be summed up in the following:

2.1. Increasing x from 0, the root locus starts from the poles.

2.2. In case of $x \to \infty$ the root locus ends in the zeros.

2.3. The number of the separable parts coincides with the number of poles or that of zeros depending on which is the larger.

2.4. On the root locus are those parts of the real axis at the right of which the sum of the poles and zeros on the real axis is odd if x is positive or this sum is even in case of x negative.

2.5. The angle of departure of the root locus from the k-th pole is

$$\theta_k = \sum_{j} \operatorname{arc}(p_{pk} - p_{0j}) - \sum_{i \neq k} \operatorname{arc}(p_{pk} - p_{pi}) = \begin{cases} 180^\circ \pm h \cdot 360^\circ & \text{if } x > 0 \\ 0 \pm h \cdot 360^\circ & \text{if } x < 0 \end{cases} \tag{7}$$

2.6. Let the designated number of poles be by n and that of the zeros by m. If $n > m$, we have $n - m$ asymptotes whose angle of slope is

$$\varphi_A = \frac{180^\circ}{n-m} \pm \frac{k \cdot 360^\circ}{n-m} \qquad \text{if} \quad x > 0 \tag{8a}$$

and

$$\varphi_A = \pm \frac{k \cdot 360^\circ}{n-m} \qquad \text{if} \quad x < 0 \tag{8b}$$

In case of $n = m$ the above theorem can be applied after transformation $x + 1 = u$.

2.7. The centroid at which the asymptotes intersect is

$$p_A = \frac{\sum_{i=1}^{n} Re\, p_{pi} - \sum_{j=1}^{m} Re\, p_{0j}}{n-m} \tag{9}$$

2.8. The coalescent points p can be derived [8] from equation (10)

$$A(p)B'(p) - A'(p)B(p) = 0\,. \tag{10}$$

$A'(p)$ and $B'(p)$ mean derivatives of $A(p)$ resp. $B(p)$ with respect to p. From this after transformation the equation

$$\sum_{i=1}^{n} \frac{1}{p - p_{pi}} - \sum_{j=1}^{m} \frac{1}{p - p_{0j}} = 0 \tag{11}$$

can be obtained which is also suitable for the graphic attempt.

2.9. With the parameter transformation $x = \frac{1}{u}$ the roles of poles and zeros can be inverted.

2.10. In the range $-\infty < x < +\infty$ the roots belonging to any two parameter values can be regarded as zeros resp. poles, the geometrical location of the root loci remains unchanged.

In the following section will be examined how the above rules will develop in case of quadratic equation. The single points, however, will be dealt with in an order differring from the above.

3. Root loci in quadratic case [7]

In case of quadratic expression the determination of the root locus becomes a task substantially more complicated. Namely, express x from the characteristic equation

$$x^2 A(p) + x B(p) + C(p) = 0 \, . \tag{12}$$

Then

$$x_{1,2} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \, . \tag{13}$$

It can be seen that a resolution to factors is, in general, not possible, therefore the phase criterion (5) and the amplitude criterion (6) cannot be written up more and for the investigation of the properties of the quadratic root locus a new method is required. Let us investigate first the cases in which the quadratic expression can directly be reduced to a linear one, and then give an obvious interpretation how the construction can be traced, in general, back to the construction of linear root loci. Then it will be investigated how the rules 2.1.—2.9. will be modified in case of quadratic equation. The characteristic equation will be regarded as given by the roots of $A(p)$, $B(p)$, $C(p)$ p_{ai}, p_{bi}, p_{ci} as well as by the constant factors of A_0, B_0, C_0.

3.1. Cases reduceable to linear expression

The characteristic equation can be reduced to linear form by parameter transformation in the following cases:

3.1.1. One of the terms in the equation vanishes.

a) Let $A(p) \equiv 0$ be
Then $B(p) + xC(p) = 0$.

b) If $B(p) \equiv 0$, then
$A(p) + x^2C(p) = A(p) + uC(p) = 0$.

c) If $C(p) \equiv 0$, then
$A(p) + xB(p) = 0$.

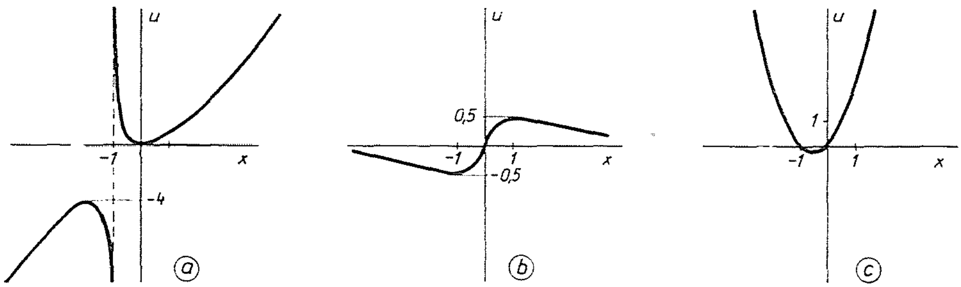

Fig. 2

In all three cases a linear characteristic equation was obtained.

3.1.2. Two coefficients coincide.

We have three cases:

a) $A(p) \equiv B(p)$.
Hence we can obtain $A(p) + uC(p) = 0$, where $u = \dfrac{x^2}{1+x}$

b) $A(p) \equiv C(p)$.
Hence $A(p) + uB(p) = 0$, where $u = \dfrac{x}{1+x^2}$.

c) $B(p) \equiv C(p)$.
Hence $A(p) + uB(p) = 0$, where $u = x + x^2$.

The character of the functions $u(x)$ is represented in *Figs. 2a, b* and *c.* All these functions are characterized by the fact that the linear parameter u obtained after transformation cannot take all values if x varies from $-\infty$ to $+\infty$ but it takes even twice the other values. Thus on the locus twofold portions may occur and there may be conversion points where $\dfrac{du}{dx} = 0$. This fact may become important in the investigation of pole sensitivity.

3.2. *Derivation of the general quadratic root loci from the linear root loci*

It can easily be understood that the quadratic characteristic equation can be made to correspond to such linear characteristic equation whose poles or zeros vary with the variation of the parameter. Namely, it can be written that

$$A(p)+xB(p)+x^2C(p)=\begin{cases}[A(p)+x[B(p)+xC(p)] & (14\text{a})\\ [A(p)+x^2C(p)]+xB(p) & (14\text{b})\\ [A(p)+xB(p)]+x^2C(p) & (14\text{c})\end{cases}$$

E.g., in (14a) the roots of $A(p)$ are the poles, while the zeros are the roots of the equation

$$B(p)+xC(p)=0\,. \tag{15}$$

The variation of these roots again takes place along linear root loci. Any point of the quadratic root locus can be obtained consequently by means of repeated use of linear loci. First the zeros in terms of (15) are determined and the value of the parameter pertaining to them is calculated, then on the root loci of the pole-zero arrangement thus obtained the points belonging to the parameter calculated previously are sought for. The construction based on this principle is convenient in case of networks of a lower degree in p. However, it must not be forgotten that expression (15) is characterized not only by its roots but also by the coefficient of the term with the highest exponent which also varies with x in the case where the degree of $C(p)$ is not lower than that of $B(p)$. An example of the construction is given in the Appendix.

3.3. *The rules remaining valid unchanged*

Part of the rules described in section 2 remain valid unchanged or nearly unchanged also for quadratic root loci. Also the proof can be given similarly.

In the following it will be assumed that all three terms of the characteristic equation exist.

3.3.1. On beginning to increase x from 0, the locus departs from the roots of $A(p)$.

3.3.2. In case of $x\to\infty$ the locus ends in the roots of $C(p)$.

3.3.3. The number of the separable parts coincides with the highest of the degrees of $A(p)$, $B(p)$ and $C(p)$.

3.3.4. With the transformation $x=\dfrac{1}{u}$ $A(p)$ and $C(p)$ can be inverted. It is not sufficient to interchange the roots but A_0 and C_0 must also be exchanged. Namely, we already have three constant coefficients one of which can be choosen to be equal to unity while of the two others only one can be "worked in" into the parameter.

In terms of analogy the roots of $A(p)$ can be called poles while those of $C(p)$ zeros.

3.3.5. The angle of departure of the root locus from the k-th pole is

$$\Theta_k = \sum_i \text{arc}\,(p_{ak} - p_{bi}) - \sum_j \text{arc}\,(p_{ai} - p_{ak}) +$$

$$+ \begin{cases} 180^\circ \pm k \cdot 360^\circ, & \text{if } \dfrac{xB_0}{A_0} > 0 \qquad (16a) \\ 0^\circ \pm k \cdot 360^\circ, & \text{if } \dfrac{xB_0}{A_0} < 0. \qquad (16b) \end{cases}$$

Namely, in case of $x \to 0$ we have $x^2 \ll x$ and it is sufficient to take only the first two terms into account.

The determination of the other properties requires, however, more consideration.

3.3.6. Investigation of points of the real axis. In the following it will be determined what parts of the real axis belong to the root locus. For this $p = \sigma$ will be substituted in to the computation formula and it will be looked for under which conditions for σ it gives a real result concerning x.

For the real value of p $A(p)$, $B(p)$ and $C(p)$ are also real. Since

$$x = \frac{-B \pm \sqrt{B^2 - 4CA}}{2C}$$

the condition of the real solution is

$$B^2 \geq 4CA \qquad (17)$$

Let us now consider the various sign variations:

a) $AC > 0$, i.e. A und C have the same sign. Then the condition of the solution is

$$|B|^2 \geq 4|C||A|. \qquad (18)$$

The discriminant will be lower than B^2, therefore a term smaller than B must be added to $-B$ or extracted from it. In both cases the sign of x is the same as that of $\dfrac{-B}{C}$.

b) $AC < 0$, i.e. the two signs are different. The condition (17) is fulfilled automatically and after evolution we obtain a term greater than $|B|$. This always results in a positive and a negative x irrespective of the sign of B.

In the following with the purpose of decreasing the number of sign variations without impairing the generality it will be assumed that the parameter is positive while the signs of A_0, B_0 and C_0 may be arbitrary.

Summing up, for the points on the real axis the following theorems can be stated:

Theorem 3.3.6.1. — The onefold parts of the root locus are those for which the signs of $A(p)$ and $C(p)$ are opposite. This can be determined in the known way from the signs of A_0 and C_0 as well as from the number of poles [roots of $A(p)$] and zeros [roots of $C(p)$] being to the right of the point considered. (See Appendix.)

Theorem 3.3.6.2. — The points at which the signs of $A(p)$ and $C(p)$ are the same and the sign of $B(p)$ opposed to them, are twofold points of the root locus if

$$|B|^2 \geq 4|C||A|. \tag{18}$$

Theorem 3.3.6.3. — Other points of the real axis cannot be on the root locus.

3.3.7. Graphic estimate of the parameter. The parameter value pertaining to any point p_1 of the locus can be determined from the complex numbers $A(p_1)$, $B(p_1)$ and $C(p_1)$. These can be obtained by multiplying the coefficients with the vectors drawn from the roots to p_1. In their knowledge the computing formula should be used which would be, however, a tedious work for complex coefficients. Therefore, a simple graphic method based on relations (19) and (20) will be described.

$$x_1 + x_2 = -\frac{B(p)}{C(p)} \tag{19}$$

$$x_1 x_2 = \frac{A(p)}{C(p)} . \tag{20}$$

Since x_1 is a positive real quantity, the phase angle of x_2 is the difference of the phase angles of $A(p_1)$ and $C(p_1)$. Let us determine $-\frac{B(p_1)}{C(p_1)}$ in the known way. It must be the sum of two complex numbers with a known arc. Thereafter the construction is shown in *Fig. 3*. This method cannot be used if $B(p_1)/C(p_1)$ and $A(p_1)/C(p_1)$ are also real but then all numbers occurring in the computing formula are also real.

3.3.8. Asymptotes. The problem is to determine the angle and the intersection point of the asymptotes. This can be done by assuming a very large p and taking its two highest powers into account only. It is known from the theory of the linear root loci that if

$$x \approx \pm \frac{p^n + A_1 p^{n-1}}{p^m + B_1 p^{m-1}} \tag{21}$$

then we have $n-m$ asymptotes and they pass through the point p_a

$$p_a = \frac{B_1 - A_1}{n - m} \, . \tag{22}$$

In quadratic case retained in the computing formula only the highest powers in denominator and numerator, the form corresponding to (21) can

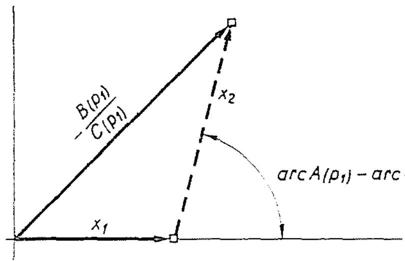

Fig. 3

be obtained. However, for the determination of the highest powers several cases must be treated separately. Let the degrees of $A(p)$, $B(p)$, $C(p)$ be designated in turn by a, b, c.

3.3.8.1. $2b > a + c$

Then

$$B^2 \geq 4AC$$

$$x_{1,2} = \frac{-B \pm B\sqrt{1 + \dfrac{4AC}{B^2}}}{2C} \approx \frac{-B \pm B\left(1 - \dfrac{2AC}{B^2}\right)}{2C}$$

Hence

$$x_1 = \frac{-A}{B}$$

$$x_2 = \frac{-B^2 + AC}{CB}$$

With respect to x_1 for the asymptotes we obtain the same expression as in the linear case, consequently also the direction and the intersection point of the $a - b$ asymptotes can be determined in the same way.

For x_2 two cases must be separated:

a) $2b > a + c + 1$

Then in the numerator AC can be neglected besides B^2:

$$x_2 \approx \frac{-B}{C}.$$

Therefore, we have the same asymptotes as in the linear case $B + Cx_2 = 0$.

b) $2b = a + c + 1$

After arrangement

$$x_2 = -\frac{B_0}{C_0}\,\frac{p^{2b} + \left(2B_1' - \dfrac{A_0 C_0}{B_0^2}\right) p^{2b-1}}{p^{b+c} + (B_1' + C_1')\,p^{b+c-1}}$$

where

$$B_1' = \frac{B_1}{B_0} \quad \text{and} \quad C_1' = \frac{C_1}{C_0}.$$

In first approximation

$$x_2 = \frac{-B_0}{C_0}\,p^{b-c}$$

Hence we have $b - c$ asymptotes, namely, for the cases $x_2 \to \infty$ or $x_2 \to 0$ depending on whether $b > c$ or $b < c$. That is, the number of the asymptotes as well as their angle are the same as in case a.

If $b = c$, in cases a. and b. p goes to infinity at the limiting process $x \to \dfrac{-B_0}{C_0}$. This is interesting for us only if B_0 and C_0 have opposed signs. The analysis of the asymptotes can be performed by setting $u = x + \dfrac{B_0}{C_0}$

The point of intersection is

$$p_A = \frac{C_1' + \dfrac{A_0 C_0}{B_0^2} - B_1'}{b - c}.$$

With the omission of the derivation, the results obtained for the other cases will be given below.

3.8.2. Using the notation

$$2b = a + c$$

$$D_0 = \pm\sqrt{B_0^2 - 4A_0 C_0}$$

$$E_0 = \frac{-B_0 \pm \sqrt{B_0^2 - 4C_0 A_0}}{2C_0}$$

for the directions we obtain

$$p_{1,2} \to \sqrt[b-c]{\frac{-x}{E_0}}$$

These can be arbitrary as E_0 can assume a complex value of any arc. Of course, they are at the angle of $\frac{360°}{b-c}$ from one another.

The centroid is

$$p_A = \frac{C_1' + \dfrac{B_1 E_0 + A_0 C_1' + A_1}{D_0 E_0}}{b-c} .$$

3.8.3. $2b < a + c$

a) $2b = a + c - 1$

It can be shown that in this case there are no asymptotes, the root loci go to parabolas.

b) $2b = a + c - 2$

The angle of slope is computed as follows

$$p \to \sqrt[a-c]{\frac{-x^2 C_0}{A_0}}$$

The point of intersection is

$$p_A = \frac{C_1' - A_1' \pm B_0 \sqrt{\dfrac{-1}{A_0 C_0}}}{a-c}$$

If A_0 and C_0 have the same sign, for p_A two conjugate complex numbers will be obtained. From both $a - c$ asymptotes depart, but half of them pertain to negative parameter values.

c) $2b < a + c - 2$

The asymptotes coincide with the asymptotes of the curves of the equation $A(p) + x^2C(p) = 0$.

3.9. Coalescent points

Let us consider the root locus as the mapping, determined by the characteristic equation, of one of the lines of plane x, the positive real axis, on the plane p. A breakaway point can be at the point where the mapping loses its conform property, i.e. it becomes $\frac{dx}{dp} = 0$ or infinite.

$$\frac{dx}{dp} = \frac{A'(p) + xB'(p) + x^2 C'(p)}{B(p) + 2xC(p)} = 0. \tag{23}$$

The primes designate the derivation with respect to p. For the coalescent points the following two equations can be written

$$A' + xB' + x^2 C' = 0 \tag{24}$$

$$A + xB + x^2 C = 0 \tag{12}$$

From these, eliminating x, resp. x^2 we obtain

$$x^2 = \frac{A'B - AB'}{CB' - BC'} \qquad x = \frac{A'C - AC'}{BC' - BC'} .$$

Hence

$$(A'C - AC')^2 = (A'B - AB')(B'C - BC'). \tag{25}$$

Equation (25) must be satisfied by all coalescent points. Since, however, in the derivation x it was not assumed to be real, the equation may have a solution which is not on the root locus. Even the solution on the root locus is not necessarily a coalescent point since the squaring can bring about also false roots.

Equation (25) can be of very high degree. If, e.g., $a = 3$, $c = 2$, an eighth-degree equation must be solved. This, however, is facilitated by the circumstance that both sides of the equation can be written in a radical form. The solution is facilitated by the fact that the coefficients of the equation correspond to the form (10) used in determining the coalescent point of linear root loci, therefore, the root factors in case of lower degrees can often be written directly. If the radical solutions are already known, the coalescent point is approximated by an iterative method, by multiplication of distances. An example for the construction is shown in the Appendix.

It can easily be demonstrated that for the angle between the directions of arrival and departure the same statements hold as in the linear case, i.e. this angle is 90° in case of double coalescent point.

3.10. Sensitivity

The root sensitivity can be obtained directly from (23).

$$S_x^p \triangleq \frac{dp}{dx} = \frac{B(p) + 2xC(p)}{A'(p) + xB'(p) + x^2 C'(p)} \tag{26}$$

The general analysis of this is very complicated. In the given cases it may be more convenient to determine the roots belonging together and to regard them as starting points with parameter transformation. Namely, for $x = 0$

$$S_x^p(x = 0) = \frac{B(p)}{A'(p)}.$$

This agrees with the root sensitivity of the linear equation $A(p) + xB(p) = 0$.

In the coalescent points the root sensitivity is also very high for this time. It is remarkable that S_x^p can also be zero where the numerator of (26) is 0. Hence, it can be understood that in these points the discriminant is 0. To them belong the return points.

4. Appendix

Let us investigate the input impedance of the network shown in Fig. 4.

$$Z_{\text{in}} = \frac{0.1}{p}\left[(p + 10) + 20q + q^2 10(p^2 + 1)\right] \tag{27}$$

We shall determine the root locus of zeros when q varies from 0 to $+\infty$. In terms of the root arrangement shown in Fig. 5 the following statements can be made.

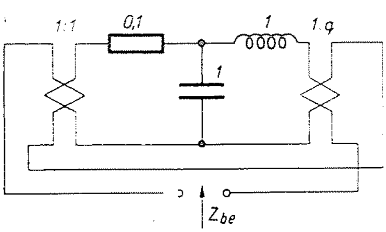

Fig. 4

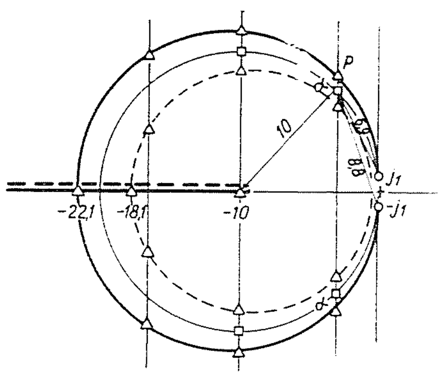

Fig. 5

a) The onefold part of the root locus is the portion of the real axis to the right of the point $\sigma = 10$. Namely, both A_0 and C_0 are positive, therefore the sum of poles and zeros to the right of the root locus must be odd.

b) According to 3.3.4. and 3.3.5. the root locus arrives in the zero lying on j with an angle of 90°.

c) There is only one asymptote, namely the negative real axis (in 3.8.3. the case of $2b < a + c - 2$).

d) For the coalescent point

$$(A'C - AC')^2 = (A'B - AB')(CB' - BC') .$$

The roots of the left side are the coalescent points of the root locus $A + uC = 0$. Since this root locus is a circle, in terms of the construction we have

$$\sigma_{E1} = 0.05 \qquad \text{and} \qquad \sigma_{E2} = -20 .$$

The constant multiplier can be computed as

$$k = A_0 C_0 (a - c) = 10 \,.\, (-1)$$

The first factor of the right side in independent of frequency, its value is 20.

The second factor of the right side is $-20 \cdot 10 \cdot 2p = -400p$. Substituting it into (25) and dividing the equation by 100 we obtain

$$(p - 0.05)^2 (p + 20)^2 = -80p \tag{26}$$

In connection with its solution let us analyse two cases.

α. $\qquad p \approx 0$

Then $p - 0.05^{2} \cdot 400 = -80p$

Its solutions are

$$p_{1,2} = -0.05$$

This means that in the point $p = -0.05$ there is a twofold coalescent point if this is located on the root locus.

β. $\qquad p \gg 0.05$

In this case

$$p^2 (p + 20)^2 = -80p$$

By iterative method we obtain the roots

$$p_3 = -18.1 \qquad p_4 = -22.1 \qquad p_5 = -0.1 .$$

p_5 is a false root as it does not satisfy the condition (28).

The construction method described in 3.2. can be used for the construction of the single points. For this purpose the characteristic equation will be arranged as follows

$$\frac{p+10}{10q^2} + (p^2+1) + \frac{2}{y} = 0 .$$

Hence

$$(p+10)v + (p^2+1) + z = 0$$

where

$$z = 6.33 \sqrt{v} .$$

This is the equation of a linear root locus whose poles migrate in a given way along an arc, then along the real axis (thin line in Fig. 5). Let the poles be located in the point $= -3 + j7.4$. To this pertains the parameter $v = \frac{6.9 \cdot 8.8}{10} = 6.07$. The new root locus has two poles, at the points α and $\bar{\alpha}$, and the point of this root locus, corresponding to the parameter $z = 6.33 \sqrt{6.07} = 15.6$ is to be determined. The root locus is a vertical straight line. At point P the parameter is

$$z = (\omega - \omega_1)(\omega + \omega_1)$$

and from this

$$\omega = \sqrt{z + \omega_1^2} = 8.4 .$$

The points of the root locus pertaining to negative q can be obtained in wholly similar way, the difference only being in the sign of z.

The points designated by a triangle have been determined by this method.

If q is negative, according to the sign convention the sign of B_0 must be exchanged. The root locus points may also be points of the real axis to the right of $\sigma = 10$ for which the condition of the discriminant is fulfilled. For the limiting points we can obtain

$$p_6 = 0$$

$$p_7 = -0.1$$

$$p_8 = -9.9 .$$

From our results it is obvious that the point $p_4 = -22.1$ is the coalescent point belonging to the positive parameter q, while $p_3 = -18.1$ is that pertaining to the negative q. In Fig. 5 the root locus was drawn with a thick line and its supplement by a dotted one. It can be seen that the supplementary root loci also reach the real axis at the point $p = -0.05$. They branch here, then returning at the points 0 and -0.1 they meet again at the point -0.05.

Summary

The network functions depend in biquadratic form on the ratio of the ideal transformer, the resistance of the ideal gyrator and the conversion coefficient of the negative impedance converter. The analysis of the variation of poles and zeros requires the generalisation of the root loci for the quadratic case.

The quadratic equation can in some cases be directly reduced to linear equation by means of parameter transformation. The connection with the linear root loci is shown here also in generality which may be used in the determination of the single points. For the starting and ending points of the quadratic root locus, for the angle of slope at these points and for the number of parts a wholly similar relation holds as for the linear case. For the points lying at the real axis simple rules can be found. A graphic method for the determination of the parameter is given. Concerning the asymptotes several cases ought to be separated. For the coalescent points an equation of high degree is obtained which is, however, suitable for iterative solutions. The new properties are that twofold portions are possible, the asymptotes can in some cases be arbitrary and the root sensitivity may also be zero.

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