ULTRASONIC TESTING OF INHOMOGENEITIES IN HETEROGENEOUS MEDIA

By

I. BIHARI and J. SZILÁRD

Department of Mathematics, Polytechnical University, Budapest
(Received March 9, 1967)

Presented by Prof. Dr. Gy. ALEXITS

Introduction

Roughly speaking the inhomogeneities of heterogeneous media may be of two types. They can fill in whole parts of the body within which the characteristics of the medium influencing the propagation of the waves considered vary continuously or they exhibit a granulated (polycrystalline) structure, inside of which these characteristics are constant, but vary with the kind of grains. Correspondingly, we speak about inhomogeneities distributed continuously or discretely.

In some respect the problem is similar in regard to electromagnetic waves and mechanical (acoustical and thermal) waves, respectively, only the appropriate characteristics of the media are to be taken into account. Of course, the former problem will not be dealt with here.

In the following treatment normally ultrasonic waves are considered (practical examples), but as the phenomena involved depend only on the relative size of the wavelength and inhomogeneities, the results obtained can naturally be applied also on lower frequencies, larger grains and on higher frequencies (hypersonic and thermal phenomena), smaller grains, respectively.

An example of the continuous distribution is the sea-water with its continuously varying salinity and temperature, certain geological formations, structure of the earth layers, etc.

The structure of metals provides an example of the discrete distribution. But here belong the suspensions, emulsions, fogs, smokes (aerosols) too, within which grains are suspended, imbedded in a bulk medium. Of this type is the blood consisting of red blood-cells, animal and plant tissues, etc.

Beside these two main types of inhomogeneities also transitional and mixed forms are possible, however the investigation of these presumes that of the two basic forms.

Not so long ago the inner structure of a medium could be investigated only after its destruction which is to be avoided. Methods are needed for testing without destruction. One of these is testing by X. rays However, its range of applicability is limited. The ultrasonic waves can be applied in a broader area. They are mechanical waves of high frequency, the propagation properties of which are determined by the mechanical features of the medium, i.e. density and elastic moduli. Reaching another medium the velocity of the propagation can vary and on the boundary refraction and reflection may occur. Just these properties of the ultrasound are used for testing material defects, i.e. macroscopic discontinuities, inhomogeneities. However the question arises, what occurs, when the ultrasound does not reach a few discontinuities of large size, but the medium contains a large number of discontinuities having small size (smaller than the wavelength). By these the ultrasonic wave will be scattered, the particles become sources radiating in every direction and the examination of the scatter furnishes a possibility to obtain certain informations concerning the interior of the material without destroying it.

The procedure dealt with in the present work is based on the pulse-echo method. By a transmitter electric pulses will be formed and these are led to a piezoelectric crystal-plate which transforms the electric oscillations into mechanical ones and radiates them as ultrasonic waves into the examined matter. Then acting as receiver, it picks up the returning echos, retransforms them into electrical signals, which can be — after amplification — studied on an oscilloscope screen. There the echo signals appear at places corresponding to travel time (which is proportional to the path covered) of ultrasonic pulse.

The ultrasonic testing probe, which contains the piezo-crystal, the electromechanical transducer, is attached to the object by a coupling medium. In this arrangement the transmitter acts as a receiver as well, the advantage of which is that only one probe is needed. Usually, the opposite side of the object may not be reached and the separation of the scattered wave from the direct wave is a difficult problem.

The problem of continuously distributed inhomogeneities for liquids and gases has been treated in [3], while for solid bodies in [1]. The last paper contains the basic formulae and their application on a hemispherical layer. The problem of discrete distribution was treated in [5].

The first part of the present paper proceeds on the lines of [1] by determining the scattering (attenuation) coefficients [1](not to be confounded with the absorption coefficients) both for pressure waves and shear waves (in the sequel P and S waves) provided that $ka \ll 1$. Here $k = \frac{2\pi f}{c}$, f is the frequency, c the velocity of the propagation ($c = \alpha$ for P waves and $c = \beta$ for S waves), a means any of the later defined correlation distances. A further purpose is to obtain the "scattering formula" expressing the scattered energy depending on the direction and other quantities.[1]

The problem of discrete inhomogeneities will be discussed in the second part.

[1] The determination of these quantities makes the newness of the first part.

It is obvious that the results may be only of statistical character (not as testing rough material defects). The inhomogeneities of a solid body are strictly determined being the medium immobile providing that thermal agitations are disregarded and the phenomenon is considered macroscopically. (The irradiation, however, may influence the medium.) Fluids as moving media mean a different case. With this in mind the distribution of the inhomogeneities is well determined and not at random. However, under conditions regarded as quite identical (which after all are not identical) the irradiated waves will be scattered differently and so this scatter is a random process. This circumstance will be expressed by saying: the inhomogeneities have a random distribution, from which the accidental regular distribution of the inhomogeneities is to be distinguished. This may perhaps be described by a special formula.

The random character of the measurement will be imputed to the structure of the inhomogeneities of the medium. However, this is correct, because the latter may be examined and recognized through the former only. The successive measurements give a statistical ensemble concerning the functions (of position) $\lambda(\bar{r})$, $\mu(\bar{r})$, $\varrho(\bar{r})$ (Lamé "constants" and density), i.e. every measurement could provide at most a function-triplet $\lambda(\bar{r})$, $\mu(\bar{r})$, $\varrho(r)$, if it would be possible at all. In fact, only some quantities composed of certain statistical characteristics of these three random processes may be compared to measured data.

Consequently, our basic hypothesis is that λ, μ, ϱ have determined probability distributions, each of them is an *ergodic* and spatially homogeneous and isotropic process. Let us denote their auto- and cross-correlation functions by

$$N_{\lambda\lambda} = N_\lambda\,,\ N_{\lambda\mu}\,,\ N_{\mu\mu} = N_\mu\,,\ N_{\lambda\varrho}\,,\ N_{\mu\varrho}\,,\ N_{\varrho\varrho} = N_\varrho\,.$$

By definition they depend on the mutual distance $|\bar{r}_1 - \bar{r}_2|$ of the two positions only. On the other hand the ergodicity furnishes a value equal to the mean value of the ensemble, moreover by means of a unique local mean value formed by a function representing the process. E.g.

$$N_\varrho(R) = N_\varrho = \lim \frac{1}{V}\int_V \varrho(\bar{r})\,\varrho(\bar{r}+\bar{R})\,dv = E(\varrho(\bar{r})\,\varrho(\bar{r}+\bar{R}))$$

(expected value), where the integration has to be extended over the whole medium which is regarded as infinite, $R = |\bar{R}|$ and $P(\bar{r})$ is a point of dv.

This will not be needed in connection with the above correlation functions, because another mean value built up of them (the mentioned scattering coefficient) will be determined and compared to the measured value.

The wave equation and its solution

The obstacle derives from the circumstance that the waves propagating in solid bodies may be characterized by vectors only, because both P and S waves can arise. At a measuring process each of them may be irradiated into the material. On the other hand, in agreement with the physical model both of these arise from *any* kind of incident wave. Viz. the direction of the wave-vector will be changed by the scattering sources compared to the original direction of propagation. E.g. if the vector were perpendicular to this, then it will not be, but it will have a component parallel to this too, which exactly means the arising of a P wave.

The vector u of the elastic displacement satisfies the following wave equation (s. e.g. [1], p. 339)

$$\bar{F}(x,y,z,t;\ \lambda,\mu,\varrho;\ \bar{u}) \equiv -\varrho\frac{\partial^2\bar{u}}{\partial t^2} + \nabla[(\lambda+2\mu)\nabla\cdot\bar{u}] - \nabla\times\mu\nabla\times u + \tag{1}$$

$$+ 2(\nabla\mu\cdot\nabla)\bar{u} - 2(\nabla\mu)\nabla\cdot\bar{u} + 2(\nabla\mu)\times\nabla\times\bar{u} = 0.$$

Let us take an infinite, isotropic medium (the matter may be assumed as infinite in respect to the echo, in which we are interested, because it arrives back earlier than that coming from the farther situated boundaries), the characteristics λ, μ, ϱ of which have everywhere the constant values λ_0, μ_0, ϱ_0 except a volume V where

$$\lambda = \lambda_0 + \delta\lambda, \quad \mu = \mu_0 + \delta\mu, \quad \varrho = \varrho_0 + \delta\varrho \tag{2}$$

and $\delta\lambda$, $\delta\mu$, $\delta\varrho$ are random functions of the positions and

$$\frac{\delta\lambda}{\lambda_0} \ll 1, \quad \frac{\delta\mu}{\mu_0} \ll 1, \quad \frac{\delta\varrho}{\varrho_0} \ll 1 \qquad \text{(small inhomogeneities)}.$$

Let us assume that a wave $\bar{u}_0$ falls in V from the outside. The resulting total vector-field is composed of $\bar{u}_0$ and the scattered wave $\bar{u}_1$

$$\bar{u} = \bar{u}_0 + \bar{u}_1\,. \tag{3}$$

Then $|\bar{u}_1| \ll |\bar{u}_0|$. Replacing (2) and (3) into (1) and cancelling the terms higher in order than one, the following wave equation will be obtained for $\bar{u}_1$

$$\frac{\partial^2\bar{u}_1}{\partial t^2} - \alpha^2\nabla\nabla\cdot\bar{u}_1 + \beta^2\nabla\times\nabla\,\bar{u}_1 = \frac{1}{\varrho_0}\bar{F}(x,y,z,t;\ \delta\lambda,\delta\mu,\delta\varrho;\ \bar{u}_0) =$$

$$= \frac{1}{\varrho_0}\bar{F}_0\,, \tag{4}$$

where

$$\alpha^2 = \frac{\nu_0}{\varrho_0}, \qquad \beta^2 = \frac{\mu_0}{\varrho_0}, \qquad \gamma = \lambda + 2\mu.$$

For the scattered wave, equation (4) is an inhomogeneous linear (elastic) wave equation, the solution of which for infinite medium fulfils the equation (s. loc. cit.)

$$4\pi\varrho_0 \frac{\partial^2 \bar{u}_1}{\partial t^2} = \frac{1}{\alpha^2}\int_V \left[\frac{\bar{r}}{r^2}\frac{\partial^2 F_r}{\partial t^2} + \frac{3\alpha\bar{r}}{r^3}\frac{\partial F_r}{\partial t} + \frac{3\alpha^2 r}{r^4}F_r - \frac{\alpha}{r^2}\frac{\partial \bar{F}}{\partial t} - \frac{\alpha^2}{r^3}\bar{F}\right] d\tau - \tag{5}$$

$$- \frac{1}{\beta^2}\int_V \left\langle \frac{\bar{r}}{r^2}\frac{\partial^2 F_r}{\partial t^2} + \frac{3\beta\bar{r}}{r^3}\frac{\partial F_r}{\partial t} + \frac{3\beta^2\bar{r}}{r^4}F_r - \frac{1}{r}\frac{\partial^2 F}{\partial t^2} - \frac{\beta}{r^2}\frac{\partial \bar{F}}{\partial t} - \frac{\beta}{r^3}\bar{F}\right\rangle d\tau$$

where

$$\bar{r} = [\xi - x, \eta - y, \zeta - z], \quad d\tau = d\xi\, d\eta\, d\zeta, r = |\bar{r}|$$

and the integrations are to be extended to V; F_r is the scalar projection of $\bar{F}$ on $\bar{r}$. The brackets [ ] and ⟨⟩ mean retardations corresponding to the velocities of the propagation of the S and P waves. These are just α and β. Therefore, the argument t must be replaced by $t - \frac{\alpha}{r}$ and $t - \frac{\beta}{r}$, respectively. x, y, z are the coordinates of the observation point P.

If $\bar{u}_0$ and $\bar{F}(\bar{u}_0)$ are harmonic with a time dependence of the form $e^{-i\omega t}$ equation (5) will be formed as

$$4\pi\varrho_0\,\omega^2\,\bar{u}_1 = \int_V \left\{\left(\frac{k_\alpha^2}{r} + \frac{3ik_\alpha}{r^2} - \frac{3}{r^3}\right)\frac{F_r\bar{r}}{r} + \left(-\frac{ik_\alpha}{r^2} + \frac{1}{r^3}\right)\bar{F}\right\} e^{ik_\alpha r}\, d\tau -$$

$$- \int_V \left\{\left(\frac{k_\beta^2}{r} + \frac{3\,ik_\beta}{r^2} - \frac{3}{r^3}\right)\frac{F_r\bar{r}}{r} + \right. \tag{6}$$

$$\left. + \left(-\frac{k_\beta^2}{r} - \frac{ik_\beta}{r^2} + \frac{1}{r^3}\right)\bar{F}\right\} e^{ik_\beta r}\, d\tau,$$

where

$$k_\alpha = \frac{\omega}{\alpha}, \quad k_\beta = \frac{\omega}{\beta}, \quad \omega = 2\,\pi f.$$

This may be rearranged to the form (s. loc. cit.)

$$\bar{u}_1 = \frac{1}{4\,\pi\varrho_0\,\omega^2}\int_V \left(\nabla_\xi \frac{e^{ik_\alpha r}}{r}\right)(\nabla_\xi \cdot \bar{F})\, d\tau - \tag{7}$$

$$- \frac{1}{4\,\pi\varrho_0\,\omega^2}\int_V \left(\nabla_\xi \frac{e^{ik_\beta r}}{r}\right) \times (\nabla_\xi \times \bar{F})\, d\tau$$

too, where the first term corresponds to the scattered P wave, the second one to the scattered S wave.

1. Continuously distributed inhomogeneities

1.1 *The scattered SS wave*

Let the (along axis x) incident plane-polarized monochromatic ("monotone") plane S wave be of the form

$$\bar u_0 = A\bar e_z\, e^{i(k_\beta x - \omega t)}, \quad \bar e_z = [0, 0, 1] \tag{8}$$

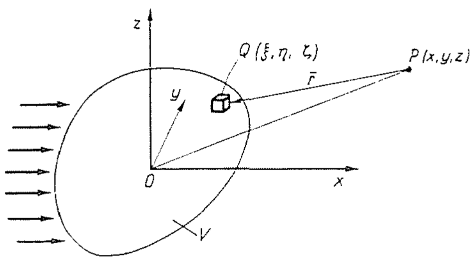

Fig. 1

the size of V small to $\overline{OP}$ (P = observer), large to the wavelength, and 0 a point of V. The directon of the oscillation is the z axis.

We will determine the scattered S wave in $P(x, y, z)$. Let us remark that for an actual measurement the incident wave is not a plane wave, rather a slightly divergent spherical one with an angle of divergency below 10°. However, this will be approached by a plane wave. — Furthermore the applied wave is a pulse consisting of 5—10 waves (and so it is not monochromatic), which can easily be studied on the screen of an oscilloscope. — In the present treatment we remain at the approach by a monochromatic plane wave.

At first the vector $\bar F(u_0)$ playing a role in (7) must be determined. This may be rearranged to the form

$$\bar F = -\varrho\ddot{\bar u}_0 + \nu\nabla\nabla\cdot\bar u_0 + (\nabla\cdot\bar u_0)(\nabla\nu) - \mu(\nabla\times\nabla\times\bar u_0) +$$
$$+ (\nabla\mu)\times(\nabla\times\bar u_0) + 2(\nabla\mu\cdot\nabla)\bar u_0 - 2(\nabla\mu)(\nabla\cdot\bar u_0); \tag{9}$$
$$(\mu = \mu_0 + \delta\mu\ .\ \text{etc.})$$

Replace (8) into (9). The corresponding terms in turn can be determined:

1° $\quad -\varrho\ddot{\bar u}_0 = \varrho\omega^2\bar u_0 = \varrho\omega^2 f\bar e_z,\ f = f(x, t) = Ae^{i(k_\beta x-\omega t)}$

2° $\quad \nabla\cdot\bar u_0 = 0\ ,\ \nu\nabla^2\cdot\bar u_0 = \nu\nabla(\nabla\cdot\bar u_0) = 0$

3° $\quad (\nabla\cdot\bar u_0)(\nabla\nu) = 0$

4° $\nabla\times\bar{u}_0 = -ikf\bar{e}_y, \ \bar{e}_y = [0,1,0], k = k_\beta$

$$\nabla\times\nabla\times\bar{u}_0 = k^2 f\bar{e}_z, \ -\mu\nabla\times\nabla\times\bar{u}_0 = -\mu k^2 f\bar{e}_z$$

5° $\nabla\mu\times(\nabla\times\bar{u}_0) = ikf(\mu_z\bar{e}_x - \mu_x\bar{e}_z), \ \mu_x = \dfrac{\partial\mu}{\partial x} = \dfrac{\partial(\delta\mu)}{\partial x}$, etc.

6° $\nabla\mu\cdot\nabla = \mu_x\dfrac{\partial}{\partial x} + \mu_y\dfrac{\partial}{\partial y} + \mu_z\dfrac{\partial}{\partial z}$

$$2(\nabla\mu\cdot\nabla)\bar{u}_0 = 2\mu_x\frac{\partial}{\partial x}(f\bar{e}_z) = 2ikf\mu_x\bar{e}_z$$

7° $-2(\nabla\mu)\nabla\cdot\bar{u}_0 = 0.$

Thus

$$\bar{F} = f\left[(\omega^2\varrho - \mu k^2 + ik\mu_x)\bar{e}_z + ik\mu_z\bar{e}_x\right]. \tag{10}$$

But

$$k = \frac{\omega}{\beta}, \ \beta^2 = \frac{\mu_0}{\varrho_0},$$

whence

$$\omega^2\varrho - \mu k^2 = k^2(\beta^2\varrho - \mu) = k^2\left[\beta^2(\varrho_0 + \delta\varrho) - (\mu_0 + \delta\mu)\right] = k^2(\beta^2\delta\varrho - \delta\mu)$$

and so

$$\bar{F} = kf\{[k(\beta^2\delta\varrho - \delta\mu) + i\mu_x]\bar{e}_z + i\mu_z\bar{e}_x\}. \tag{11}$$

Let the abbreviation

$$\beta^2\varrho - \mu = \sigma \tag{12}$$

be introduced, then

$$\bar{F} = kf\left[(k\delta\sigma + i\mu_x)\bar{e}_z + i\mu_z\bar{e}_x\right] = [\varphi, 0, \psi] \tag{12'}$$

where

$$\varphi = ik\mu_z f, \quad \psi = kf(k\delta\sigma + i\mu_x), \quad \mu_x = \frac{\partial\mu}{\partial x} = \frac{\partial\delta\mu}{\partial x}.$$

Expression $\Delta\times\bar{F}$ in (7) will have the form

$$\nabla\times\bar{F} = [\psi_y, \varphi_z - \psi_x, -\varphi_y] = kf[\mathscr{A}, \mathscr{B}, \mathscr{C}].$$

Here

$$\mathscr{A} = k\sigma_y + i\mu_{xy}$$

$$\mathscr{B} = -ik^2\delta\sigma + k\varepsilon_x + i(\mu_{zz} - \mu_{xx}) \ (\varepsilon = \mu - \sigma \text{ abbreviation}) \tag{12''}$$

$$\mathscr{C} = -i\mu_{yz}.$$

Furthermore

$$\nabla_\xi\left(\frac{e^{ikr}}{r}\right) = \frac{1 - ikr}{r^3}e^{ikr}\bar{r}, \ \bar{r} = [\xi - x, \eta - y, \zeta - z].$$

Then by (7) the x component of the scattered SS wave

$$u_{ss_x} = \frac{Ak}{4\pi\varrho_0\omega^2}\int\limits_V \frac{1-ikr}{r^3}\left[(\eta-y)\mathcal{C}-(\zeta-z)\mathcal{B}\right]e^{i[k(\xi\cdot r)-\omega t]}\,d\tau \qquad (13)$$

where ξ, η, ζ are the coordinates of the scattering element. The integration is extended to V being $\mathcal{A}=\mathcal{B}=\mathcal{C}=0$ elsewhere. To determine the square of the absolute value of u_{s_x} multiply it by its conjugate. So we have

$$|u_{ss_x}|^2 = \frac{A^2k^2}{16\pi^2\varrho_0^2\omega^2}\int\limits_{V_1}\int\limits_{V_2}\frac{(1-ikr_1)(1+ikr_2)}{r_1^3r_2^3}(f_1f_2^*)\,e^{ik(\xi_1-\xi_2+r_1-r_2)}\,d\tau_1\,d\tau_2. \qquad (14)$$

Here each of the integrals are extended to the same $V_1=V_2=V$ volume and their elements, points are distinguished by the subscripts 1 and 2, and f is not identical with the above f. Its value

$$f=(\eta-y)\mathcal{C}-(\zeta-z)\mathcal{B}$$

and

$$f_1f_2^* = \left[(\eta_1-y)\mathcal{C}_1-(\zeta_1-z)\mathcal{B}_1\right]\left[(\eta_2-y)\mathcal{C}_2^*-(\zeta_2-z)\mathcal{B}_2^*\right]. \qquad (15)$$

Taking into account that the size of V is small to $\overline{OP}$, we take in $f_1f_2^*$ $\eta_1-y\sim\sim\eta_2-y$, etc., even neglect ξ, η, ζ beside x, y, z. Obtaining

$$f_1f_2^* = (y\mathcal{C}_1-z\mathcal{B}_1)(y\mathcal{C}_2^*-z\mathcal{B}_2^*) =$$
$$= y^2\mathcal{C}_1\mathcal{C}_2^*+z^2\mathcal{B}_1\mathcal{B}_2^*-yz(\mathcal{B}_1\mathcal{C}_2^*+\mathcal{C}_1\mathcal{B}_2^*).$$

The formulae of u_{ss_y} and u_{ss_z} are similar, only $f_1f_2^*$ are replaced by

$$g_1g_2^* = z^2\mathcal{A}_1\mathcal{A}_2^*+x^2\mathcal{C}_1\mathcal{C}_2^*-zx(\mathcal{C}_1\mathcal{A}_2^*+\mathcal{A}_1\mathcal{C}_2^*)$$

and

$$h_1h_2^* = x^2\mathcal{B}_1\mathcal{B}_2^*+y^2\mathcal{A}_1\mathcal{A}_2^*-xy(\mathcal{A}_1\mathcal{B}_2^*+\mathcal{B}_1\mathcal{A}_2^*)$$

respectively, the sum of which is

$$S=f_1f_2^*+g_1g_2^*+h_1h_2^*=(y^2+z^2)\mathcal{A}_1\mathcal{A}_2^*+$$
$$+(z^2+x^2)\mathcal{B}_1\mathcal{B}_2^*+(x^2+y^2)\mathcal{C}_1\mathcal{C}_2^*-yz(\mathcal{B}_1\mathcal{C}_2^*+\mathcal{C}_1\mathcal{B}_2^*)-$$
$$-zx(\mathcal{A}_1\mathcal{C}_2^*+\mathcal{C}_1\mathcal{A}_2^*)-xy(\mathcal{A}_1\mathcal{B}_2^*+\mathcal{B}_1\mathcal{A}_2^*).$$

Corresponding to the ensembles of λ, μ, ϱ, the average (expected) value of $|u_{SS}|^2$ is as follows:

$$|u_{ss}|^2 = \frac{A^2k^2}{16\pi^2\varrho_0^2\omega^4}\int\limits_{V_1}\int\limits_{V_2}\frac{(1-ikr_1)(1+ikr_2)}{r_1^3r_2^3}\langle S\rangle\,e^{ik_\beta(\xi_1-\xi_2+r_1-r_2)}\,d\tau_1\,d\tau_2 \qquad (16)$$

where $\langle S\rangle$ means the average value of S in the same sense. To compute (16) we introduce new coordinates, center of mass coordinates and relative coordinates. These are

$$x_0 = \frac{\xi_1 + \xi_2}{2}, \quad y_0 = \frac{\eta_1 + \eta_2}{2}, \quad z_0 = \frac{\zeta_1 + \zeta_2}{2}$$

$$x' = \xi_2 - \xi_1, \quad y' = \eta_2 - \eta_1, \quad z' = \zeta_2 - \zeta_1.$$

With regard to the smallness of V to $\overline{OP}$ we obtain

$$r_2 - r_1 \approx \frac{\bar{R}\bar{r}_1}{r_1}, \quad \bar{R} = [x', y', z'] \quad (s.[1], \text{p. } 340)$$

$$k(\xi_1 - \xi_2 + r_1 - r_2) = k\left[x'\left(1 + \frac{\xi_1 - x}{r_1}\right) + \right.$$
$$\left. + y'\frac{\eta_1 - y}{r_1} + z'\frac{\zeta_1 - z}{r_1}\right] = \bar{K}\bar{R} \tag{17}$$

$$\bar{K} = \left[1 + \frac{\xi_1 - x}{r_1}, \ \frac{\eta_1 - y}{r_1}, \ \frac{\zeta_1 - z}{r_1}\right] = k\left(\bar{e}_x + \frac{\bar{r}_1}{r_1}\right).$$

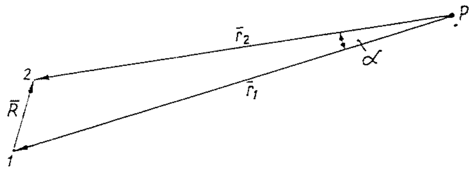

Fig. 2

For a later application we wish to remark that

$$K^2 = |\bar{K}|^2 = 2\,k^2\left(1 + \frac{\xi_1 - x}{r_1}\right). \tag{17'}$$

In (16) everywhere we take $r_1 = r_2 = r = \text{const}$, except in the exponents, where for $r_1 - r_2$ the above better approximation must be taken, since e.g. $\sin k(r_1 - r_2)$ varies more quickly. Then

$$\langle |u_{ss}|^2 \rangle = \frac{A^2 k^2}{4\pi^2 \varrho_0^2 \omega^4} \int\limits_{V_0}\int\limits_{V'} \frac{1 + k^2 r^2}{r^6} \langle S \rangle e^{-i\bar{K}\bar{R}}\, dv'\, dv_0 =$$

$$(dv' = dx'\, dy'\, dz',\ dv_0 = dx_0\, dy_0\, dz_0,\ V_0 = V' = V =)$$

$$= \frac{A^2 k^2}{4\pi^2 \varrho_0^2 \omega^4} \int\limits_{V_0} \frac{1 + k^2 r^2}{r^6}\, dv_0 \int\limits_{V'} \langle S \rangle e^{-i\bar{K}\bar{R}}\, dv' = \tag{18}$$

$$= \frac{A^2 k^4 V}{4\pi^2 \varrho_0^2 \omega^4 r^4}\left(1 + \frac{1}{k^2 r^2}\right)\mathcal{I}$$

where

$$\mathcal{J} = \int_{V'} \langle S \rangle e^{-i\bar{K}\bar{R}}\, dv'. \tag{19}$$

The correlation functions decrease rapidly with the distance $R = |\bar{R}|$. Therefore, in (19) V' may be replaced by a small sphere S_b, with radius b and centered at point 1. Then $\mathcal{J}$ can be determined term by term as follows. Corresponding to the value of S in (15) $\mathcal{J}$ consists of the terms

$$1^\circ \qquad \mathcal{J}_{\mathcal{A}} = (y^2 + z^2) \int_{S_b} \langle \mathcal{A}_1 \mathcal{A}_2^* \rangle e^{-i\bar{K}\bar{R}}\, dv'.$$

Here by (12'') (denoting $k\delta\sigma + i\mu_x$ by f; its meaning differs from the previous one)

$$\mathcal{A}_1 \mathcal{A}_2^* = \frac{\partial f_1}{\partial \eta_1} \frac{\partial f_2^*}{\partial \eta_2} = \frac{\partial^2 (f_1 f_2^*)}{\partial \eta_1 \partial \eta_2}$$

since f_1 is independent of η_2 and f_2^* of η_1. — Everyone of the correlation functions depends only on

$$R = \sqrt{(\xi_2 - \xi_1)^2 + (\eta_2 - \eta_1)^2 + (\zeta_2 - \zeta_1)^2}$$

consequently

$$-\frac{\partial N}{\partial \xi_1} = \frac{\partial N}{\partial \xi_2} = \frac{\partial N}{\partial x'}, \quad \text{etc.} \tag{20}$$

Taking into account that $(\mathcal{A}_1 \mathcal{A}_2^*)$ consists (linearly) of these correlation functions (s. later) we obtain

$$\langle \mathcal{A}_1 \mathcal{A}_2^* \rangle = -\frac{\partial^2 \langle f_1 f_2^* \rangle}{\partial y'^2}.$$

Furthermore by (17) (denoting the direction cosines of $\overline{OP}$ by L, M, N)

$$\frac{\partial}{\partial x'}(\bar{K}\bar{R}) = k\left(1 + \frac{\xi_1 - x}{r_1}\right) = k(1 - L)$$

$$\frac{\partial(\bar{K}\bar{R})}{\partial y'} = k\frac{\eta_1 - y}{r_1} = -kM$$

$$\frac{\partial(\bar{K}\bar{R})}{\partial z'} = k\frac{\zeta_1 - z}{r_1} = -kN.$$

This gives e.g.

$$\frac{\partial(e^{-iKR})}{\partial x'} = -ik(1 - L)\, e^{-i\bar{K}\bar{R}}, \text{ etc.}$$

So by repeated integrations by parts and with regard that $\delta\mu$, $\delta\varrho$, together with their derivatives, and thereby $\mathscr{A}$ too, vanish on the surface of S_b

$$\mathscr{I}_{\mathscr{A}} = r^2(M^2 + N^2) \int_{S_b} - \frac{\partial^2 \langle f_1 f_2^* \rangle}{\partial y'^2} e^{-i\bar{K}\bar{R}} dv' = \\ = r^2(M^2 + N^2) k^2 M^2 \int_{S_b} \langle f_1 f_2^* \rangle e^{-i\bar{K}\bar{R}} dv' . \tag{21}$$

Now let us inspect the value $f_1 f_2^*$ more closely

$$f_1 f_2^* = \left(k\,\delta\sigma_1 + i\frac{\partial\delta\mu_1}{\partial\xi_1}\right)\left(k\,\delta\sigma_2 - i\frac{\partial\delta\mu_2}{\partial\xi_2}\right) = \\ = k^2\,\delta\sigma_1\,\delta\sigma_2 + ik\frac{\partial(\delta\mu_1\,\delta\sigma_2)}{\partial\xi_1} - ik\frac{\partial(\delta\sigma_1\,\delta\mu_2)}{\partial\xi_2} + \frac{\partial^2(\delta\mu_1\,\delta\mu_2)}{\partial\xi_1\,\partial\xi_2}$$

whence

$$\langle f_1 f_2^* \rangle = k^2\langle\delta\sigma_1\,\delta\sigma_2\rangle - ik\frac{\partial}{\partial x'}\left(\langle\delta\mu_1\,\delta\sigma_2\rangle + \right. \\ \left. + \langle\delta\sigma_1\,\delta\mu_2\rangle\right) - \frac{\partial^2\langle\delta\mu_1\,\delta\mu_2\rangle}{\partial y'^2} .$$

Here we made use of the relation

$$\left\langle\frac{\partial(\delta\sigma\,\delta\mu)}{\partial x'}\right\rangle = \frac{\partial\langle\delta\sigma\,\delta\mu\rangle}{\partial x'} \quad \text{(s. [2], p. 102)} .$$

But

$$\langle\delta\sigma_1\,\delta\sigma_2\rangle = N_\sigma, \quad \langle\delta\sigma_1\,\delta\mu_2\rangle = \langle\delta\sigma_2\,\delta\mu_1\rangle = N_{\sigma\mu}, \quad \text{etc.}$$

so

$$\langle f_1 f_2^* \rangle = k^2 N_\sigma - 2\,ik\frac{\partial N_{\mu\sigma}}{\partial x'} - \frac{\partial^2 N_\mu}{\partial x'^2} .$$

Therefore, by repeated integrations by parts

$$\int_{S_b} \langle f_1 f_2^* \rangle e^{-i\bar{K}\bar{R}} dv' = k^2 \int_{S_b} \left[N_\sigma + 2\,(1 - L)\,N_{\mu\sigma} + (1 - L^2)\,N_\mu\right] e^{-i\bar{K}\bar{R}} dv' , \tag{22}$$

since the N's and their derivatives vanish on the surface of S_b. Let us suppose that the correlation functions have the form

$$N = n^2 e^{-\frac{R^2}{a^2}} ,$$

where n^2 and a are constant (the correlation coefficient and correlation distance, on which N decreases on its $\frac{1}{e}$ th part). Herewith it becomes necessary the evaluation of

$$\mathcal{J}' = \int_{S_b} e^{-\frac{R^2}{a^2}} e^{-i\bar{K}\bar{R}}\, dv' .$$

To this end introducing the polar coordinates R, θ, φ (θ is the angle included by $\bar{R}$ and $\bar{K}$)

$$\mathcal{J}' = \int_0^{2\pi} \int_0^{\pi} \int_0^{b} e^{-\frac{R^2}{a^2}} e^{-iKR\cos\theta} R^2 \sin\theta\, dR\, d\theta\, d\varphi, \quad K = |\bar{K}|, R = |\bar{R}| =$$

$$= \frac{4\pi}{K} \int_0^b R e^{-\frac{R^2}{a^2}} \sin KR\, dR.$$

The exponential factor decreases rapidly with R. Hence b may be replaced by $+\infty$. By integration by parts

$$\mathcal{J}' = 2\pi a^2 \int_0^{\infty} e^{-\frac{R^2}{a^2}} \cos KR\, dR$$

the value of which is as follows

$$\mathcal{J}' = \pi^{3/2} a^3 e^{-\frac{K^2 a^2}{4}} \approx \pi^{3/2} a^3 \tag{23}$$

since by (17′) and our assumption

$$\frac{K^2 a^2}{4} = \frac{k^2 a^2}{2} (1 - L) \ll 1 .$$

From (21)—(22)—(23)

$$\mathcal{J}_{\mathcal{A}} = \pi^{3/2} k^4 r^2 (M^2 + N^2) M^2 \left[n_\sigma^2 a_\sigma^2 + 2(1-L)\, n_{\mu\sigma}^2 a_{\mu\sigma}^3 + (1-L)^2 n_\mu^2 a_\mu^3 \right]$$

2° $$\mathcal{J}_{\mathcal{B}} = (z^2 + x^2) \int_{S_b} \langle \mathcal{B}_1 \mathcal{B}_2^* \rangle e^{-i\bar{K}\bar{R}}\, dv' .$$

Here

$$\mathcal{B}_1 \mathcal{B}_2^* = \left(-ik\,\delta\sigma_1 + k \frac{\partial \delta\varepsilon_1}{\partial \xi_1} + i \frac{\partial^2 \delta\mu_1}{\partial \zeta_1^2} - i \frac{\partial^2 \delta\mu_1}{\partial \xi_1^2} \right) \cdot$$

$$\cdot \left(ik\,\delta\sigma_2 + k \frac{\partial \delta\varepsilon_2}{\partial \xi_2} - i \frac{\partial^2 \delta\mu_2}{\partial \zeta_2^2} + i \frac{\partial^2 \delta\mu_2}{\partial \xi_2^2} \right) .$$

Carrying out the multiplication, using (20), introducing the correlation functions and integrating repeatedly by parts we obtain

$$\mathcal{J}_{\mathcal{B}} = r^2 k^4 (L^2 + N^2) \int_{S_b} \Big[N_\sigma + 2(1-L) N_{\sigma\varepsilon} +$$

$$+ 2 \left(N^2 - (1-L)^2\right) N_{\sigma\mu} + (1-L)^2 N_\varepsilon +$$

$$+ 2(1-L)\left(N^2 - (1-L)^2\right) N_{\varepsilon\mu} + \left((1-L)^2 - N^2\right) N_\mu \Big] e^{-i\bar{K}\bar{R}}\, dv'$$

which may be computed in the same way as $\mathcal{I}_A$. Obtaining

$$\mathcal{I}_{\mathcal{B}} = \pi^{3/2} k^4 r^2 (L^2 + N^2) \left[n_\sigma^2 a_\sigma^3 + 2(1-L) n_{\sigma\varepsilon}^2 a_{\sigma\varepsilon}^3 + \right.$$
$$+ 2\left(N^2 - (1-L)^2\right) n_{\sigma\mu}^2 a_{\sigma\mu}^3 + (1-L)^2 n_\varepsilon^2 a_\varepsilon^3 +$$
$$\left. + 2(1-L)\left(N^2 - (1-L)^2\right) n_{\varepsilon\mu}^2 a_{\varepsilon\mu}^3 + \left((1-L)^2 - N^2\right) n_\mu^2 a_\mu^3\right].$$

Similarly without detailing the steps of the calculation

3° $$\mathcal{I}_{\mathcal{C}} = \pi^{3/2} r^2 k^4 (L^2 + M^2) M^2 N^2 n_\mu^2 a_\mu^3$$

4° $$\mathcal{I}_{\mathcal{AC}} = 2\pi^{3/2} r^2 k^4 L M^2 N^2 \left[n_{\sigma\mu}^2 a_{\sigma\mu}^3 + (1-L) n_\mu^2 a_\mu^3\right]$$

5° $$\mathcal{I}_{\mathcal{BC}} = 2\pi^{3/2} r^2 k^4 M^2 N^2 \left[n_{\sigma\mu}^2 a_{\sigma\mu}^3 + (1-L) n_{\varepsilon\mu}^2 a_{\varepsilon\mu}^3 + \right.$$
$$\left. + \left(N^2 - (1-L)^2\right) n_\mu^2 a_\mu^3\right]$$

6° $$\mathcal{I}_{\mathcal{AB}} = -2\pi^{3/2} r^2 k^4 LMN \left[n_\sigma^2 a_\sigma^3 + \left((1-L) - \right.\right.$$
$$- (1-L)^2 + N^2\big) n_{\sigma\mu}^2 a_{\sigma\mu}^3 + (1-L) n_{\sigma\varepsilon}^2 a_{\sigma\varepsilon}^3 +$$
$$\left. + (1-L)^2 n_{\mu\varepsilon}^2 a_{\mu\varepsilon}^3 + (1-L)\left(N^2 - (1-L)^2\right) n_\mu^2 a_\mu^3\right].$$

Summing up the values 1° to 6° we get the value of $\mathcal{I}$ in (19), whence by (18)

$$\langle |u_{SS}|^2 \rangle = \frac{4\pi^3 \sqrt{\pi} A^2 V f^4}{\varrho_0^2 \beta^8 r^2} \mathcal{S} \tag{24}$$

where

$$\mathcal{S} = K_\sigma n_\sigma^2 a_\sigma^3 + K_\mu n_\mu^2 a_\mu^3 + K_\varepsilon n_\varepsilon^2 a_\varepsilon^3 +$$
$$+ K_{\sigma\mu} n_{\sigma\mu}^2 a_{\sigma\mu}^3 + K_{\sigma\varepsilon} n_{\sigma\varepsilon}^2 a_{\sigma\varepsilon}^3 + K_{\varepsilon\mu} n_{\varepsilon\mu}^2 a_{\varepsilon\mu}^3 .$$

The coefficients $K_\sigma, \ldots, K_{\varepsilon\mu}$ are rather involved polynomials of L, M, N. Simpler results will be obtained, if in (22) and similar formulae we return from σ, μ, ε to ϱ, λ, μ. By (12) e.g.

$$N_\sigma = \beta^4 N_\varrho - 2\beta^2 N_{\varrho\mu} + N_\mu, \quad \text{etc.}$$

So we obtain

$$\mathcal{S} = L_\varrho n_\varrho^2 a_\varrho^3 + L_\mu n_\mu^2 a_\mu^3 + L_{\varrho\mu} n_{\varrho\mu}^2 a_{\varrho\mu}^3 \tag{25}$$

with

$$\begin{aligned} \frac{L_\varrho}{\beta^4} &= M^2 + L^2 N^2 - L^2 M^2 - 2L^2 MN + L^4 \\ L_\mu &= L^2 M^2 - L^4 M^2 + L^4 N^2 + N^6 \\ \frac{L_{\varrho\mu}}{2\beta^2} &= L - LM^2 + LM^4 + 3LN^2 + 2L^3 MN - LMN^3. \end{aligned} \tag{26}$$

(24) is the "scattering formula" giving the energy of the scattered SS waves depending (besides other quantities) on the direction $[L, M, N]$. E.g. the energy

scattered in the direction of the incident beam is proportional to

$$\mathscr{S} = \beta^4 n_\varrho^2 a_\varrho^3 + 2\beta^2 n_{\varrho\mu}^2 a_{\varrho\mu}^3 \qquad (L = 1, M = N = 0)$$

1.2. *The scattering coefficient* $\varkappa_{SS}$

Let V be a thin cylindrical layer with base F, thickness dx. The energy flux of the herein incident beam is

$$E_i = \gamma F A^2 \; (\gamma = \text{const}, \; A \text{ amplitude}) \,. \tag{27}$$

The scattered energy is the surface integral

$$E_s = \gamma \oint_{F_r} \langle |u_{SS}|^2 \rangle \, dF \tag{28}$$

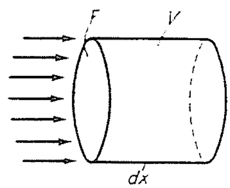

Fig. 3

extended to a sphere of radius r with $\frac{1}{k^2 r^2} \ll 1$ (viz. so the approximation $1 + \frac{1}{k^2 r^2} \approx 1$ applied above is justified). Then we have to compute a number of integrals. One of them is

$$\oint_{F_r} L_\varrho \, dF = \beta^4 \int_0^{2\pi} \int_0^{\pi} (M^2 + L^2 N^2 - L^2 M^2 - 2 L^2 MN + L^4) \, r^2 \sin\theta \, d\theta \, d\varphi$$
$$(L = \sin\theta \cos\varphi, \; M = \sin\theta \sin\varphi, \; M = \cos\theta).$$

The energy left after traversing the layer

$$E_i - E_s = E_i \, e^{-\varkappa dx} \approx E_i (1 - \alpha \, dx)$$

whence

$$\varkappa = \frac{E_s}{E_i \, dx} = \frac{E_s}{\gamma F A^2 \, dx} = \frac{E_s}{\gamma A^2 V} \,. \tag{29}$$

Finally we have

$$\varkappa_{SS} = \frac{16 \pi^4 \sqrt{\pi} f^4}{105 \varrho_0^2 \beta^8} (35 \beta^4 n_\varrho^2 a_\varrho^3 + 22 n_\mu^2 a_\mu^3). \tag{30}$$

This is the energy scattering coefficient. The amplitude scattering coefficient is half this value.

1.3. *The scattered PP wave and* α_{PP}

This means a P wave which arose from an incident P wave

$$\bar{u}_0 = A\bar{e}_x e^{i(k_\alpha x-\omega t)},\ \bar{e}_x - [1,0,0],\ k_\alpha = \frac{\omega}{\alpha},\ \omega = 2\pi f,\ \alpha^2 = \frac{\nu_0}{\varrho_0},\qquad \nu_0 = \lambda_0 + 2\mu_0. \tag{31}$$

Replacing this value into (9) we have (instead of the incorrect value on p. 340 of [1]) the value

$$\bar{F} = \left[\left(\omega^2\delta\varrho - k_\alpha^2\delta\nu + 2ik_\alpha\frac{\partial\delta\mu}{\partial x}\right)\bar{e}_x + ik_\alpha\nabla\delta\lambda\right]Ae^{i(k_\alpha x-\omega t)}. \tag{32}$$

Now the first term of (7) must be evaluated. Omitting the details we obtain

$$\langle|u_{PPx}|^2\rangle = \frac{\pi^3\sqrt{\pi}A^2Vf^4}{\varrho_0^2\alpha^8 r^2}L^2(1-L^2)P \tag{33}$$

where

$$P = \alpha^4 n_\varrho^2 a_\varrho^3 - 2\alpha^2(1-L)n_{\varrho\nu}^2 a_{\varrho\nu}^3 + L^2 n_\nu^2 a_\nu^3 + 2\alpha^2(1-L^2)n_{\varrho\lambda}^2 a_{\varrho\lambda}^3 + (1-2L)(1-L^2)n_{\lambda\nu}^2 a_{\lambda\nu}^3 + (1-L^2)n_\lambda^2 a_\lambda^3.$$

On the other hand $u_{PPx} = L\,|u_{PP}|$, therefore

$$\langle|u_{pp}|^2\rangle = \frac{1}{L^2}\langle|u_{PPx}|^2\rangle \tag{34}$$

which shows the fact that corresponding to the P character of the scattered wave it depends only on the angle included with axis x. For α_{PP} we have

$$\alpha_{pp} = \frac{8\pi^4\sqrt{\pi}f^4}{15\varrho_0^2\alpha^8}\left[10\alpha^4 n_\varrho^2 a_\varrho^3 - 50\alpha^2 n_{\varrho\nu}^2 a_{\varrho\nu}^3 + 4n_\nu^2 a_\nu^3 + 12\alpha^2 n_{\varrho\lambda}^2 a_{\varrho\lambda}^3 + 10n_{\lambda\nu}^2 a_{\lambda\nu}^3 + 6n_\lambda^2 a_\lambda^3\right] \tag{35}$$

or replacing λ by $\nu - 2\mu$ (in a correspondent former formula)

$$\alpha_{PP} = \frac{8\pi^4\sqrt{\pi}f^4}{15\varrho_0^2\alpha^8}\left[10\alpha^4 n_\varrho^2 a_\varrho^3 + 16n_\nu^2 a_\nu^3 - 24\alpha^2 n_{\varrho\mu}^2 a_{\varrho\mu}^3 - 44n_{\mu\nu}^2 a_{\mu\nu}^3 + 6n_\mu^2 a_\mu^3\right]. \tag{36}$$

1.4. *The SP wave and* α_{SP}

Starting out from (11) the first term of (7) gives (as in 1.3.)

$$\langle|u_{sp}|^2\rangle = \frac{\pi^3\sqrt{\pi}A^2Vf^4}{\varrho_0^2\alpha^6\beta^2 r^2}N^2\left[\alpha^2\beta^2 n_\varrho^2 a_\varrho^3 - 4\alpha\beta M n_{\varrho\mu}^2 a_{\varrho\mu}^3 + 4M^2 n_\mu^2 a_\mu^3\right] \tag{37}$$

and

$$\alpha_{sp} = \frac{4\,\pi^4 \sqrt{\pi}\, f^4}{15\,\varrho_0^2\, \alpha^6\, \beta^2} \left(5\,\alpha^2\, \beta^2\, n_\varrho^2\, a_\varrho^3 + 4\, n_\mu^2\, a_\mu^3 \right). \tag{38}$$

1.5. *The PS wave and* α_{PS}

By evaluating the second term of (7) with $\bar{F}$ given in (32)

$$\langle |u_{PS}|^2 \rangle = \frac{\pi^3 \sqrt{\pi}\, A^2\, V\, f^4}{\varrho_0^2\, \alpha^2\, \beta^6\, r^2} (1 - L^2)\,(\alpha^2\, \beta^2\, n_\varrho^2\, a_\varrho^3 - 4\,\alpha\,\beta\, L\, n_{\varrho\mu}^2\, a_{\varrho\mu}^3 + 4\, L^2\, n_\mu^2\, a_\mu^3) \tag{39}$$

and

$$\alpha_{PS} = \frac{8\pi^4 \sqrt{\pi}\, f^4}{15\,\varrho_0^2\, \alpha^2\, \beta^6} (5\,\alpha^2\, \beta^2\, n_\varrho^2\, a_\varrho^3 + 4\, n_\mu^2\, a_\mu^3). \tag{40}$$

Remark 1. A more exact approximation can be obtained for the values of $\langle | u |^2 \rangle$ and α provided the exponent

$$\varepsilon = \frac{K^2\, a^2}{4} = \frac{k^2\, a^2}{2} (1 - L)$$

in (23) will not be neglected. In this case in the formulae of $\langle |u|^2 \rangle$'s every term of the form $n^2 a^3$ must be multiplied by a factor of the form $e^{-\varepsilon}$. The evaluation of the α's necessitates the computation of more involved integrals containing this factor. On the other hand the results will be valid for larger ka values too. — The integrals in question are of the form

$$\mathcal{I}_k' = \oint L^k\, e^{uL}\, d\, F = \int_0^{2\pi} \int_0^{\pi} \cos^k \theta\, e^{u \cos \theta} \sin \theta\, d\, \theta\, d\varphi = 2\pi\, \mathcal{I}_k$$

$$\left(L = \cos \theta, \quad u = \frac{(k\, a)^2}{2} \right), \qquad \mathcal{I}_k = \int_0^{\pi} \cos^k \theta \sin \theta\, e^{u \cos \theta}\, d\, \theta,$$

$$k = 0, 1, 2, 3, 4, 5.$$

By integration by parts

$$\mathcal{I}_k = \frac{e^u - (-1)^k\, e^{-u}}{u} - \frac{k}{u}\, \mathcal{I}_{k-1}$$

which gives $\mathcal{I}_0$ to $\mathcal{I}_5$ in turn. For $u \gg 1$ (i.e. $ka \gg 1$) and fluids and gases we obtain the asymptotic expression

$$\alpha = 2\sqrt{\pi} \left[-4\, k^{-2} \left(\frac{n_\varrho}{\varrho} \right)^2 a_\varrho^{-3} + 4\, k^{-2} \left(\frac{n_c}{c} \right)^2 a_c^{-3} + k^2 \left(\frac{n_{\varrho c}}{\varrho c} \right)^2 a_{\varrho c} \right].$$

This formula differs greatly from the corresponding one of [3] (p. 55) viz.

$$\alpha = \sqrt{\pi}\, k^2 \left(\frac{n_c}{c}\right)^2 a_c .$$

Remark 2. The measuring process gives immediately the attenuation coefficient only

$$\varkappa = Af + Bf^4$$

the first term of which corresponds to the absorption, the second to the scatter. By measuring $\varkappa$ at two frequencies A and B can be determined and the value Bf^4 is to be compared to the calculated values α_{ss}, etc.

Remark 3. The correlation coefficients n^2 and distances a (altogether 6—6 values) appear in each formula in the same connection n^2a^3. Therefore given the four α's perhaps four such values can be determined, but separately n and a not at all. Further considerations are needed for their separation.

Remark 4. A pulse modulated incident beam is not monotone, but has an involved Fourier-spectrum. E.g. if it has a triangle shaped envelope (consisting of straight lines), then approximately

$$p_0 = f(t)\, e^{i(\omega t - kx)}, \quad f(t) = \begin{cases} 0 & , \; t \leq 0 \\ mt & , \; 0 \leq t \leq t_0 \\ m(2t_0 - t), & t_0 \leq t < 2t_0 \\ 0 & , \; t \geq 2t_0 \end{cases}$$

provided that the medium is a fluid or a gas. Then it can easily be seen that in our scattering formula A^2 must be replaced by

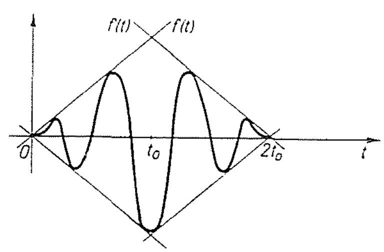

Fig. 4

$$f^2 + \frac{4}{c_0^2 k^2} f'^2$$

where c_0 is the velocity of the propagation, while the scattering coefficient remains unchanged.

2. Discretely distributed inhomogeneities

In this part we shall deal with the case of discretely distributed random inhomogeneities, where inside the grains ϱ, λ, μ are constant and on the surface they suffer a jump. Here belong the metals, which are polycrystalline materials and their structures depend, beside composition, greatly on the process of production and heat treatment. This may be very different within the same work piece according to welding and e.g. the heat treatment may be limited to certain parts, in most cases to the surface of the pieces. This can be seen very well by the microscopical investigation of the polished surface of the metals, or by X-ray or neutron ray diffraction technique to a maximal depth 1 or 10 mm, respectively. On the other hand the ultrasound may be applied for investigations of very thick specimens.

The inhomogeneities may be of two sorts according to the circumstance that either the single grains are isotropic with characteristics different from the environment or their values may be the same, but the grains have a crystalline structure and their crystallographic axes are randomly oriented. Of course, these two factors can also appear together. The problem was first treated by BHATIA [5]. The present paper gives more exact results in a different way, slightly correcting certain equations of [5] and showing that its formulae are approximately valid only for $ka \ll 1$ (a is the radius of the grain), while the formulae obtained here hold for some larger ka too.

MERKULOV [6] has expanded the work of LIFSHITS and PARKHOMOVSKII [7] to hexagonal and cubic crystals, with similar results as BHATIA [5] and BHATIA—MOORE in a more recent work [8]. However, the agreement of the results in [8] with the measurements is not better than those in [5].

PAPADAKIS [9] summed up the results in the most important work in this field and gave a method to determine the average grain size.

2.1. *Determination of the SS wave*

Let us assume V as a small sphere S_a (small compared to the wavelength) with radius a, centered at 0 and ϱ, λ, μ to be constant having values

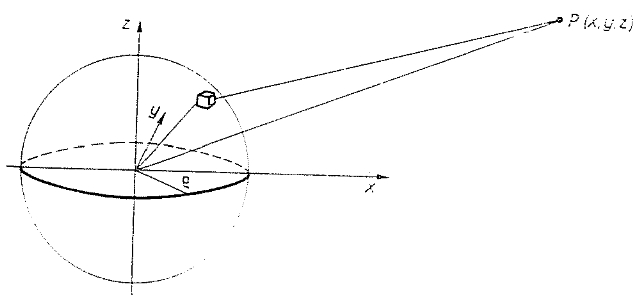

Fig. 5

ϱ_0, λ_0, μ_0 in the bulk medium and $\varrho_0 + \delta\varrho$, $\lambda_0 + \delta\lambda$, $\mu_0 + \delta\mu$ in the grains. We provisionally assume $\delta\varrho$, $\delta\lambda$, $\delta\mu$ to be continuous and vanishing on the boundaries of the grains together with their derivatives which appear in our formulae.[2] In addition we assume $\frac{\delta\varrho}{\varrho_0} \ll 1$ etc. now too.

If the incident wave is again of the form (8) and $\frac{a}{r} \ll 1$, every formula related to the SS waves so far obtained in 1.3 up to (22) remains valid, only the correlation functions N_σ, $N_{\mu\sigma}$, N_μ of (22) must be replaced by the (ensemble) averages

$$\langle\delta\,\sigma^2\rangle, \quad \langle\delta\,\sigma\,\delta\,\mu\rangle, \quad \langle\delta\,\mu^2\rangle$$

which also satisfy (20). So we obtain instead of (22)

$$\int_{S_a} \langle f_1 f_2^*\rangle\, e^{-i\bar{K}\bar{R}}\, dv' = k^2 \int_{S_a} [\langle\delta\,\sigma^2\rangle + 2(1-L)\langle\delta\,\mu\,\delta\,\sigma\rangle + (1-L^2)\langle\delta\,\mu^2\rangle]\, e^{-i\bar{K}\bar{R}}\, dv'. \tag{41}$$

Now e.g. in the integral

$$\mathcal{J}_\sigma = \int_{S_a} \langle\delta\sigma^2\rangle\, e^{-i\bar{K}\bar{R}}\, dv'$$

let $\langle\delta\sigma^2\rangle$ be substituted by its value at a suitable point of S_a. Then[3]

$$\mathcal{J}_\sigma = \langle\delta\sigma^2\rangle\,\mathcal{J}, \quad \mathcal{J} = \int_{S_a} e^{-i\bar{K}\bar{R}}\, dv'.$$

This value $\langle\delta\sigma^2\rangle$ can then be regarded as the constant value of $\delta\sigma^2$ which is valid in the grain. The value of $\mathcal{J}$ is as follows:

$$\mathcal{J} = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} e^{-iKR\cos\theta}\, R^2 \sin\theta\, dR\, d\Theta\, d\varphi = \frac{4\pi}{K}\int_0^a R\sin KR\, dR = \frac{4\pi}{K^3}(\sin Ka - Ka\cos Ka).$$

But for $\alpha \ll 1$

$$\sin\alpha - \alpha\cos\alpha = \alpha - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} - + \ldots - \alpha\left(1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} - + \ldots\right) = \frac{1}{3}\alpha^3 - \frac{1}{30}\alpha^5 + \mathcal{O}(\alpha^7).$$

Thus

$$\mathcal{J} = V\left(1 - \frac{1}{10}(Ka)^2\right) + \mathcal{O}(K^4 a^7), \quad V = \frac{4}{3}a^3\pi.$$

[2] S. the analysis of this assumption in the Remark after 2.5.

[3] Being $|\bar{K}\bar{R}| \ll 1$ $\sin \bar{K}\bar{R}$, $\cos \bar{K}\bar{R}$, is of constant sign in S_a and the mean value theorem may be applied.

If $Ka < 1$, the relative error of $\mathcal{I}$ is less than

$$\frac{10}{9}\cdot\frac{4\pi}{K^3}\,\frac{6}{7!}(Ka)^7 : \frac{4}{3}a^3\pi = \frac{(Ka)^4}{250} < \frac{1}{250} = 0.004 = 0.4\ \%$$

and this error decreases with Ka. Being $K^2 = 2k^2(1 - L)$ [s. (17')] we have

$$\mathcal{I} = V\left[1 - \frac{(ka)^2}{5}(1 - L)\right].$$

This approximation also remains valid for not too small values of ka, while (23) does not. (There the significance of a is different). The more exact calculation[4] thus carried out, gives the scattering formula for a unique grain

$$\langle|u_{ss}|^2\rangle = \frac{\pi^2 A^2 V^2 f^4}{\varrho_0^2 \beta^8 r^2}\left[1 - \frac{1}{5}(k\,a)^2(1 - L)\right]S \tag{42}$$

$$S = L_\varrho\langle\delta\,\varrho^2\rangle + L_\mu\langle\delta\mu^2\rangle + L_{\varrho\mu}\langle\delta\,\varrho\,\delta\,\mu\rangle$$

where L_ϱ, L_μ, $L_{\varrho\mu}$ are the values of (26).

2.2. *The scattering coefficient α_{ss}*

The total SS energy scattered by a unique sphere S_a is proportional to the surface integral

$$E_s = \gamma\oint\langle|u_{ss}|^2\rangle\,d\,F,\quad \gamma = \text{const} \tag{43}$$

extended to a sphere of radius r around S_a with $\frac{1}{k^2r^2} \ll 1$.

If grains of number N are situated in a layer with cross-section F, thickness dx and volume $\mathcal{V} = Fdx$, the size of which is small compared to r, then the total energy scattered by them is

$$\mathcal{E}_s = \sum E_s. \tag{44}$$

The $\langle|u_{ss}|^2\rangle$ and with it E_s are quantities proportional to V^2. If the volumes V_i $(i = 1, 2, \ldots, N)$ of the grains are nearly equal, i.e. $V_i \sim V$, then

$$\sum V_i^2 = N V^2 = N V\cdot V = (q\mathcal{V})\cdot V \tag{45}$$

where $0 < q \leq 1$, viz. the whole volume of the grains contained in $\mathcal{V}$ is a

[4] This can be regarded as the branch point between Bhatia's and our calculations.

fraction q of $\mathscr{V}$ only. The total incident energy of the beam (the flux)

$$\mathscr{E}_i = \gamma F A^2$$

and the energy remaining in the layer

$$\mathscr{E}_i - \mathscr{E}_s = \mathscr{E}_i e^{-\alpha dx} \approx \mathscr{E}_i(1 - \alpha dx)$$

whence

$$\alpha = \frac{\mathscr{E}_s}{\mathscr{E}_i dx} = \frac{\mathscr{E}_s}{\gamma A^2 F dx} = \frac{\mathscr{E}_s}{\gamma A^2 \mathscr{V}}. \tag{46}$$

Therefore first E_s must be evaluated. By (43) and (42) we have

$$E_s = \gamma \frac{4 A^2 V^2 \pi^3 f^4}{105 \varrho_0^2 \beta^8} [5 \beta^4 (7 - u) \langle \delta\varrho^2 \rangle + 22 (1 - u) \langle \delta\mu^2 \rangle + 62 \beta^2 u \langle \delta\varrho\, \delta\mu \rangle]$$

$$u = u_\beta = \frac{1}{5} (k_\beta a)^2$$

and by (45) and (46)

$$\alpha_{ss} = \frac{4 \pi^3 q V f^4}{105 \varrho_0^2 \beta^8} [5 \beta^4 (7 - u) \langle\langle \delta\varrho^2 \rangle\rangle + 22 (1 - u) \langle\langle \delta\mu^2 \rangle\rangle + 62 \beta^2 u \langle\langle \delta\varrho \delta\mu \rangle\rangle] \tag{47}$$

where $\langle\langle \delta\varrho^2 \rangle\rangle$ etc. is the average of the ensemble average $\langle \delta\varrho^2 \rangle$ over the different grains.

2.3. *The PP wave and* α_{PP}

In the sequel the argumentation may be greatly abbreviated. We obtain

$$\langle |u_{PP}|^2 \rangle = \frac{\pi^2 A^2 V^2 f^4}{\varrho_0^2 \alpha^8 r^2} (1 - L)^2 \left[1 - \frac{1}{5} (k_\alpha a)^2 (1 - L)\right] P \tag{48}$$

$$P = \alpha^4 \langle \delta\varrho^2 \rangle - 2 \alpha^2 (2 - L) \langle \delta\varrho\, \delta\nu \rangle + L^2 \langle \delta\nu^2 \rangle + 2 \alpha^2 (1 - L^2) \langle \delta\varrho\, \delta\lambda \rangle + (1 - 2 L)(1 - L^2) \langle \delta\lambda\, \delta\nu \rangle + (1 - L^2) \langle \delta\lambda^2 \rangle$$

and

$$\alpha_{PP} = \frac{8 \pi^3}{15} \frac{V q f^4}{\varrho_0^2 \alpha^8} [5 \alpha^4 (2 - 3 u) \langle\langle \delta\varrho^2 \rangle\rangle - 16 u \alpha^2 \langle\langle \delta\varrho\, \delta\nu \rangle\rangle + (16 - 21 u) \langle\langle \delta\nu^2 \rangle\rangle - 8 \alpha^2 (3 - 4 u) \langle\langle \delta\varrho\, \delta\mu \rangle\rangle - 44 (1 - u) \langle\langle \delta\mu\, \delta\nu \rangle\rangle + 2 (3 - 4 u) \langle\langle \delta\mu^2 \rangle\rangle], \quad u = u_\alpha = \frac{(k_\alpha a)^2}{5}. \tag{49}$$

2.4. *The SP wave and* $\varkappa_{SP}$

$$\langle |u_{SP}|^2 \rangle = \frac{\pi^2 A^2 V^2 f^4}{\varrho_0^2 \varkappa^6 \beta^2 r^2} N^2 \left[1 - \frac{a^2 (k_\alpha^2 + k_\beta^2)}{10} + \frac{k_\alpha k_\beta a^2}{5} L \right] P, \tag{50}$$

$$P = \varkappa^2 \beta^2 \langle \delta\varrho^2 \rangle - 4 \varkappa\beta M \langle \delta\varrho\, \delta\mu \rangle + 4 M^2 \langle \delta\mu^2 \rangle ,$$

$$\varkappa_{SP} = \frac{4 \pi^3 V q f^4}{15 \varrho_0^2 \varkappa^6 \beta^2} \left(1 - \frac{u_\alpha + u_\beta}{2} \right) (5 \varkappa^2 \beta^2 \langle\langle \delta\varrho^2 \rangle\rangle + 4 \langle\langle \delta\mu^2 \rangle\rangle) \tag{51}$$

$$u_\alpha = \frac{a^2 k_\alpha^2}{5}, \quad u_\beta = \frac{a^2 k_\beta^2}{5} .$$

2.5. *The PS wave and* $\varkappa_{PS}$

$$\langle |u_{PS}|^2 \rangle = \frac{\pi^2 A^2 V^2 f^4}{\varrho_0^2 \varkappa^2 \beta^6 r^2} \left[1 - \frac{a^2 (k_\alpha^2 + k_\beta^2)}{10} - \frac{k_\alpha k_\beta a^2}{5} L \right] (1 - L^2) P \tag{52}$$

$$P = \varkappa^2 \beta^2 \langle \delta\varrho^2 \rangle - 4 \varkappa\beta L \langle \delta\varrho\, \delta\mu \rangle + 4 L^2 \langle \delta\mu^2 \rangle$$

$$\varkappa_{PS} = \frac{4 \pi^3 V q f^4}{105 \varrho_0^2 \varkappa^2 \beta^6} S \tag{53}$$

$$S = \left(1 - \frac{u_\alpha + u_\beta}{2} \right) (70 \varkappa^2 \beta^2 \langle\langle \delta\varrho^2 \rangle\rangle + 56 \langle\langle \delta\mu^2 \rangle\rangle) + $$
$$+ 28 \varkappa^2 k_\alpha^2 a^2 \langle\langle \delta\varrho\, \delta\mu \rangle\rangle .$$

Formulae (42), (48), (50), (52) hold for a unique grain. The scattering formulae valid for a medium of volume $\mathscr{V}$ containing many small grains of volume V will be obtained from these provided V^2 is replaced by $q\mathscr{V}V$ and $\langle \delta\varrho^2 \rangle$ etc. by $\langle\langle \delta\varrho^2 \rangle\rangle$ etc.

Remark. In 2.1 it was assumed that $\delta\varrho$, $\delta\lambda$, $\delta\mu$ with all their derivatives playing a role in the calculation are continuous and vanish on the boundaries of the grains. Thereafter (at the end of the evaluation) we have taken $\delta\varrho$, etc. constant, which would involve $\delta\varrho = \delta\lambda = \delta\mu = 0$ within the grains too, which is impossible. In fact, instead of this "working hypothesis" the following more exact hypothesis must be assumed. Let $\delta\varrho$ be zero on the surface of the grain and depending on the distance from the center only, on the other hand its derivative (or derivatives) large near the boundary and vanishing elsewhere but in such a way, that on a line parallel to the x axis let $\delta\varrho$ be of opposite sign leaving the grain and entering it. Then e.g. in

$$\mathcal{J} = \int_{S_a} (\delta\varrho)_{x'x'}\, e^{-iKR}\, dv'$$

the expression obtained by integration by parts vanishes

$$\left[(\delta\varrho)_{x'}\, e^{-i\overline{K}\overline{R}}\right]_{x'_1}^{x'_2} = 0$$

because $\overline{K}\overline{R} = KR \cos\theta$ and x'_1 and x'_2 belong to the same value of θ. The same statement holds for similar integrals.

It would also be a conceivable assumption that the first, second and third derivatives vanish on the boundary, but among the higher derivatives there are non-vanishing ones too.

3. Discussion of the results

3.1. *Case of metals*

Our results [(47), (49)] slightly differ from those of Bhatia as the following comparison shows:

$$\varkappa_{SS} = \frac{8\,\pi^3 V f^4}{15\,\varrho_0^2\,\beta^8}\,[5\,\beta^4\,\langle\langle\delta\varrho^2\rangle\rangle + 3\,\langle\langle\delta\mu^2\rangle\rangle]\ \text{(Bhatia)}$$

$$\varkappa_{PP} = \frac{4\,\pi^3 V f^4}{15\,\varrho_0^2\,\alpha^8}\,[5\,\alpha^4\,\langle\langle\delta\varrho^2\rangle\rangle + 15\,\langle\langle\delta\nu^2\rangle\rangle - 40\,\langle\langle\delta\mu\,\delta\nu\rangle\rangle + 32\,\langle\langle\delta\mu^2\rangle\rangle]\ \text{(Bhatia)}.$$

Disregarding the factor q and some numerical coefficients, these formulae are almost identical to (47) and (49) provided u is taken as zero. The more interesting is the circumstance that in formulae

$$\varkappa_{SP} = \frac{4\,\pi^3 V f^4}{15\,\varrho_0^2\,\alpha^5\,\beta^3}\,(5\,\alpha^2\,\beta^2\,\langle\langle\delta\varrho^2\rangle\rangle + 4\,\langle\langle\delta\mu^2\rangle\rangle)$$

$$\varkappa_{PS} = \frac{8\,\pi^3 V f^4}{15\,\varrho_0^2\,\alpha^3\,\beta^5}\,(5\,\alpha^2\,\beta^2\,\langle\langle\delta\varrho^2\rangle\rangle + 4\,\langle\langle\delta\mu^2\rangle\rangle)$$

of BHATIA even the constants are the same as in (51) and (53) but the denominators differ slightly (in the ratio $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$ respectively). Recalculating the data of [4] (pp. 417, 423, 424), we partly obtain a better approximation. Taking pure metals or some alloys $\langle\langle\delta\varrho^2\rangle\rangle$ can be assumed to be zero, the scatter depends only on the anisotropy and the grain size and our formulae may be simplified, because now (s. [5], p. 21)

$$\langle\langle\delta\nu^2\rangle\rangle = \frac{16}{9}\,\langle\langle\delta\mu^2\rangle\rangle = \frac{4}{3}\,\langle\langle\delta\nu\,\delta\mu\rangle\rangle \tag{54}$$

and so for an incident S wave without correction [taking $u_\alpha = u_\beta = 0$ in (47) and (51)]

$$\alpha_{SS} = \frac{4\,\pi^3\,Vqf^4}{15\,\varrho_0^2\,\beta^5}\,3\,\langle\langle\delta\mu^2\rangle\rangle\,, \quad \alpha_{SP} = \frac{4\,\pi^3\,Vqf^4}{15\,\varrho_0^2\,\alpha^6\,\beta^2}\,4\,\langle\langle\delta\mu^2\rangle\rangle$$

$$\alpha_S = \alpha_{SS} + \alpha_{SP} = \frac{4\,\pi^3\,Vqf^4}{15\,\beta^4}\,\frac{\langle\langle\delta\mu^2\rangle\rangle}{\mu_0^2}\left[1 + \frac{4}{3}\left(\frac{\beta}{\alpha}\right)^6\right] \tag{55}$$

and with correction corresponding to the actual values of u_α and u_β

$$\alpha_{S_c} = \frac{4\,\pi^3\,Vqf^4}{15\,\beta^4}\,\frac{\langle\langle\delta\mu^2\rangle\rangle}{\mu_0^2}\left[\frac{22}{7}(1 - u_\beta) + 4\left(1 - \frac{u_\alpha + u_\beta}{2}\right)\left(\frac{\beta}{\alpha}\right)^6\right] \tag{56}$$

1° For aluminium

$$\frac{\langle\langle\delta\mu^2\rangle\rangle}{\mu_0^2} = 3.3\cdot 10^{-3}\,, \quad \frac{\langle\langle\delta\nu^2\rangle\rangle}{\nu_0^2} = 3\cdot 10^{-4}$$

$$\nu_0 = 11.20\cdot 10^{11}\,, \; \mu_0 = 2.62\cdot 10^{11} \text{ dyne/cm}^2$$

$$\varrho_0 = 2.71 \text{ g/cm}^3, \; q \approx 1$$

and a test sample with $2a = 0.130$ mm (grain's diameter) (55) gives

$$\frac{\alpha_s}{2f^4} = 6.17\cdot 10^{-30} \text{ neper/cm/cycle}^4,$$

$$\frac{\alpha_{S_c}}{2f^4} = 6\cdot 10^{-30}\,, \; (f = 3\cdot 10^6)$$

while the measured value is $9.4\cdot 10^{-30}$, and that of [5] is $9.9\cdot 10^{-30}$ which is a better one.

For an incident P wave

$$\alpha_{PP} = \frac{16\,\pi^3\,qVf^4}{15\,\varrho_0^2\,\alpha^8}\,[8\,\langle\langle\delta\nu^2\rangle\rangle - 22\,\langle\langle\delta\mu\,\delta\nu\rangle\rangle + 3\,\langle\langle\delta\mu^2\rangle\rangle]$$

$$\alpha_{PS} = \frac{16\,\pi^3\,Vqf^4}{15\,\varrho_0^2\,\alpha^2\,\beta^6}\,2\,\langle\langle\delta\mu^2\rangle\rangle$$

and by (54)

$$\alpha_P = \alpha_{PP} + \alpha_{PS} = \frac{16\,\pi^3\,Vqf^4}{15\,\alpha^4}\,\frac{\langle\langle\delta\nu^2\rangle\rangle}{\nu_0^2}\left[-\frac{109}{16} + \frac{9}{8}\left(\frac{\alpha}{\beta}\right)^6\right]$$

$$\alpha_{P_c} = \frac{16\,\pi^3\,Vqf^4}{15\,\alpha^4}\,\frac{\langle\langle\delta\nu^2\rangle\rangle}{\nu_0^2}\left[-\frac{109}{16} + \frac{15\,u_\alpha}{4} + \right.$$

$$\left. + \frac{9}{8}\left(1 - \frac{u_\alpha + u_\beta}{2}\right)\left(\frac{\alpha}{\beta}\right)^6\right]$$

which gives for the same sample as above

$$\frac{\alpha_P}{2f^4} = 0.905 \cdot 10^{-30}$$

(in [5] $1.46 \cdot 10^{-30}$)

and

$$\frac{\alpha_{P_c}}{2f^4} = 0.89 \cdot 10^{-30} \text{ and } 0.714 \cdot 10^{-30}$$

(for $f = 3 \cdot 10^6$ and $9 \cdot 10^6$ respectively) which agrees better with the measured value $0.695 \cdot 10^{-30}$ than that of [5]. Herewith the discrepancy mentioned in [5] (p. 21) may be regarded as explained.

At the same time the question arises whether the measured value $\frac{\alpha_s}{2f^4} = 9.4 \cdot 10^{-30}$ is not too high. It seems better (according to the above results) to have a value under $6 \cdot 10^{-30}$.

Table 1

	$10^{30} \cdot \alpha_p/2f^4$			$10^{30} \cdot \alpha_s/2f^4$		
	measured	Bhatia	Present work	measured	Bhatia	Present work
Al	0.695	1.46	0.905 0.89 ($f = 3 \cdot 10^6$) 0.714 ($f = 9 \cdot 10^6$)	9.4	9.9	6.17 6.10 ($f = 3 \cdot 10^6$)
Mg	0.46	0.362	0.582 0.525 ($f = 10^7$)	—	—	—

2° For magnesium (s. [4], pp. 417 and 524)

$$\nu_0 = 5.88 \cdot 10^{11}, \ \mu_0 = 1.77 \cdot 10^{11}, \ \frac{\langle\langle \delta\nu^2 \rangle\rangle}{\nu_0^2} = 2.22 \cdot 10^{-4}, \ \varrho_0 = 1.75, \ q \approx 1$$

and for the given sample $2a = 0.1$ mm. Correspondingly

$$\frac{\alpha_P}{2f^4} = 5.82 \cdot 10^{-31}$$

$$\frac{\alpha_{P_c}}{2f^4} = 5.25 \cdot 10^{-31} \quad (\text{for } f = 10^7)$$

while the measured value (s. [4], p. 425) is $4.6 \cdot 10^{-31}$ (Bhatia's value is $3.62 \cdot 10^{-31}$). For lack of a measured value α_s the corresponding theoretical α_s has not been evaluated.

3.2. *Remarks concerning liquids and "mixed" media*

The deductions and formulae, which concern only P waves, are also valid for fluids and gases, only the material constants must have appropriate values: $\mu = \delta\mu = 0$, $\nu = \lambda$, $\delta\nu = \delta\lambda$, where $\lambda = \lambda_0 + \delta\lambda$ is reciprocal to the adiabatic compressibility. Here mode conversion cannot take place. Scatter is determined by $\delta\varrho$, $\delta\lambda$ and the droplet size.

However, none of the formulae so far obtained are valid quantitatively for "mixed" media as solid particles suspended in fluids or gases, fluid or gas filled pores in solids, gas bubbles in liquids, or fogs and a few special cases, like emulsions of mercury, etc., because our main assumptions $\left(\frac{\delta\varrho}{\varrho_0} \ll 1, \text{etc.}\right)$ are not satisfied in these cases.

Even in these cases the results may be of some use, but only in a qualitative sense, showing the character of the scatter functions.

Returning to liquids we have now for the velocity of the propagation

$$c^2 = \frac{\lambda}{\varrho}, \quad \lambda = c^2 \varrho, \quad \delta\lambda = 2\, c_0\, \varrho_0\, \delta c + c_0^2\, \delta\varrho$$

whence by (49) the (unique) scattering coefficient will have the form

$$\begin{aligned} \varkappa = \frac{16\,\pi^3 V q f^4}{15\, c^4} \Bigg[13\,(1 - 2\,u) \frac{\langle\langle \delta\varrho^2 \rangle\rangle}{\varrho_0^2} + \\ + 2\,(16 - 21\,u) \frac{\langle\langle \delta c^2 \rangle\rangle}{c_0^2} + 2\,(16 - 29\,u) \frac{\langle\langle \delta\varrho\, \delta c \rangle\rangle}{\varrho_0\, c_0} \Bigg] \\ \left(u = \frac{(ka)^2}{5}, \quad k = \frac{2\,\pi f}{c} \right) \end{aligned} \tag{57}$$

while the scattering formula (49) is as follows

$$\langle |\bar{u}|^2 \rangle = \frac{\pi^2 A^2 V^2 f^4}{c^4 r^2} (1 - L)^2 \left[1 - \frac{(ka)^2}{5} (1 - L) \right] P \tag{58}$$

where

$$\begin{aligned} P = (2\,L^3 - 3\,L^2 + 3) \frac{\langle \delta\varrho^2 \rangle}{\varrho_0^2} + 4\,[1 + (1 - L^2)(1 - 2\,L)] \frac{\langle \delta c^2 \rangle}{c_0^2} + \\ + 4\,(2\,L^3 - 2\,L^2 - L + 2) \frac{\langle \delta\varrho\, \delta c \rangle}{c_0\, \varrho_0}. \end{aligned}$$

Here $L = \cos\,\theta$, θ is the angle of the scatter.

According to the measurements the term containing $\frac{\langle\langle \delta c^2 \rangle\rangle}{c_0^2}$ or $\frac{\langle \delta c^2 \rangle}{c_0^2}$ dominates. The influence of the variation of c is larger than that of ϱ.

If in formula (36), which relates to continuously distributed inhomogeneities, a similar transformation is carried out, we get for fluids and gases

$$\varkappa = \frac{k^4 \sqrt{\pi}}{15} \left[13 \left(\frac{n_\varrho}{\varrho} \right)^2 a_\varrho^3 + 32 \left(\frac{n_c}{c} \right)^2 a_c^3 + 32 \left(\frac{n_{c\varrho}}{c\varrho} \right)^2 a_{c\varrho}^3 \right], \; k = \frac{2\pi f}{c} \tag{59}$$

and by the above remark

$$\alpha \approx \frac{32}{15} \sqrt{\pi}\, k^4 \left(\frac{n_c}{c} \right)^2 a_c^3 \tag{60}$$

which is double the value on p. 55 of [3] provided that $\left(\frac{n_c}{c} \right)^2$ is identified with the mean square value of the fluctuation of the refraction coefficient. In fact

$$m = \frac{c_0}{c}, \quad \frac{\delta m}{m} = - \frac{\delta c}{c}$$

$$\frac{\langle \delta m^2 \rangle}{m^2} = \frac{\langle \delta c^2 \rangle}{c^2} = \frac{n_c^2}{c^2}.$$

Summary

The authors attempt to treat the problem of scatter of elastic waves in heterogeneous media from a fairly general standpoint. The treatment is separated for media by continuously and by discretely distributed inhomogeneities (I. and II. part respectively). The novelty of the I. part consists of obtaining a general scattering formula (not limited to a given shape as in [1]) giving the dependence of the scattered energy (beside other quantities) on the direction and the scattering coefficients too, which are more suitable for the measurement. As to the II. part the wave equation was solved in a higher approximation giving more precise results than the former works (s. [5]).

At first, continuous wave operation (monotone or "monochromatic" waves) will be discussed, but later it is shown that pulse modulation does not affect the results.

Only the following assumptions are made: *a)* the inhomogeneities are approximately spherical and of the same size and kind: *b)* ka or $\frac{2a}{\lambda}$ is small compared to 1. In the second part for 0.5% accuracy it is sufficient (theoretically) to have $ka \leq 1$, i. e. $\frac{2a}{\lambda} \leq 0.32$ as a more exact analysis shows[5]; *c)* the parent material and the inhomogeneities do not greatly differ either in density or in elastic moduli. The results are valid for solids, liquids and gases, only for the latter two media the rigidity modulus is zero.

The formulae obtained are compared with those of CHERNOV, KNOPOFF and HUDSON, and BHATIA.

In the literature measurements with all the necessary details are very rare, so up till now only experiments of MASON and MCSKIMIN were used to check the theory, and a good agreement was found.

[5] S. the value $\mathcal{T}$ in 2.1.

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Imre BIHARI
János SZILÁRD
Budapest XI., Sztoczek u. 2—4. Hungary