# THE STATISTICAL ANALYSIS OF MULTI-RATE SAMPLED-DATA CONTROL SYSTEMS

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## Introduction

Those sampled-data control systems are denominated as multi-rate systems which contain several sampling organs of different sampling times. KRANC and FAN TSHUN-VUJ have analysed these systems, their results can be found also in the monographs dealing with sampled-data control systems (see References). Sampled-data control systems are special systems with variable parameter, in which changes in time are caused by the introduction of the sampling organs. Difficulties originating herefrom are evaded with the aid of the $z$ transformation analysis in such a way, that the intervals between the sampling instants are omitted and the system is examined only at the sampling instants when the switches are always "closed", consequently the system is invariant. In the case of multi-rate systems the situation is more complicated. If there are two sampling frequencies and one of them is an integral multiple of the other, the systems can be examined relatively simply by introducing the $z_n$ transformation. For this case KRANC has given a method for the synthesis, too [3]. In other cases the situation is much more complicated, the sampling organs are then decomposed with the aid of sampling organs having a sampling period corresponding to the lowest common multiple of the sampling periods and of ideal leading and delay blocks, and the systems are examined according to the principle of superposition, in each case separately [6], though closed formulae can also be deduced. If we want, however, to perform the statistical analysis of these systems, it is advisable to go back to the conception of the linear system with variable parameter.

Let the system contain $k$ pieces of sampling organs, having the sampling periods $T_1, T_2, \ldots T_k$, further let $T$ denote the lowest common multiple of these sampling periods. Each sampling organ is substituted by sampling organs of period $T$, connected in parallel. For the substitution of the $j$-th sampling organ $n_j = T/T_j$ pieces of sampling organs should be used, together with $e^{\pm slT_j}$ $l = 0, 1, \ldots n_j - 1$ ideal leading and delay blocks *(Fig. 1)*. Upon having transformed the system in this way, we have obtained a periodically varying parameter system with period $T$. In the present paper, on the basis of the above perception, the variance of the error arising at the output of the multi-frequency sampled-data control system is determined for the case that a stationary ergodic signal is acting at the input side.

For the sake of simplicity, in the following we shall examine those systems where the input can be sampled redundantly. Let us assume that the period of sampling the input is $T_1 = T/n_1$ (*Fig. 2*). This implies that during the structural variance period $T$ of the system, signals reach the system $n_1$ times. If the system is examined at the instants $p \cdot T/n_1$, $p \cdot T/n_1 + T$, $p \cdot T/n_1 + 2T$, $p \cdot T/n_1 + 3T$, etc, where $p$ is fixed, $p = 0, 1, 2, \ldots n_1 - 1$,

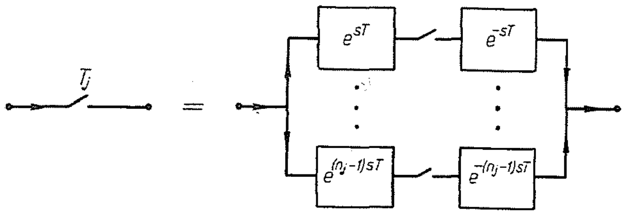


*Fig. 1*

then the system will be invariant, as a final result. Let $w_p(t)$ denote the output produced by the signal $r^*(t) = \delta(t - p \cdot T/n_1)$. Then, in the case of an input $r(t)$, the output signal is

$$c(t) = \sum_{r=0}^{n_1-1} \sum_{k=0}^{\infty} r\left(t - kT - p\frac{T}{n_1}\right) w_p(kT)\,. \tag{1}$$

In the following the output will be examined at the instants $t = kT$ $(k = 0, 1, 2, \ldots)$, for the sake of applying the $z$ transformation. Formula (1) is, by the way, the simplest generalization of the weighting function theorem for systems with periodically changing parameters [1].

## The solution of the problem

Let us now regard the arrangement shown on *Fig. 3*. Here, the sum of the useful signal $s(t)$ and that of the noise $n(t)$ acts at the input side of the system characterized by the values $T$, $n_1$ $W_p(z)$ $(p = 0, 1, \ldots n_1 - 1)$. The useful signal is also acting at the input of the system $W_d(s)$ which is to be approximated ideally. The difference of the output signals of the two systems is formed in intervals $T$, these are squared and the average value formed. One of the criteria of the merit of our system may be this average value $\overline{e^2(nT)}$.

Further we suppose that $s(t)$ and $n(t)$ form an ergodic stationary process and we determine, on the basis of this assumption, the value $e^2(nT)$.

First we should introduce some symbols and definitions. Let $g(t)$ and $f(t)$ denote ergodic stationary stochastic processes. Let us define the following correlation series:

$$\Phi_{ff}(mT) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} f(nT) f(nT+mT) \tag{2}$$

and

$$\Phi_{gf}(mT) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} g(nT) f(nT+mT). \tag{3}$$

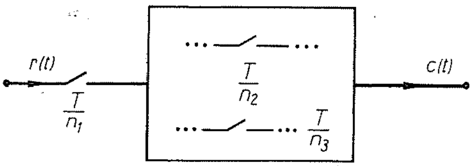


*Fig. 2*

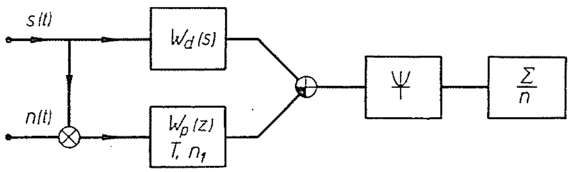


*Fig. 3*

It can be proved that

$$\Phi_{ff}(mT) = \varphi_{ff}(\tau)\big|_{\tau=mT} \tag{4}$$

and

$$\Phi_{gf}(mT) = \varphi_{gf}(\tau)\big|_{\tau=mT} \tag{5}$$

where $\varphi_{ff}(\tau)$ and $\varphi_{gf}(\tau)$ are the correlation functions known from the analysis of continuous systems.

Be

$$S_{ff}(z) = \sum_{m=-\infty}^{\infty} \Phi_{ff}(mT)\, z^{-m}$$

and

$$S_{gf}(z) = \sum_{m=-\infty}^{\infty} \Phi_{gf}(mT)\, z^{-m}$$

It follows from formulae (4) and (5) that

$$S_{ff}(z) = \mathfrak{Z}\{S_{ff}(s)\}$$

and

$$S_{gf}(z) = \mathfrak{Z}\{S_{gf}(s)\}$$

where $S_{ff}(s)$ and $S_{gf}(s)$ are the corresponding power density spectra.

Turning hereafter to our problem:

$$\overline{e^2(nT)} = \overline{(c(nT) - d(nT))^2} = \overline{c^2(nT)} - \overline{c(nT)\,d(nT)} - \\ - \overline{d(nT)\,c(nT)} + \overline{d^2(nT)}.$$

Upon considering (2) and (3),

$$\overline{e^2(nT)} = \Phi_{cc}(0) - \Phi_{cd}(0) - \Phi_{dc}(0) + \Phi_{dd}(0),$$

or if the $z$ transforms are given, by using the inversion formula,

$$\overline{e^2(nT)} = \frac{1}{2\pi j}\int_{\Gamma}\left[S_{cc}(z) - S_{cd}(z) - S_{dc}(z) + S_{dd}(z)\right]\frac{dz}{z} \qquad (6)$$

where $\Gamma$ denotes the unit radius circle. Let us determine the individual terms of this expression.

At the beginning we should take the summing according to equation (1) and $k$ formally, let us extend it to $-\infty$, considering that in the case of $w_p(t) = 0 \;\; t < 0$:

$$c(nT) = \sum_{p=0}^{n_1-1}\sum_{k=-\infty}^{\infty} r\left(nT - kT - p\frac{T}{n_1}\right) w_p(kT).$$

The expression for $c(nT + mT)$ is written similarly, afterwards we form the product $c(nT)\,c(nT + mT)$ and take the average of these products according to $n$. On arranging the sums in the usual way, we obtain

$$\Phi_{cc}(mT) = \sum_{p=0}^{n_1-1}\sum_{i=0}^{n_1-1}\sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} w_p(kT)\,w_i(jT)\lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N} r\left(nT - \right. \\ \left. - kT - p\frac{T}{n_1}\right) r\left(nT + mT - jT - i\frac{T}{n_1}\right).$$

Let us introduce the notation

$$\Phi r_p r_i(mT) = \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N} r\left(nT - p\frac{T}{n_1}\right) r\left(nT + mT - i\frac{T}{n_1}\right) \qquad (7)$$

thus

$$\Phi_{cc}(mT) =$$
$$= \sum_{p=0}^{n_1-1} \sum_{i=0}^{n_1-1} \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} w_p(kT)\, w_i(jT)\, \Phi r_i r_i (mT + kT - jT). \qquad (8)$$

It is evident from formula (7), that

$$\Phi r_p r_i (mT) = \varphi_{rr}\left(\tau - (i-p)\frac{T}{n_1}\right)\Bigg|_{\tau=mT}. \qquad (9)$$

It follows, on the other hand, from (9) that

$$S_{r_p r_i}(z) = \mathfrak{Z}\left\{S_{rr}(s)\, e^{-S(i-p)\frac{T}{n_1}}\right\}.$$

Let us determine the transform of (8)

$$S_{cc}(z) = \sum_{m=-\infty}^{\infty} \Phi_{cc}(mT)\, z^{-m} \qquad (10)$$

On substituting the expression for $\Phi_{cc}(mT)$ from formula (8) to (10), introducing the new summation index $q$ by the definition $q = m + k - j$ and rearranging the equation, we obtain

$$S_{cc}(z) = \sum_{p=0}^{n_1-1} \sum_{i=0}^{n_1-1} \sum_{k=-\infty}^{\infty} w_p(kT) z^{k} \sum_{j=-\infty}^{\infty} w_i(jT) z^{-j} \sum_{q=-\infty}^{\infty} \Phi r_p r_i (qT)\, z^{-q}$$

that is

$$S_{cc}(z) = \sum_{p=0}^{n_1-1} \sum_{i=0}^{n_1-1} W_p(z^{-1})\, W_i(z)\, S_{r_p r_i}(z). \qquad (11)$$

As in this case $r(t) = s(t) + n(t)$, thus

$$\varphi_{rr}(\tau) = \varphi_{ss}(\tau) + \varphi_{sn}(\tau) + \varphi_{ns}(\tau) + \varphi_{nn}(\tau).$$

We should depart from this equation on determining both $\Phi r_p r_i(mT)$ and $S_{r_p r_i}(z)$. If signal and noise are not correlated, or there is no noise, the equations become more simple.

On determining $S_{cd}(z)$ we similarly depart from the expression for the time domain.

$$\Phi_{cd}(mT) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} c(nT)\, d(nT+mT) =$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \sum_{p=0}^{n_1-1} \sum_{k=-\infty}^{\infty} r\left(nT - kT - p\frac{T}{n_1}\right) \cdot$$

$$\cdot\, w_p(kT) \int_{-\infty}^{\infty} w_p(t)\, s(nT+mT-t)\, dt.$$

On rearranging in the usual manner and performing the limit transition, we find that

$$\Phi_{cd}(mT) = \sum_{p=0}^{n_1-1} \sum_{k=-\infty}^{\infty} w_p(kT) \int_{-\infty}^{\infty} w_d(t)\, \varphi_{rs}(mT+kT-t)\, dt\,, \tag{12}$$

where

$$\varphi_{rs}(\tau) = \varphi_{ss}(\tau) + \varphi_{ns}(\tau)$$

and

$$\varphi_{r_p s}(\tau) = \varphi_{ss}\left(\tau + p\frac{T}{n_1}\right) + \varphi_{ns}\left(\tau + p\frac{T}{n_1}\right).$$

On determining the correlation between the transforms in the usual way from equation (12),

$$S_{cd}(z) = \sum_{p=0}^{n_1-1} w_p(z^{-1})\, \mathfrak{Z}\left\{w_d(s)\, S_{r_p s}(s)\right. . \tag{13}$$

By a similar train of thought,

$$S_{dc}(z) = \sum_{p=0}^{n_1-1} W_p(z)\, \mathfrak{Z}\left\{W_d(-s)\, S_{sr_p}(s)\right\} \tag{14}$$

Finally let us determine $S_{dd}(z)$. Since

$$d_n(nT) = \int_{-\infty}^{\infty} w_d(t)\, s(nT-t)\, dt\,,$$

thus

$$\Phi_{dd}(mT) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \int_{-\infty}^{\infty} w_d(t)\, s(nT-t)\, dt \int_{-\infty}^{\infty} w_d(\tau)\, s(nT+mT-\tau) d\tau.$$

On rearranging and performing the limit transition,

$$\Phi_{dd}(mT) = \int_{-\infty}^{\infty} w_d(t)\,dt \int_{-\infty}^{\infty} w_d(\tau)\,d\tau\,\varphi_{ss}(mT + t - \tau)\,.$$

Finally the correlation between the transforms is

$$S_{dd}(z) = \mathfrak{Z}\left\{W_d(s)\,W_d(-s)\,S_{ss}(s)\right\}. \tag{15}$$

Thus, our problem has been solved. Let us now briefly summarize the method of calculating the variance $\overline{e^2(nT)}$. A system characterized by the values $T$, $n_1$, $w_p(nT)$ is given. We specify the ideal system to be approximated, on which no noise is acting, by the transfer function $W_d(s)$. By using the statistical characteristics of signal and noise, we determine the cross and autocorrelation functions. The value of the variance is given by formula (6). The expressions for its individual members are to be found in equations (11) (13), (14) and (15). We would obtain a more complete image of the performance of the system, if we had not examined the average square sum of the sampled value of the error, but had integrated the square of the error continuously and on dividing this by the length of the integration interval, we had formed the average. It is, however, evident that this process would imply a longer calculation work.

The above formulae can be used directly for the numerical calculation of the variance. The various cross and autocorrelation functions are frequently given empirically in diagrams and not in the form of analytical expressions. Since their value is practically zero above a certain value, therefore the $z$ transforms calculated after eventual displacement will be finite combinations of positive and negative exponent powers of $z$. These polynomials are substituted in the suitable formulae and afterwards the correlation (6) is evaluated.

A special case deserves separate discussion, namely, the determination of the variance for such a system, in which a multi-frequency compensator is employed for compensation and this is the reason for the system being a multi-rate one (*Fig. 4*).

Since on such an occasion we are able to choose the higher sampling frequency as an integral multiple of the lower sampling frequency ($T_1/T_2 = n$), the $z_n$ transformation calculus may be employed. In this case the output (see e.g. [6]) is

$$C(z_n)_n = \frac{D(z_n)_n\,G(z_n)_n\,R(z_n^n)}{1 + \mathfrak{Z}\{D(z_n)_n\,G(z_n)_n\}}\,,$$

where e.g.

$$G(z_n)_n = \sum_{k=0}^{\infty} g\left(k\,\frac{T}{n}\right) z_n^{-k}\,,$$

and $g(t)$ is the weighting function of $G$

$$R(z_n^n) = \sum_{k=0}^{\infty} r(kT) z^{-k} \Big|_{z=z_n^n}$$

further $\mathfrak{Z}\{D(z_n)_n G(z_n)_n\}$ denotes the $z$ transform formed from the $z_n$ transform. It is easily conceivable (see [6]) that

$$\mathfrak{Z}\{D(z_n)_n\, G(z_n)_n\} = \frac{1}{2\pi j} \int_{\Gamma} \frac{D(z_n)_n\, G(z_n)_n}{1 - z_n^n\, z^{-1}} \cdot \frac{dz_n}{z_n} .$$

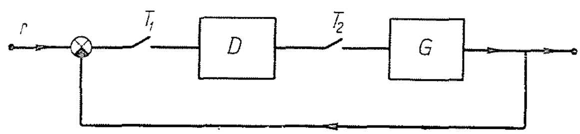


*Fig. 4*

In this case, in turn, we may calculate the average of the equal errors taken not at the intervals $T_1 = T$, but at $T_2 = T/n$, which is regarded as being more reliable from the aspect of the error.

By repeating the considerations at the beginning of the present section

$$\overline{e^2\left(m\,\frac{T}{n}\right)} = \frac{1}{2\pi j} \int_{\Gamma} \left[S_{cc}(z_n)_n - S_{cd}(z_n)_n - S_{dc}(z_n)_n + S_{dd}(z_n)_n\right] \frac{dz_n}{z_n} .$$

Considering that the final result of the analysis is obtained as the transfer function of a one-frequency system ($f = n/T$), the standard correlations elaborated for the calculation of the variance in one-frequency systems can be used.

Accordingly

$$S_{cc}(z_n)_n = W(z_n)_n\, W(z_n^{-1})_n\, S_{rr}(z_n^n)$$

where

$$W(z_n)_n = \frac{D(z_n)_n\, G(z_n)_n}{1 + \mathfrak{Z}\{D(z_n)_n\, G(z_n)_n\}} .$$

In the denominator $z_n^n$ should be similarly substituted in place of $z$,

$$S_{rr}(z_n^n) = S_{rr}(z)_{z=z_n^n}$$

$S_{rr}(z)$ can be calculated on the basis of the sampling time $T$ just as previously.

Similarly

$$S_{cd}(z_n)_n = W(z_n)_n \mathfrak{Z} \{W_d(z) S_{rs}(s)\},$$

and

$$S_{dc}(z_n)_n = W(z_n)_n \mathfrak{Z}_n \{W_d(-s) S_{rs}(s)\},$$

finally

$$S_{dd}(z_n)_n = \mathfrak{Z}_n \{W_d(s) W_d(-s) S_{ss}(s)\}.$$

## Illustrative example

Let us examine the arrangement shown on *Fig. 5*. Impulses obtained by sampling a stationary stochastic ergodic signal are fed with a sampling period $T/2 = 1/2$ sec to a one-capacity proportional element. The output is

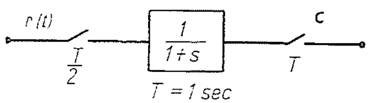


*Fig. 5*

sampled by a sampling period of $T = 1$ sec. We have to determine the average $\overline{c^2(nT)}$, if the $\varphi_{rr}(\tau)$ autocorrelation function, or the two-sided Laplace transform $S_{rr}(s)$ of this is known. According to the preceding,

$$\overline{c^2(nT)} = \Phi_{cc}(0) = \frac{1}{2\pi j} \int_\Gamma S_{cc}(z) \frac{dz}{z} \tag{16}$$

where the definitions of the functions $\Phi_{cc}(mT)$ and $S_{cc}(z)$ are:

$$\Phi_{cc}(mT) = \varphi_{cd}(\tau)|_{\tau=mT}$$

and

$$S_{cc}(z) = \sum_{m=-\infty}^{\infty} \Phi_{cc}(mT) z^{-m}.$$

In the present case the "structural variance period" of the system is $T = 1$ sec. During this time two signals enter the system from the input side, accordingly we have two "weighting series", $w_0(kT)$ and $w_1(kT)$, respectively. Let us determine these and their $z$ transforms, respectively. On the basis of *Fig. 6*, $w_0(0) = 1$, $w_0(1) = e^{-1}$, $w_0(2) = e^{-2}$, etc. and $w_1(0) = 0$, $w_1(1) = e^{-1/2} = e^{1/2} e^{1}$, $w_1(2) = e^{-3/2} = e^{1/2} e^{-2}$, etc.

The $z$ transforms of these weighting series are

$$W_0(z) = \sum_{k=0}^{\infty} w_0(kT)z^{-k} = \sum_{k=0}^{\infty} e^{-k} z^{-k} = \frac{1}{1 - e^{-1} z^{-1}} \tag{17}$$

Similarly

$$W_1(z) = \sum_{k=0}^{\infty} w_1(kT)z^{-kT} = \sum_{k=1}^{\infty} e^{1/2} e^{-k} z^{-k} = \frac{e^{-1/2} z^{-1}}{1 - e^{-1} z^{-1}}. \tag{18}$$

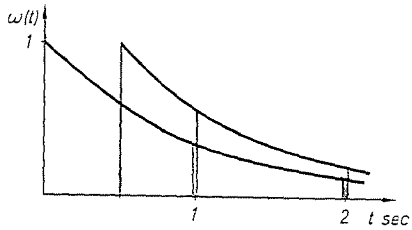


*Fig. 6*

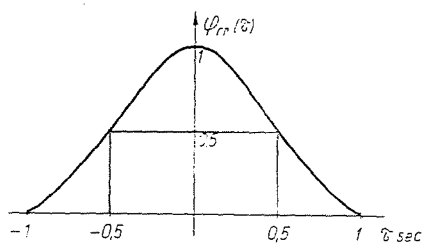


*Fig. 7*

To facilitate further calculation work, we assume, that the values of the (empirically obtained) auto-correlation function of the process $r(t)$ at the places which are of interest for us are 0, 0.5, 1, 0.5, 0 (see *Fig. 7*). The value of $S_{cc}(z)$ will be determined with the aid of equation (11). Accordingly

$$S_{cc}(z) = \sum_{p=0}^{1} \sum_{i=0}^{1} W_p(z^{-1}) W_i(z) S_{r_p r_i}(z). \tag{19}$$

In the above expression, on the basis of the preceding

$$S_{r_1 r_1}(z) = 1 \tag{20}$$

$$S_{r_0 r_0}(z) = 1 \tag{21}$$

$$S_{r_1 r_0}(z) = 0.5z + 0.5 \tag{22}$$

and

$$S_{r_1 r_0}(z) = 0.5 + 0.5z^{-1}. \tag{23}$$

We have taken into consideration here that $T = 1$ sec. Hereafter, by simply substituting the calculated values of (17), (18), (20), (21), (22) and (23) into (19),

$$S_{cc}(z) = \frac{1}{1-e^{-1}z} \cdot \frac{1}{1-e^{-1}z^{-1}} + \frac{e^{1/2}z}{1-e^{-1}z} \cdot \frac{e^{-1/2}z^{-1}}{1-e^{-1}z^{-1}} +$$

$$+ \frac{1}{1-e^{-1}z} \cdot \frac{e^{-1/2}z^{-1}}{1-e^{-1}z^{-1}} (0{,}5 + 0{,}5z^{-1}) +$$

$$+ \frac{e^{-1/2}z}{1-e^{-1}z} \cdot \frac{1}{1-e^{-1}z^{-1}} (0{,}5 + 0{,}5z) . \tag{24}$$

Since the above expression has a common denominator the addition could be performed. As, however, the aim is the calculation of residues within the unit circle, we should remain by this form for the sake of good arrangement. Substitute (24) into (16) and determine the sum of the residues. According to calculations not detailed here, $\overline{c^2(nT)} = 6.14$.

The dimension and unit of $\overline{c^2(nT)}$ depends on the autocorrelation function of the input signal. Calculations have been performed here in absolute values. The concrete calculation of residues, though not burdened with theoretical difficulties, is nevertheless lengthy and contains many error potentialities. It is evident from this that the calculation of the variance is a tedious work even in the case of relatively simple systems and signals of simple statistical characteristics.

Finally I wish to thank for the help given me by my professor, Dr. György Fodor, during the composition of the present paper.

## Summary

In the article the variance of the error at the output of multi-frequency linear contro systems has been determined on the condition, that the signal and noise acting at the input are stationary and ergodic. A closed formula was obtained for the error which can be most advantageously evaluated with the aid of the residue theorem. The method can be employed in generalized cases, too. For that significant case, however, when two sampling frequencies occur in the system and one is an integral multiple of the other, a separate formula has been deduced.

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