THE SINGULAR PERTURBATION THEORY OF DIFFERENTIAL EQUATIONS IN CONTROL THEORY

by

GY. BÜTI

Department for Theoretical Electricity, Polytechnical University, Budapest

(Received September 30, 1965)

Presented by Prof. Dr. K. SIMONYI

Introduction

The engineer examining a control system is always compelled to disregard some circumstances and to make several simplifications, as the dynamical systems occurring in practice are too complicated to be examined by considering each of its characteristics. Negligences and idealizations are naturally permissible only if real conditions are depicted by the faithfully idealized model. This sphere of problems is closely connected with the perturbation theory of differential equations. The engineer making negligences and idealizations empirically, is frequently justified by this theory. On many occasions, however, this is not the case, since our differential equation sometimes becomes singular in a certain sense, in consequence of negligences and idealizations, which may have serious consequences. In this case it is already difficult to decide under which conditions the problem becomes exact, not to speak of stability, etc. In the present paper an attempt is made to answer questions raised by this problem, with the aid of the mathematical apparatus described herein.

The aim of the paper is to apply the results from the field of the singular perturbation theory of differential equations in control engineering, and of the asymptotic expansion of the solution of singularly perturbated differential equations. This theme has been the subject of intensive mathematical research work in the last 15 to 20 years. A good summary of work performed so far can be found in CESARY's book [3]. The physical applications of this mathematical apparatus can be found sporadically in the theory of non-linear oscillations [1], [2], [7].

In their paper [12], MILLER and MURRAY refer to certain mathematical articles, in which some results of this theory (which was in the initial stage at that time) can be found. Lately KOKOTOVITCH and RUTMAN [4] made suggestions for the application of this apparatus in control theory, for some problems of sensitivity analysis. Apart from these two articles (in which only references can be found), the present paper is (to our knowledge) the first summary of the methods and results of this mathematical apparatus which seem to be of interest from the aspect of control theory.

The theory of asymptotic expansions has a much greater tradition in physical applications. Especially the asymptotic expansion of the solution of linear differential equations of the second order has been examined, and the results obtained were successfully employed, first of all, in quantum mechanics, but in other sections of physics as well [3], [13], [6], [5], etc. This apparatus may have a very significant role in solving technical calculation problems. The other aim of the paper is to show this possibility.

Qualitative considerations

Let us consider the following system of differential equations:

$$\Omega(\mu)\dot{z} = H(z, t, \mu), \tag{1}$$

where

$$\Omega(\mu) = \begin{bmatrix} \mu^{h_1} & 0 \dots & & \\ 0 & \mu^{h_2} & & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \mu^{h_n} \end{bmatrix}, \tag{2}$$

$z = \{z_1 \dots z_n\}$, $H = \{H_1 \dots H_n\}$, the H_i values are bounded, differentiable, and μ is a small positive parameter. h_i is equal to zero or one and let us assume that $h_i = 1$, $i = 1, \dots s$ and $h_i = 0$, $i = s + 1, \dots n$. In this case formula (1) can be rewritten in the form

$$\mu\dot{x} = F(x, y, t, \mu), \quad \dot{y} = G(x, y, t, \mu), \tag{3}$$

where $x_i = z_i$, $F_i = H_i$; $1 \leqslant i \leqslant s$, $y_i = z_i$, $G_i = H_i$; $s + 1 \leqslant i \leqslant n$.

Examine the solution of the system of differential equations (3) in the case of $\mu \to 0$, with the initial conditions $x|_{t=t_0} = x^\circ$, $y|_{t=t_0} = y^\circ$. If $\mu = 0$, we obtain from (1) the system of differential equations

$$\begin{aligned} 0 &= F(x, y, t, 0) = F(x, y, t) \\ \dot{y} &= G(x, y, t, 0) = G(x, y, t), \end{aligned} \tag{4}$$

that has the number of dimensions $n' = n - s$.

If (3) is written in the form

$$\dot{x} = \frac{F(x, y, t, \mu)}{\mu}, \quad \dot{y} = G(x, y, t, \mu), \tag{5}$$

it is evident that the system of differential equations has a singularity in the case of $\mu = 0$. This is the reason for the denomination by singular perturbation.

Let us examine, on which conditions can the solution of the system of differential equations (1) be approximated by the solution of the system of differential equations (4) and how can this solution of (4) be determined.

Let F designate the subspace $F(x, y, t) = 0$ of the n-dimension phase space. If the time figures at the right sides of the system of differential equations (4) explicitly, F is changing in time. In the following we shall

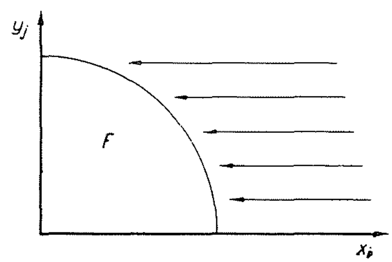

Fig. 1. Trajectories in the range of "rapid motion"

declare, following ANDRONOV, that a point (x^*, y^*) belongs to the surroundings O $[g(\mu)]$ of F at the instant t^*, if $|F(x^*, y^*, t^*)| \leqslant g(\mu)$.

Let us examine the part of the phase space outside the $O[\mu^\alpha]$ $(0 < \alpha < 1)$ surroundings of F. Since

$$|F(x, y, t)| \geqslant O[\mu^\alpha], \tag{6}$$

that is

$$|\dot{x}| \geqslant O[\mu^{\alpha-1}] \tag{7}$$

thus if μ is sufficiently small, x will change very rapidly. Let us denominate this part of the phase space as the range of "rapid motion".

In this range

$$\left|\frac{dy_j}{dx_i}\right| = \mu \left|\frac{G_j(x, y, t)}{F_i(x, y, t)}\right| \leqslant O[\mu^{1-\alpha}] \to 0, \tag{8}$$

$$\text{if } \mu \to +0$$

thus if μ is sufficiently small, we obtain *Fig.* 1 by representing the trajectories characterizing the movement of the system in the projection $y_j - x_i$.

If we now examine the movement in the interval $\Delta t \leqslant O[\mu^{1-\alpha}]$, y changes only in the order of magnitude of $\mu^{1-\alpha}$, accordingly we may declare that the movement is taking place in the surroundings of the subspace $y =$ $=$ const.

Introduce the "rapid time" by the definition

$$\tau = \frac{t - t_0}{\mu}, \tag{9}$$

then we may examine, in place of (3), the system of differential equations

$$\begin{aligned} \frac{dx}{d\tau} &= F(x, y, \mu\tau + t_0, \mu) \\ \frac{dy}{d\tau} &= \mu G(x, y, \mu\tau + t_0, \mu) \cong 0. \end{aligned} \tag{10}$$

If, however, we examine the part of the phase space inside the surroundings $O[\mu]$ of F, then we may consider the system of differential equations (4) in place of (3), since velocities are limited in the case of $\mu \to +0$ too. Let us denominate this movement as "slow motion" and these surroundings of F as the range of "slow motion".

Let us examine the possible motions in the complete phase space.

a. It may occur that all the trajectories of the rapid motion enter the small surroundings of F. Then the system will here move in the following as well, since these trajectories do not leave the surroundings. If the system is originally in the range of rapid motion, then it will be reached by rapid motion, during a time of Δt the border of the range of slow motion. It can be proved, that $\Delta t \leq O\left[\mu \ln \frac{1}{\mu}\right]$. From now on it will here move according to (4). In this case we may say that the x_i are such phase variables, which have no significant role outside the short interval Δt in the system. This means from the engineering point of view, that certain parameters of the system (which are in connection with the x_i phase variables) have no significant influence on the system. These parameters will be named the parasitic parameters. Let us examine what are the criteria of this state.

Consider the system of differential equations (10) valid for the rapid motion. For the sake of simplicity we assume that the right side of (10) does not depend explicitly on time, that is

$$\frac{dx}{d\tau} = F(x, y, \mu), \tag{11}$$

and now this system of differential equations will be examined in the complete phase space. The subspace $F(x, y, \mu) = 0$ is a state of equilibrium for rapid motion. Examine whether it will be stable. As is known, this question can be answered with the aid of the system of differential equations

$$\frac{d\xi_i}{d\tau} = \sum_{k=1}^{s} \frac{\partial F_i(x, y, \mu)}{\partial x_k} \xi_k \qquad i = 1 \ldots s, \tag{12}$$

which is valid for the first variation and where $\{x\,y\} \in F$ are now parameters. The stability of (12) can be decided by examining (e.g. with the aid of the Routh—Hurwitz criterion) the roots of the characteristic equation

$$\det. \left| \frac{\partial F_i}{\partial x_j} - \lambda E \right| = 0. \tag{13}$$

If all the roots have a negative real part, with respect to all the $\{x, y\} \in F$, the points of F form the stable equilibrium points of rapid motion.

In this case in turn all the trajectories are entering F, i.e. we have thereby obtained the necessary and sufficient condition of the possibility of examining our system by the system of differential equations, after the time Δt. This means technically that after the time Δt the effect of the above mentioned disturbing parameters can be left out of consideration.

b. Let us assume that $F = F^- + F^+$, where F^- denotes those points where (13) has also roots with positive real part. According to the precedings the system may not remain continuously in F^-. Either it enters F^+ and then the movement takes place there according to the laws of slow motion, or it leaves into the range of rapid motion. This latter is the case with systems performing relaxation oscillations.

Let us examine what happens if the system moving in F^+ reaches the boundary K between F^+ and F^-. Since the roots of (13) are continuously depending on the parameters $\{x, y\} \in F$ and in F^+ the real part of the roots is negative, while in F^- the real part is positive, we shall obtain at the boundary K either a purely imaginary pair of roots, or a root of zero value. We shall examine only this last mentioned case. Upon substituting the value $\lambda = 0$ into (13) we obtain the condition that the Jacobian determinant for F is zero. Thus all the conditions applying to K are

$$\begin{aligned} D(x, y) &= \frac{\partial(F_1 \ldots F_s)}{\partial(x_1 \ldots x_s)} = 0, \\ F_i(x, y) &= 0, \qquad i = 1 \ldots s. \end{aligned} \tag{14}$$

Accordingly K is the $n' - 1 = n - s - 1$ dimension subspace of the phase space. By differentiating in (14) the equations $F_i(x, y) = 0$ with respect to t, afterwards using the original system of equations, we find that

$$\sum_{j=1}^{s} \frac{\partial F_i}{\partial x_j} \dot{x}_j + \sum_{k=1}^{n'} \frac{\partial F_i}{\partial y_k} G_k = 0 \qquad i = 1, \ldots s. \tag{15}$$

From this

$$\dot{x}_i = \frac{D_i(x, y)}{D(x, y)} \qquad i = 1, \ldots s, \tag{16}$$

where we obtain $D_i(x, y)$ by substituting the i-th column of the Jacobian determinant by the column vector $-\sum_{k=1}^{n'} \frac{\partial F_i}{\partial y_k} G_k$. However, it can be seen

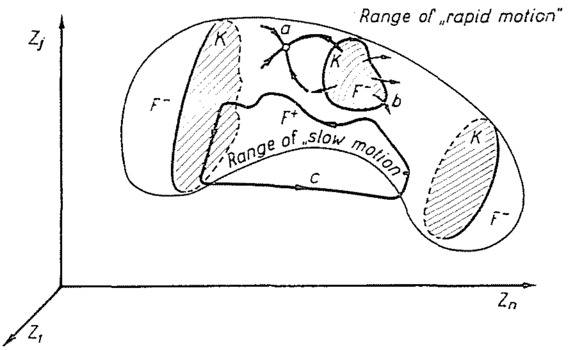

Fig. 2. Trajectories in the complete phase space. *a)* Stable equilibrium point, *b)* instable range of "slow motion", *c)* closed trajectory of the relaxation oscillation

in (16) that the $\dot{x}_i$ values become ∞ at the boundary K. It is conceivable that D, and with it all the $\dot{x}_i$, are changing signs on passing the boundary, thus the system cannot pass over to F^-. Accordingly the trajectories are accommodating tangentially to K and the system moves to the range of rapid motion, afterwards here rapidly (the more rapidly, the smaller is μ) again towards the range of slow motion *(Fig. 2)*.

In the complete phase space closed trajectories may be built up (relaxation oscillations) from these trajectories, and stable equilibrium states may exist. The literature discussing such phenomena (from the aspect of non-linear oscillations) is already very extensive [1], [14]. In the present paper only those aspects of the theory have been examined which are most important in control engineering, neglecting thereby the special problems of the examination of non-linear oscillations.

The singular perturbation of a system of differential equations

We have seen that if $\mu \to 0$, the differential equation

$$\Omega(\mu)\,\dot{z} = H(z, t, \mu) \qquad z|_{t=t_0} = z^0\,, \tag{17}$$

i.e. the equation

$$\begin{aligned} \mu\,\dot{x} &= F(x, y, t, \mu) \quad x|_{t=t_0} = x^0 \\ \dot{y} &= G(x, y, t, \mu) \quad y|_{t=t_0} = y^0 \end{aligned} \tag{18}$$

has a singularity. Therefore, we cannot generally expect the solution of (17) to be expanded, similarly to the "regular" perturbation calculation, in a convergent power series with respect to μ. This problem, as we have mentioned, was discussed in the last 15 to 20 years relatively often (frequently in such a way that the effect of the "great" parameter $1/\mu = \lambda$ has been examined in the differential equation), and the problem has a very large literature in mathematics. In clarifying the problem, WASSILJEWA [8], [10], [11] has done much work, who, partly from the results of TIHONOW [9] and others, has proved that the solution of (17) can be expanded in an asymptotic series with respect to the powers of μ, on certain conditions. In the following the pertinent results are described.

Let us consider the systems of differential equations (17) and (18), respectively, in a D domain. Since this system of differential equations is not linear in the general case, the equation

$$F(x, y, t, 0) = 0 \tag{19}$$

may have several solutions. Let $x = \varphi(y, t)$ designate some of the solutions of (19). The equations

$$\begin{aligned} 0 &= F(x, y, t, 0) = F(x, y, t) \\ \dot{y} &= G(x, y, t, 0) = G(x, y, t); \quad y|_{t=t_0} = y^0 \end{aligned} \tag{20}$$

or

$$\begin{aligned} x &= \varphi(y, t) \\ \dot{y} &= G(x, y, t) \qquad y|_{t=t_0} = y^0 \end{aligned} \tag{21}$$

are called a degenerated system of differential equations and the solution will be designated by $\bar{z}(t)$ (concretely by $\bar{x}(t)$ and $\bar{y}(t)$).

The hypersurface $x = \varphi(y, t)$ is called isolated, if there exists such a value $\varepsilon > 0$, that the equation $F(x, y, t) = 0$ has no solution beyond $x = \varphi(y, t)$ in the subspace $|\,z - \varphi(y, t)\,| < \varepsilon$. The equation of rapid motion ordered to (18) has the form of

$$\frac{dx}{d\tau} = F(x, y, \mu\tau + t_0, \mu), \tag{22}$$

or since the right side of (22) depends "regularly" on μ, we may regard in place of (22) also the equation

$$\frac{dx}{d\tau} = F(x, y^*, t^*). \tag{23}$$

By comparing the preceding, $y^* = y^0$, $t^* = t_0$, now y^* and t^* are handled as parameters.

The isolated "curve" $x = \varphi(y, t)$ is called stable, if the points $x = \varphi(y^*, t^*)$ ordered to the values y^* and t^*, pertaining to any D, are asymptotically stable points for the equation (23). We designate by D_φ and call the influence domain of the stable curve $x = \varphi(y, t)$ the set of those points $\{x^*, y^*, t^*\}$ which have the characteristic that the solution pertaining to the initial condition x^* tends to $\varphi(y^*, t^*)$. Hereafter we may declare the following theorem, the content of which has been discussed in detail in the qualitative considerations of this paper [9].

If the solution $x = \varphi(y, t)$ of the equation $F(x, y, t) = 0$ is an isolated stable curve in the bounded closed domain D and if the point determined by the initial conditions of the system of differential equations (18) falls into the influence domain of $\varphi(x, t)$ ($\{x^0, y^0, t_0\} \in D_\varphi$), further if the solution $\bar{y}(t)$ and $\bar{x}(t)$ of the degenerated system of differential equations (21) falls into D in the domain $t_0 \leqslant t \leqslant T_0$, then the solution $z(t, \mu)$ of the system of differential equations (17) tends to the solution $\bar{z}(t)$ of the system of differential equations (21), in the case of $\mu \to 0$. By writing this in detail,

$$\lim_{\mu \to 0} x(t, \mu) = \bar{x}(t) = \varphi(\bar{y}, t); \quad t_0 < t \leqslant T_1 < T_0 \tag{24}$$

and

$$\lim_{\mu \to 0} y(t, \mu) = \bar{y}(t); \quad t_0 \leqslant t \leqslant T_1 < T_0 . \tag{25}$$

It should be noted that convergence (25) is uniform in the domain $t_0 \leqslant t \leqslant T_1$, convergence (24) in turn will be uniform only in the domain $t_0 < t_1 \leqslant t \leqslant T_1$. This is in connection with the fact that $\bar{x}(t)$ has a discontinuity at the point t_0, as we have already seen in the preceding qualitative considerations. It should further be mentioned that the stability of $\varphi(\bar{y}, t)$ is a very essential requirement, as is also evident from the foregoing.

It follows from this theorem that if μ is sufficiently small, the solution $z(t, \mu)$ can be well approximated by the solution $\bar{z}(t)$, if we omit the small surroundings of t_0 for the variables x. This fact has already been evaluated qualitatively, in the following we are going to examine how the approximative expression for the solution can be determined by series expansion with respect to the powers of μ.

Determination of the asymptotic series expansion

Let us consider the systems of differential equations (17) and (18), respectively. We assume in the case of this system of differential equations the convergences (24) and (25). Afterwards, by introducing the new variable $\tau = \dfrac{t - t_0}{\mu}$, the "rapid" time, equations (17) and (18), respectively, can be rewritten in the forms

$$\frac{dz}{d\tau} = (\mu E - \Omega(\mu))\, H(z, t_0 + \mu\tau, \mu); \quad z|_{\tau=0} = z^0, \tag{26}$$

and

$$\begin{aligned} \frac{dx}{d\tau} &= F(x, y, t_0 + \mu\tau, \mu); & x|_{\tau=0} &= x^0 \\ \frac{dy}{d\tau} &= \mu\, G(x, y, t_0 + \mu\tau, \mu); & y|_{\tau=0} &= y^0, \end{aligned} \tag{27}$$

respectively. (E denotes the unit matrix.) We try to determine the solution of this system of differential equations formally by the series

$$\overset{1}{z}(\tau) = \overset{1}{z}_0(\tau) + \mu \overset{1}{z}_1(\tau) + \mu^2 \overset{1}{z}_2(\tau) + \dots \tag{28}$$

Since the system of differential equations depends regularly on μ, we may proceed according to the well known rules of the perturbation calculus. We obtain systems of differential equations which can be solved recursively, from which the first two are the following.

$$\begin{aligned} &0. & \frac{d\overset{1}{x}_0}{d\tau} &= F(\overset{1}{x}_0, \overset{1}{y}_0, t_0, 0); & \overset{1}{x}_0|_{\tau=0} &= x^0 \\ & & \frac{d\overset{1}{y}_0}{d\tau} &= 0; & \overset{1}{y}_0|_{\tau=0} &= y^0 \\ &1. & \frac{d\overset{1}{x}_1}{d\tau} &= \overset{1}{F}_{x0} \overset{1}{x}_1 + \overset{1}{F}_{y0} \overset{1}{y}_1 + \overset{1}{F}_{t0} + \overset{1}{F}_{\mu 0}; & \overset{1}{x}_1|_{\tau=0} &= 0 \\ & & \frac{d\overset{1}{y}_1}{d\tau} &= G(\overset{1}{x}_0, \overset{1}{y}_0, t_0, 0); & \overset{1}{y}_1|_{\tau=0} &= 0, \end{aligned} \tag{29}$$

where the first lower index indicates the partial derivation, while the second lower and upper indices indicate, that $\overset{1}{z}_0$, or $\overset{1}{z}_1$, etc. has been substituted in the argumentum of the function, as e.g. $\overset{1}{F}_{y0} = F_y(\overset{1}{x}_0, \overset{1}{y}_0, t_0, 0)$.

The initial conditions, apart from the 0-th equation, are all zero. The whole system of equations (29) (apart from the 0-th system) is linear, and also the order of the system of differential equations to be solved in a single step have been reduced by this recursive solution (n' and s, respectively). Beyond this, the equations for $\overset{1}{y}_i$ can be solved by a simple integration.

We will try hereafter to find the solution of (17) and (18) in the form of the series

$$\overset{2}{z}(t) = \overset{2}{z}_0(t) + \mu\, \overset{2}{z}_1(t) + \mu^2\, \overset{2}{z}_2(t) + \dots \tag{30}$$

formally, with the aid of the known methods of the perturbation calculus. The form of the 0-th equation is

$$0 = F(\overset{2}{x}_0, \overset{2}{y}_0, t), \text{ that is } \overset{2}{x}_0 = \varphi(\overset{2}{y}_0, t) \tag{31}$$

$$\frac{d\overset{2}{y}_0}{dt} = G(\overset{2}{x}_0, \overset{2}{y}_0, t) \qquad \overset{2}{y}_0\big|_{t=t_0} = y^0 ,$$

i.e. the degenerated equation as defined in (20) and (21). The first equation is

$$\frac{d\overset{2}{x}_0}{dt} = \overset{2}{F}_{x0}\, \overset{2}{x}_1 + \overset{2}{F}_{y0}\, \overset{2}{y}_1 + \overset{2}{F}_\mu$$

$$\frac{d\overset{2}{y}_1}{dt} = \overset{2}{G}_{x0}\, \overset{2}{x}_1 + \overset{2}{G}_{y0}\, \overset{2}{y}_1 + \overset{2}{G}_\mu \tag{32}$$

and the systems of differential equations obtained in this way are similarly linear from the 1st term on, they can be solved recursively. The situation is different in so far, that in the first group of the equations there will be no differential equation and $\overset{2}{x}_k$ should be expressed from them algebraically. The initial conditions are determined by the following special formula:

$$\overset{2}{y}{}_k^0 = \overset{2}{y}_k\big|_{t=t_0} = \frac{(-1)^k}{k!} \int_0^\infty \tau^k \frac{d^k}{d\tau^k} \overset{1}{G}_{(k-1)}\, \tau\, d\tau, \qquad k = 1, 2, \dots \tag{33}$$

where $\overset{1}{G}_{(k-1)}(\tau)$ is a term of the formal series expansion

$$G(\overset{1}{x}_0 + \mu\, \overset{1}{x}_1 + \mu^2\, \overset{1}{x}_2 + \dots, \overset{1}{y}_0 + \mu\, \overset{1}{y}_1 + \mu^2\, \overset{1}{y}_2 + \dots, \mu\tau + t_0, \mu) =$$

$$= \overset{1}{G}_{(0)}(\tau) + \mu\, \overset{1}{G}_{(1)}(\tau) + \mu^2\, \overset{1}{G}_{(2)}(\tau) + \dots \tag{34}$$

We have thus determined the terms of the series (30). Let us consider a third formal expansion of the system of differential equations (17). We obtain this by expanding the terms of the previously determined series (30) according to the powers of $(t - t_0)$, and afterwards by rearranging the double series obtained in this way with respect to the increasing powers of μ and $(t - t_0)$. Accordingly

$$z = \overset{2}{z}_{00} + \overset{2}{z}_{10}(t - t_0) + \overset{2}{z}_{01}\mu + \ldots + \overset{2}{z}_{ij}(t - t_0)^i \mu^j + \ldots \tag{35}$$

The partial sums of the series (28), (30) and (35) up to the n-th power are designated by $\overset{1}{(z)}_n$, $\overset{2}{(z)}_n$ and $\overline{(\overset{2}{z})}_n$, and in $\overset{1}{(z)}_n$ in place of the variable τ we have again substituted $t = \mu\tau + t_0$. Consider hereafter the expression

$$Z_n = \overset{1}{(z)}_n + \overset{2}{(z)}_n - \overline{(\overset{2}{z})}_n . \tag{36}$$

Wassiljewa has proved, that in the case of a sufficiently small μ, (36) is the n-th partial sum of the (uniformly) asymptotic series expansion of the solution of problem (17) in the interval $t_0 \leqslant t \leqslant T_1$, that is

$$|z(t, \mu) - Z_n| < C\,\mu^{n+1} \tag{37}$$

where C is a constant, independent of n and t.

Wassiljewa has also given another formulation of these results. By introducing the function

$$P_n(z) = \overset{1}{z}_n(\tau) - \sum_{i=0}^{n} \tau^i \overset{2}{z}_{i,n-i} \tag{38}$$

she has proved, that in the case of sufficiently small μ,

$$\left| \frac{d^k}{d\tau^k} P_\mu(z) \right| < C\, e^{-\varkappa\tau} \qquad k = 0, 1, \ldots \tag{39}$$

where C and $\varkappa > 0$ are constants independent of n. It is evident from definition (38) that

$$Z_n = \sum_{k=0}^{n} \mu^k \left(\overset{2}{z}_k(t) + P_k(z)\right). \tag{40}$$

By comparing this, however, with (39), we can see that the partial sum $(z)_n$ can be used for the approximation of $z(t, \mu)$, beyond the small surroundings of t_0, that is

$$|z(t, \mu) - \overset{2}{(z)}_n| < C\mu^{n+1}; \qquad t_0 < t_1 \leqslant t \leqslant T_1 . \tag{41}$$

It is similarly conceivable that inside the small surroundings of t_0 the partial sum $\overset{1}{(z)}_n$ can be used for the approximation of $z(t, \mu)$. Descriptively we may say of (36) that if t is in the small surroundings of t_0, then $\overset{2}{(z)}_n$ and $\overline{(\overset{2}{z})}_n$ are compensating each other and $z(t, \mu)$ is approximated by the expression for the "rapid motion", while if t is outside the small surroundings of t_0, then $\overset{1}{(z)}_n$ and $\overline{(\overset{2}{z})}_n$ will compensate each other and the expression for the "slow motion" will be valid for $z(t, \mu)$. We have thereby solved the problem of the "connection" of the rapid and slow motion. Otherwise it is conceivable that with the aid of the function $P_n(z)$ the initial conditions (33) can be written in the form

$$\overset{2}{y}_k\big|_{t=t_0} = \int_0^\infty P_{k-1}(\overset{1}{G})\, d\tau \qquad k = 1, 2, \ldots \tag{42}$$

The examination of the performance of the system of differential equations in the domain $t_0 \leqslant t < \infty$

This means a qualitatively new problem to a certain extent, if we include the whole domain $t \geqslant t_0$ in our considerations. However, under certain conditions our results can be extended to this case as well. Let us assume, that the (only) solution of the equation $F(x, y, t) = 0$ in the domain D (D is bounded as regards to x and y, but includes the half-line $t_0 \leqslant t < \infty$), is $x = \varphi(y, t)$. Assuming that the points $x^* = \varphi(y^*, t^*)$ are asymptotically stable points for the "rapid motion" equation of the form (23), that is the real part of the roots of the equation

$$\det\left|\frac{\partial F_i}{\partial x_j} - \lambda E\right| = 0 \tag{43}$$

is smaller than zero in the interval $t_0 \leqslant t < \infty$. Let us further assume that also the real part of the roots of the equation

$$\det (A - \lambda E) = 0 \tag{44}$$

is smaller than zero, where

$$A = \left[\frac{\partial G_i}{\partial y_j} - \frac{\partial G_i}{\partial x_j} \cdot \left|\frac{\partial F_i}{\partial x_j}\right|^{-1} \cdot \frac{\partial F_i}{\partial y_j}\right]_{z=\bar{z}} . \tag{45}$$

This latter condition is stipulated in the interval $T \leqslant t < \infty$, where T may have any value, but is fixed.

Under these conditions relations (24) and (25) will be valid also in the intervals $t_0 < t < \infty$ and $t_0 \leqslant t < \infty$, respectively, and the formulae for series expansion are valid for the complete interval $t_0 \leqslant t < \infty$.

Summary

In practice the engineer is always compelled to disregard certain circumstances and to make idealizations when examining dynamical systems. In some cases — although we feel empirically the idealizations to be "small" — these idealizations may cause modifications of such type in the system of differential equations describing the performance of the system, that our model will perform substantially differently from the original system. The aim of this paper is to describe the mathematical apparatus suitable for the examinations of problems of this kind. First the phenomena are examined partly qualitatively, afterwards the problem is formulated exactly as well. Hereafter the construction of approximative solutions is discussed, finally the conditions are examined, under which we may include in the examinations the complete domain $t_0 \leqslant t < \infty$ of the independent variable.

References

1. Андронов, А. А., Витт, А. А., Хайкин, С. Э.: Теория колебаний. Физ—Мат. 1960.
2. Minorsky, N.: Nonlinear Oscillations. Van Nostrand 1962.
3. Чезари, Л.: Асимптотическое поведение и устойчивость решений обыкновенных дифференциальных уравнений. Мир 1964.
4. Кокотович, П. В., Рутман, Р. С.: Чувствительность систем автоматического управления. Авт. и Телем. **26** № 4. (1965).
5. Erdélyi, A.: Asymptotic Expansions, Dover 1955.
6. Bellman, R.: Perturbation Techniques in Mathematics, Physics and Engineering. Holt, Rinehart and Winston, Inc, New York etc., 1964.
7. Дородницын, А. А.: Асимптотическое решение уравнения Ван дер Поля. Прикл. Матем. и мех. **11.** № 3.
8. Васильева, А. Б.: Асимптотические методы в теории обыкновенных дифференциальных уравнений с малыми параметрами при старших производных. Ж. Выч. Мат. и Мат. Физ. **3** № 4. (1963).
9. Тихонов, А. Н.: Системы дифференциальных уравнений, содержащие малые параметры при производных. Матем. сб. **32** № 3. (1952).
10. Васильева, А. Б.: Построение равномерного приближения к решению системы дифференциальных уравнений с малыми параметрами при старшей производной. Матем. сб. **50** № 1. (1960).
11. Васильева, А. Б.: Асимптотика решений некоторых задач для обыкновенных нелинейных дифференциальных уравнений с малыми параметрами при старших производных. Успехи мат. наук **18** № 3. (1963).
12. Miller, K. S., Murray, F. J.: The Mathematical Basis for the Error Analysis of Differential Analysers. MITJ Math. and Phys., No. 2—3. 1953.
13. Bates, D. T.: Quantum Theory. Acad. Press 1961.
14. Боголюбов, Н. Н., Митропольский, Ю. А.: Асимптотические методы в теории нелинейных колебаний. Гостехиздат. 1955.

Gyula Büti, Budapest XI., Egry József u. 18—20. Hungary.