DETERMINATION OF THE NONLINEARITY ON THE BASIS OF ITS DESCRIBING FUNCTION

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Introduction

The mathematical theory of optimal processes provides a possibility for the materialisation of control systems which realise the maximal potentialities of the systems. In the classical control theory this led to the synthesis of systems which possess the optimal dynamic properties. A means of realising such systems is the nonlinear correction. In spite of what was said above, in linear systems traditional frequency methods of correction have a decisive role. Methods based on these principles satisfy practical requirements and their realisation (as regards methods and means) is relatively simple.

Frequency methods are being applied also for the synthesis of nonlinear systems [5]. The result of the synthesis is a nonlinear correction of the system. This trend of nonlinear correction may assume an important role in the future.

In conjunction with the problem of synthesis of nonlinear systems the task arises to find the inverse describing function, i.e. finding the nonlinear characteristic from its describing function.

An analytical approach to the solution of the problem of the inverse describing function was proposed by GIBSON and DITADA [1], but — as the authors of this paper remark — the operations expressed by the equations "are in general very difficult to perform in practical cases, if not impossible". For this reason GIBSON and DITADA propose the use of numerical methods of solution.

In earlier works [2, 3, 4] a generalised method for determining the describing functions in the case of piecewise-linear, nonlinear systems with finite discontinuities, is described. This method permits direct solution of the problem of finding the inverse describing function. The solution is approximate.

The approximate character of the solution is due to the fact that the nonlinear characteristic is sought for (substituted) by a family of piecewise-linear nonlinear characteristics with finite discontinuities. Accordingly, the given describing function is quantized and the describing function of the nonlinear characteristic to be determined will only coincide with the given values in selected points.

The possibility of applying the approximate method in conjunction with the given problem is also confirmed by the fact that the method of describing functions is approximate in itself.

It is pointed out that in his book GIBSON [5] mentions a method, the basic idea of which is to some extent similar to that presented in this paper.

I. Generalised method for determining the describing function

The describing function indicates how the base frequency of a periodic output signal with some nonlinearity relates to its input signal in the case of a harmonic input signal.

In the complex form the describing function can be written as follows:

$$\overline{Y}_n = b + ja \tag{1.1}$$

Let us assume that the nonlinearity is characterized by the following function:

$$y = f(x) \tag{1.2}$$

where

y — output signal of nonlinearity,
x — input signal of nonlinearity.

Let us assume that the input signal of nonlinearity will be:

$$x = x_0 + \tilde{x} \tag{1.3}$$

where

$\tilde{x} = B \sin \omega t = B \sin \varphi$ (1.4)
x_0 — constant component
$\tilde{x}$ — harmonic component
B — amplitude
ω — frequency
$\varphi = \omega t$ — phase angle

The coefficients of the describing function can be determined as follows:

$$b = \frac{1}{\pi B} \int_0^{2\pi} f(x) \sin \varphi \, d\varphi \tag{1.5}$$

$$a = \frac{1}{\pi B} \int_0^{2\pi} f(x) \cos \varphi \, d\varphi \tag{1.6}$$

The nonlinear characteristics at the input signal equ. (1.3) can be considered in a very general case [3, 4] as either having different branches for the phase quarters of the harmonic component ($\tilde{x} = B \sin \varphi$), as is shown in Fig. 1. Therefore the nonlinearity is characterised by the characteristics $f_1(\tilde{x})$, $f_2(\tilde{x})$, $f_3(\tilde{x})$, $f_4(\tilde{x})$ within the limits

$$\varphi = 0 - \frac{\pi}{2}; \quad \frac{\pi}{2} - \pi; \quad \pi - \frac{3}{2}\pi; \quad \frac{3}{2}\pi - 2\pi \text{ — respectively.}$$

To obtain a similar method of investigation for all the phase quarters it is advisable to turn the branches $f_3(x)$ and $f_4(x)$ by 180° with respect to the origin of the coordinates (central projection). The so obtained branches

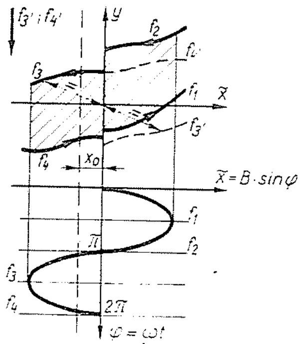

Fig. 1

of the characteristics will be denoted by

$$f_{3'}(x) \text{ and } f_{4'}(x)$$

For odd single valued characteristics, if

$$x_0 = 0:$$
$$f_1(\tilde{x}) = f_2(\tilde{x}) = f_{3'}(\tilde{x}) = f_{4'}(\tilde{x}) \tag{1.7}$$

For odd two valued characteristics, if

$$x_0 = 0:$$

$$f_1(\tilde{x}) = f_{3'}(\tilde{x}) \tag{1.8}$$

$$f_2(\tilde{x}) = f_{4'}(\tilde{x})$$

For odd single valued characteristics, if $x_0 \neq 0$, and in the case of non-symmetrical single valued characteristics:

$$f_1(\tilde{x}) = f_2(\tilde{x}) \qquad (1.9)$$
$$f_{3'}(\tilde{x}) = f_{4'}(\tilde{x})$$

In other cases (odd two valued characteristic, if $x_0 \neq 0$ and two valued non-symmetrical characteristic), the branches of the characteristic differ.

In the following only the piecewise linear characteristics with finite discontinuities will be considered. For the branches of the characteristics we

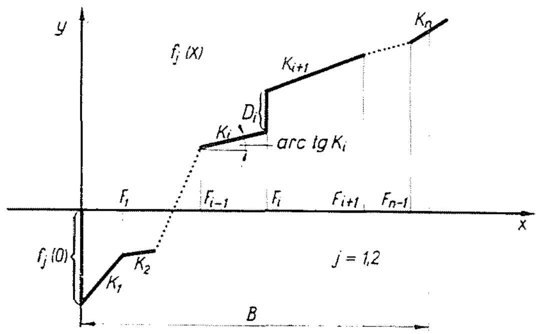

Fig. 2

will apply the designations as shown in Fig. 2 which is of a quite general type. By suitable selection of the parameters $f(0)$; F_i; K_i; D_i; $(i = 1, 2, 3, \ldots n)$ any shape of the branches of the characteristics can be realized.

The branch of the characteristic shown in Fig. 2 can be sub-divided into "elementary" characteristics. To obtain generally applicable results we investigated the characteristic $f_j(\tilde{x}) - (j = 1, 2, 3', 4')$. The "elementary" characteristic will be:

$$\begin{aligned}
f^{D_0}(\tilde{x}) &= f(0) && \tilde{x} > 0 \\
f^{i}(\tilde{x}) &= \begin{cases}(K_i - K_{i+1})\,\tilde{x} & x < F_i \\ (K_i - K_{i+1})F_i & \tilde{x} > F_i\end{cases} \\
f^{D_i}(\tilde{x}) &= \begin{cases}0 & \tilde{x} < F_i \\ D_i & \tilde{x} > F_i\end{cases} \\
f^{n}(\tilde{x}) &= K_n\,\tilde{x} && F_{n-1} < B \leqslant F_n
\end{aligned} \qquad (1.10)$$

By means of the "elementary" characteristic it is possible to construct the branch of the nonlinear characteristic:

$$f_j(\tilde{x}) = f^{D_0}(\tilde{x}) + f^{n}(\tilde{x}) + \sum_{i=1}^{n-1}\left[f^{i}(\tilde{x}) + f^{D_i}(\tilde{x})\right]. \qquad (1.11)$$

(For the purpose of simplicity it is not indicated in the righthand part of the formulae that the parameters relate to the j-th branch of the parameters.)

In this way the branch of the nonlinear characteristic was substituted by a sum of simple elements which can be superimposed (these elements correspond to the relay characteristic, linear characteristic, linear characteristic with saturation and relay characteristics with dead zone).

Such a decomposition permits investigating any piecewise linear nonlinearity by a unified method.

By introducing a new variable it can easily be proved [3, 4] that the components of the coefficients of the describing function can be determined in a single way from the branches of the characteristics. Therefore, on the basis of the formulae (1.5) and (1.6) we obtain:

$$b = \frac{1}{\pi B} \int\limits_0^{\pi/2} [f_1(\tilde{x}) + f_2(\tilde{x}) + f_{3'}(\tilde{x}) + f_{4'}(\tilde{x})] \sin \varphi \, d\varphi = b_1 + b_2 + b_{3'} + b_{4'} = \Sigma b_j$$

$$(j = 1, 2, 3', 4') \tag{1.12}$$

where

$$b_j = \frac{1}{\pi B} \int\limits_0^{\pi/2} f_j(\tilde{x}) \sin \varphi \, d\varphi \tag{1.13}$$

and

$$a = \frac{1}{\pi B} \int\limits_0^{\pi/2} [f_1(\tilde{x}) - f_2(\tilde{x}) + f_{3'}(\tilde{x}) - f_{4'}(\tilde{x})] \cos \varphi \, d\varphi = a_1 - a_2 + a_{3'} - a_{4'} \tag{1.14}$$

where

$$a_j = \frac{1}{\pi B} \int\limits_0^{\pi/2} f_j(\tilde{x}) \cos \varphi \, d\varphi \qquad (j = 1, 2, 3', 4') \tag{1.15}$$

By inserting equ. (1.11) into (1.13) and the integration taking into consideration equ. (1.10), we obtain the following generalised formula:

$$b_j = \frac{f(0)}{\pi B} + \frac{1}{4} \left\{ K_n + \sum_{i=1}^{n-1} \left[K_i - K_{i+1}) \, k\left(\frac{B}{F_i}\right) + \frac{D_i}{F_i} \, l\left(\frac{B}{F_i}\right)\right]\right\}. \tag{1.16}$$

The function which figures in equ. (1.16) can be determined from the following expressions:

$$k\left(\frac{B}{F_i}\right) = \frac{2}{\pi} \left(\text{arc sin} \frac{F_i}{B} + \frac{F_i}{B} \sqrt{1 - \frac{F_i^2}{B^2}} \right) \tag{1.17}$$

$$l\left(\frac{B}{F_i}\right) = \frac{4}{\pi} \cdot \frac{F_i}{B} \sqrt{1 - \frac{F_i^2}{B^2}} \tag{1.18}$$

From the equ. (1.12) and (1.16) the coefficient "b" can be determined for a nonlinear characteristic of any shape. By inserting the function $k\left(\frac{B}{F_i}\right)$ and $l\left(\frac{B}{F_i}\right)$ expressed by equ. (1.17) and (1.18), in the obtained expressions the coefficient "b" is obtained in its conventional form. The functions $k\left(\frac{B}{F_i}\right)$ and $l\left(\frac{B}{F_i}\right)$ can be calculated in advance and tabulated. By means of these tables the numerical values of the coefficient "b" which are necessary for using graphical methods can easily be found. These tables are contained, for instance, in an earlier work [2, 3] (it is pointed out that the function $k\left(\frac{B}{F_i}\right)$ was used for the first time by K. MAGNUS for describing functions).

In a similar manner the following generalised formula can be derived:

$$a_j = \frac{1}{2\pi}\left\{\frac{2f(0)}{B} + K_n + \sum_{i=1}^{n-1}\left[(K_i - K_{i+1})\left(2\frac{F_i}{B} - \frac{F_i^2}{B^2}\right) + 2\frac{D_i}{F_i}\left(\frac{F_i}{B} - \frac{F_i^2}{B^2}\right)\right]\right\}. \qquad (1.19)$$

The coefficient "a" can be determined by means of the formulae (1.14) and (1.19).

It is pointed out that in a similar manner it is possible to obtain the generalised formulae for determining the constant of the component and the coefficients of the higher harmonics [4].

In the case of symmetrical oscillations and single valued characteristics:

$$b = \frac{4f(0)}{\pi B} + K_n + \sum_{i=1}^{n-1}\left[(K_i - K_{i+1})\, k\left(\frac{B}{F_i}\right) + \frac{D_i}{F_i}\, l\left(\frac{B}{F_i}\right)\right] \qquad (1.20)$$

$$(F_{n-1} < B \leqq F_n)$$

$$a = o \qquad (1.21)$$

In the case of two valued characteristics:

$$b = 2b_1 - 2b_2 \qquad (1.22)$$

The component b_j ($j = 1, 2$) is determined from equ. (1.16),

$$a = 2a_1 - 2a_2 \qquad (1.23)$$

The component a_j ($j = 1, 2$) is determined by means of equ. (1.19).

Equs. (1.20)—(1.23) are correct even for such nonlinear characteristics where the shape depends on the amplitude or the frequency of the input signal.

In the following only the case of symmetrical oscillations will be investigated. In practice this case is of decisive importance. The generalised formulae also allow the investigation of non-symmetrical oscillations.

In the following it is assumed that the shape of the nonlinear characteristics does not depend on the amplitude of the input signal.

II. Determination of the nonlinearity from the describing function

1. *Single valued inverse characteristics*

In the case of symmetrical oscillations the coefficients of the describing functions unequivocally determine the appropriate nonlinear characteristic.

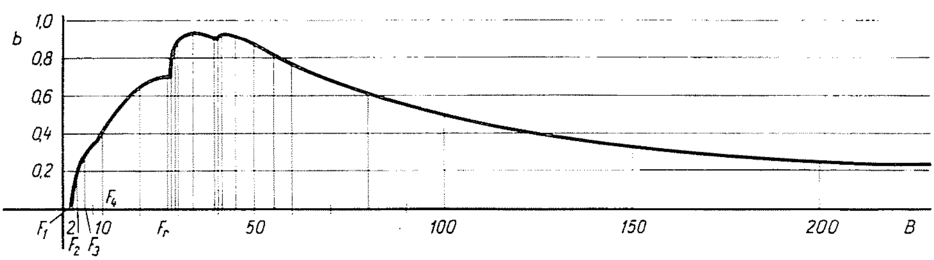

Fig. 3

These coefficients are functions of the amplitude B of the input signal of nonlinearity.

If the coefficient $a = a(B) = 0$, then the nonlinear characteristic is single valued.

The shape of the appropriate nonlinear characteristic depends on the form of the function $b = b(B)$.

a) Let us assume that $b = b(B)$ is a continuous, single valued function with continuous first derivatives. In this case $f(0) = D_i = 0$; $(i = 1, 2 \ldots n-1)$ (see Appendix 1).

On the basis of equ. (1.20) the inverse nonlinear characteristic is determined as follows:

Let us assume that the function $b(B)$ has the form shown in Fig. 3 (in accordance with the condition of continuity of the first derivatives of the functions in points where this condition is not fulfilled, we assume small rounding offs).

Let us separate the horizontal axis of Fig. 3 with suitable selected points. This permits determining the bond points of the piecewise linear nonlinear characteristics.

According to formula (1.20) we consider a sub-series of the selected points one by one.

Let us assume that $B = F_1$. According to formula (1.20):

$$K_1 = b(F_1) \tag{2.1}$$

For $B = F_2$

$$K_2 + (K_1 - K_2)\, k\left(\frac{F_2}{F_1}\right) = b(F_2)$$

Therefrom:

$$K_2 = \frac{b(F_2) - K_1 k\left(\frac{F_2}{F_1}\right)}{1 - k\left(\frac{F_2}{F_1}\right)} \tag{2.2}$$

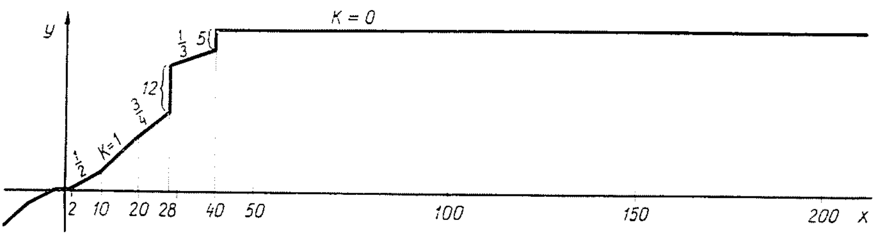

Fig. 4

The value of K_1 is known from equ. (2.1) and therefore K_2 can be determined by means of equ. (2.2).

By continuing the described procedure in this manner it is possible to determine the nonlinear characteristic in the range of interest to us. Thus, for $B = F_3$:

$$K_3 = \frac{b(F_3) - (K_1 - K_2)\, k\left(\frac{F_3}{F_1}\right) - K_2 k\left(\frac{F_3}{F_2}\right)}{1 - k\left(\frac{F_3}{F_2}\right)} \tag{2.3}$$

Generally, for $B = F_h$:

$$K_h = \frac{b(F_h) - \sum_{i=1}^{h-2} (K_i - K_{i+1})\, k\left(\frac{F_h}{F_i}\right) - K_{h-1} k\left(\frac{F_h}{F_{h-1}}\right)}{1 - k\left(\frac{F_h}{F_{h-1}}\right)} \tag{2.4}$$

The nonlinear characteristic is fully determined by the bend points $(F_1, F_2 \ldots)$ and the corresponding slope of the linear sections.

The describing function of the obtained nonlinear characteristic in the bend points will coincide accurately with the values of the given describing function.

In principle the accuracy of the method can be increased to any degree by increasing the number of linear sections. In most practical cases even a small number of sections gives satisfactory results. If the accuracy requirements are more stringent very accurate results can be obtained by using digital computers. The given method is very convenient for computer calculations.

It is advisable to sub-divide the horizontal axis of the graph of the function $b(B)$ in accordance with the shape of the function. This means that in points where the changes in the describing function are large, it is advisable to take the selected points near to each other. If the first derivative of the describing function is not a continuous function, then it is necessary to change over to characteristics with finite discontinuities to obtain results which reflect this discontinuity.

b) If the nonlinear characteristic does not begin in the origin of the coordinates ($y = 0$; $x = 0$), i. e. $f(0) \neq 0$, then, as can be seen from equ. (1.20),

$$b(B) \to \infty, \quad \text{if} \quad B \to 0 \tag{2.5}$$

The values $f(0)$ can be determined as follows. Inside the first linear section we select one further point $(0 < F_1' < F_1)$. According to equ. (1.20):

$$\begin{aligned} \frac{4f(0)}{\pi F_1'} + K_1 &= b(F_1') \\ \frac{4f(0)}{\pi F_1} + K_1 &= b(F_1) \end{aligned} \tag{2.6}$$

From equ. (2.6) we can determine $f(0)$ and K_1.

In those points for which the first derivative of the describing function has an infinite discontinuity, the non-linear characteristic has a finite discontinuity. The value of the finite discontinuity D_r can be determined in a manner similar to that of the value of $f(0)$.

Let us assume that the first derivative of the function $b(B)$ has an infinite discontinuity at the point $B = F_r$. We select on the $(r + 1)$-th linear section one more point

$$(F_r < F'_{r+1} < F_{r+1}).$$

According to equ. (1.20) we obtain two equations for determining the two unknown values (D_r and K_{r+1}).

A characteristic feature of this method is that the length of the individual sections can be chosen arbitrarily. This means that in quantizing the describing function the value are taken in arbitrary points. The selection of points F'_1 and F'_{r+1} plays a similar role. From the selection of these points depends the value of the obtained parameters which ensure in these points a given value of the describing function.

In reality in the case of an arbitrary selection $f(0)$ and D_r $(r = 1 \ldots m)$, a given value of the describing function in any point can be ensured by an appropriate selection of the slope of the linear sections. (Of course, the changes in the nonlinear characteristics may be very sharp in this case.) For determining these finite discontinuities a method has been proposed for which it was necessary that the describing function should have given values also in other points. The values of these finite discontinuities can also be determined according to the formulae [6]

$$f(0) = \frac{\pi}{4} F_r \, b(F_r)$$

where F_r — a very small amplitude for which the value of the describing function is still given.

$$D_r = \frac{F_r}{l(c)} \left[b(cF_r) - b(F_r) \right] \tag{2.8}$$

where $(c - 1)$ is a certain small number, particularly such a number that for the amplitude $B = cF_r$ we are still at the steep section of the function $b(B)$. Ultimately, taking the established values $f(0)$ and D_i into consideration, the following expression is obtained according to formula (1.20) for determining the inclination of the h-th section if the amplitude is $B = F_h$:

$$K_h = \frac{b(F_h) - \sum_{i=1}^{h-2} \left[(K_i - K_{i+1})\, k\left(\frac{F_h}{F_i}\right) + \frac{D_i}{F_i}\, l\left(\frac{F_h}{F_i}\right) \right] - K_{h-1}\, k\left(\frac{F_h}{F_{h-1}}\right) - \frac{4f(0)}{\pi F_h}}{1 - k\left(\frac{F_h}{F_{h-1}}\right)} \tag{2.9}$$

2. *Two valued inverse characteristics*

If $a = a(B) \neq 0$, the inverse nonlinear characteristic is two valued. The piecewise linear inverse characteristic can be determined by means of the formulae (1.16), (1.19) and (1.22), (1.23).

The shape of the nonlinear characteristic depends on the shape of the functions which determine the coefficients of the describing function.

a) Let us assume that if $B = 0$, then $a(0)$ and $b(0)$ will be finite values (up to the point F_1). From the equs. (1.16), (1.19) and (1.22), (1.23) it can be seen that in this case

$$f_1(0) = f_2(0) = 0 \tag{2.10}$$

From the same formulae we obtain:

$$\begin{aligned} b(0) &= \frac{1}{2}(K_1^{(1)} + K_1^{(2)}) \\ a(0) &= \frac{1}{\pi}(K_1^{(1)} - K_1^{(2)}) \end{aligned} \tag{2.11}$$

(the index in superscript brackets denotes the branch of the characteristics to which the parameters relate). From the formula (2.11) we obtain:

$$\begin{aligned} K_1^{(1)} &= \frac{2b(0) + \pi a(0)}{2} \\ K_1^{(2)} &= \frac{2b(0) - \pi a(0)}{2} \end{aligned} \tag{2.12}$$

Let us assume that $a(b)$ and $b(B)$ are such functions that the inverse nonlinear characteristic has no finite discontinuities.

Applying the same procedure as for the single valued characteristics we sub-divide the horizontal axes of the functions $a(b)$ and $b(B)$ by the same selected points, and we study one point after the other.

For the amplitude $B = F_2$ we obtain according to formulae (1.16) and (1.22):

$$b(F_2) = \frac{1}{2}\left[K_2^{(1)} + K_2^{(2)} + (K_1^{(1)} - K_2^{(1)} + K_1^{(2)} - K_2^{(2)})\, k\left(\frac{F_2}{F_1}\right)\right] \tag{2.13}$$

and according to formulae (1.19) and (1.23) we obtain:

$$a(F_2) = \frac{1}{\pi}\left[K_2^{(1)} - K_2^{(2)} + (K_1^{(1)} - K_2^{(1)} - K_1^{(2)} + K_2^{(2)})\left(2\frac{F_1}{F_2} - \frac{F_1^2}{F_2^2}\right)\right] \tag{2.14}$$

Since $K_1^{(1)}$ and $K_1^{(2)}$ are known from equ. (2.12), $K_2^{(1)}$ and $K_2^{(2)}$ can be determined from equs. (2.13) and (2.14).

Generally, for the amplitude $B = F_h$ we obtain:

$$b(F_h) = \frac{1}{2}\left[K_h^{(1)} + K_h^{(2)} + \sum_{i=1}^{h-1} (K_i^{(1)} - K_{i-1}^{(1)} + K_i^{(2)} - K_{i-1}^{(2)})\, k\left(\frac{F_h}{F_i}\right)\right] \quad (2.15)$$

$$a(F_h) = \frac{1}{\pi}\left[K_h^{(1)} - K_h^{(2)} + \sum_{i=1}^{h-1} (K_i^{(1)} - K_{i-1}^{(1)} - K_i^{(2)} + K_{i+1}^{(2)})\left(2\frac{F_i}{F_h} - \frac{F_i^2}{F_h^2}\right)\right] \quad (2.16)$$

From previous calculation steps the values

$$K_1^{(j)},\ K_2^{(j)}, \dots K_{h-1}^{(j)} \quad (j = 1.2)$$

of the equs. (2.15) and (2.16) are known, and therefore from these $K_h^{(1)}$ and $K_h^{(2)}$ can be determined.

b) The influence of finite discontinuities of nonlinear characteristics reflect strongly on the coefficients of the describing function in the same manner as for single valued characteristics.

If $f_1(0) \neq 0$ and $f_2(0) \neq 0$ (or only one of these), then, as can be established from equs. (1.16), (1.19) and (1.22), (1.23), either the function $a(B)$ or the function $b(B)$ or both these functions will have an infinite discontinuity for the amplitude $B = 0$.

A finite discontinuity of the nonlinear characteristic produces an infinite discontinuity in the first derivative of the function $b(B)$ (if $D_r^{(1)} \neq -D_r^{(2)}$). It is clear from Appendix 2 that a finite discontinuity in the characteristic produces a finite discontinuity in the first derivative of the function $a(B)$ (if $D_r^{(1)} \neq D_r^{(2)}$).

Therefore, from the shape of the functions $a(B)$ and $b(B)$ it is easy to detect the presence of finite discontinuities in the inverse characteristics. Similarly, as in the case of single valued characteristics $f_1(0)$, $f_2(0)$ and $D_r^{(1)}$, $D_r^{(2)}$ can be determined according to the formulae (1.16), (1.19), and (1.22), (1.23). Selecting one more point inside the given linear section we obtain further two equations which are necessary for determining the unknown parameters.

It was pointed out that for determining single valued inverse characteristics it is possible to select arbitrarily the values of the finite discontinuities. This fact remains correct even in the case of two valued inverse characteristics. Thus, the following values can be assumed [6]:

$$\begin{aligned} f_1(0) &= \frac{\pi F_v}{4}\left[b(F_v) + a(F_v)\right] \\ f_2(0) &= \frac{\pi F_v}{4}\left[b(F_v) - a(F_v)\right] \end{aligned} \quad (2.17)$$

In other locations the finite discontinuities can be determined from the following equations:

$$D_r^{(1)} - D_r^{(2)} = \frac{\pi F_r^2}{2}\left(\frac{da}{dB}\bigg|_{F_{r+0}} - \frac{da}{dB}\bigg|_{F_{r-0}}\right) \tag{2.18}$$

$$D_r^{(1)} + D_r^{(2)} = \frac{2F_r}{l(c)}\left[b(cF_r) - b(F_r)\right] \tag{2.19}$$

Ultimately taking finite discontinuities into consideration, we obtain the following equations for determining $K_h^{(1)}$ and $K_h^{(2)}$ from the equs. (1.16),

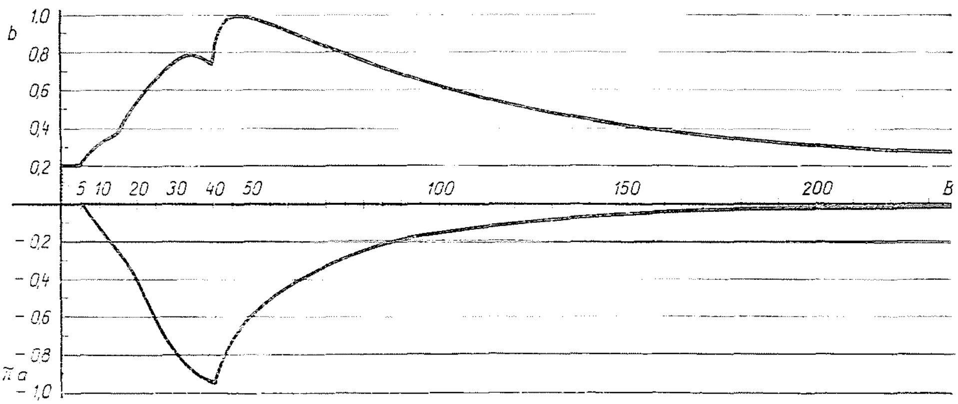

Fig. 5

(1.19) and (1.22), (1.23):

$$b(F_h) = \frac{2[f_1(0) + f_2(0)]}{\pi F_h} + \frac{1}{2}\Bigg\{K_h^{(1)} + K_h^{(2)} + \sum_{i=1}^{h-1}\Bigg[\left(K_i^{(1)} - K_{i+1}^{(1)} + K_i^{(2)} - K_{i+1}^{(2)}\right) k\left(\frac{F_h}{F_i}\right) + \frac{D_i^{(1)} + D_i^{(2)}}{F_i}\, l\left(\frac{F_h}{F_i}\right)\Bigg]\Bigg\} \tag{2.20}$$

and

$$a(F_h) = \frac{1}{\pi}\Bigg\{\frac{2[f_1(0) - f_2(0)]}{F_h} + K_h^{(1)} - K_h^{(2)} + \sum_{i=1}^{h-1}\Bigg[\left(K_i^{(1)} - K_{i+1}^{(1)} - K_i^{(2)} + K_{i+1}^{(2)}\right)\left(2\frac{F_i}{F_h} - \frac{F_i^2}{F_h^2}\right) + 2\,\frac{D_i^{(1)} - D_i^{(2)}}{F_i}\left(\frac{F_i}{F_h} - \frac{F_i^2}{F_h^2}\right)\Bigg]\Bigg\}. \tag{2.21}$$

This technique of determining inverse nonlinear characteristics is fully applicable even in the case of characteristics, the shape of which are frequency dependent. If the shape of the nonlinear characteristic depends on the amplitude of the input signal, then for a single specific amplitude we obtain from the generalised formulae two equations at the most for determining a much larger number of unknowns. Therefore, there is no single solution of the problem.

By means of the described method of determining the inverse nonlinear characteristic the nonlinearity which describes the function illustrated in Fig. 3

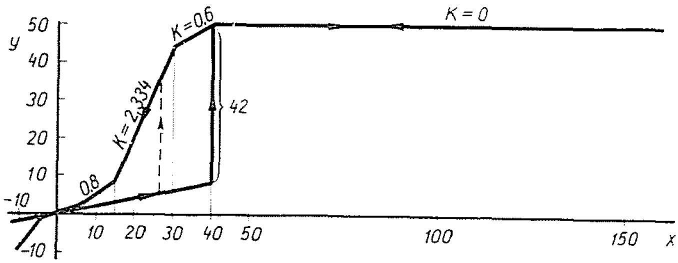

Fig. 6

is found in a form shown in Fig. 4. A detailed solution of this example is described in an earlier work [6], where also the example of determination of a two valued inverse characteristic (Fig. 6) from known coefficients of the describing function (Fig. 5) is solved.

Conclusions

A generalised method for determining the coefficients of the describing function outlined at the beginning of this paper permits direct recording of the describing function of any piecewise linear function with finite discontinuities by means of formal steps. The thus obtained formulae have a simple structure due to the presence in these of special functions. The numerical values of the coefficients of the describing functions can be easily determined by means of tables.

The generalised method for determining the coefficients of the describing function will provide a direct possibility for determining the nonlinearity from its describing function. The thus determined piecewise linear nonlinear characteristic with finite discontinuities will have exact given values of the describing function only in selected points. This method is applicable for single valued as well as for two valued inverse characteristics. The method is relatively simple to apply and suitable for solving engineering problems.

Appendix 1

In equ. (1.20) the continuity of the function $b(B)$ at the point $B = 0$ corresponds to the condition $f(0) = 0$. It can be seen from the same formula that since $k(1) = 1$ and $l(1) = 0$, for amplitudes $B > 0$, the describing function of any piecewise linear characteristic with finite discontinuities will be a continuous function.

From formulae (1.17) and (1.18) we obtain:

$$\frac{dk\left(\frac{B}{F_i}\right)}{dB} = -\frac{4}{\pi}\frac{F_i}{B^3}\sqrt{B^2 - F_i^2}$$

and

$$\frac{dl\left(\frac{B}{F_i}\right)}{dB} = \frac{4}{\pi}\frac{F_i(2F_i^2 - B^2)}{B^3\sqrt{B^2 - F_i^2}}$$

As can be seen from these expressions

$$\lim_{B \to F_i}\frac{dk}{dB} = 0, \qquad \lim_{B \to F_i}\frac{dl}{dB} = \infty$$

For the amplitudes $B > F_i$ the derivatives $\frac{dk}{dB}$ and $\frac{dl}{dB}$ are continuous functions.

On the basis of what has been said above the presence of a finite discontinuity at the origin of the coordinate $[f(0)]$ produces infinite discontinuity of the function $b(B)$ for $B = 0$ and finite discontinuities at other points (D_i) cause an infinite discontinuity of the first derivative of the function $b(B)$.

Appendix 2

The first derivative of the coefficient $a_j\,(B)$ can be determined from formula (1.19). Since

$$\lim_{B \to F_i}\frac{d\left(2\frac{F_i}{B} - \frac{F_i^2}{B^2}\right)}{dB} = 0$$

and

$$\lim_{B \to F_i} \frac{d\left(\frac{F_i}{B} - \frac{F_i^2}{B}\right)}{dB} = \frac{1}{F_i}$$

therefore at the amplitudes $B > 0$ the function $\frac{da_j}{dB}$ will be continuous except for those points where the nonlinear characteristic has a finite discontinuity. In the indicated points the function $\frac{da_j}{dB}$ has a finite discontinuity. According to equ. (1.19) the magnitude of these discontinuities will be

$$D_r^{(j)} = \pi F_r^2 \left(\frac{da}{dB}\bigg|_{F_{r-0}} - \frac{da}{dB}\bigg|_{F_{r-0}} \right) .$$

Summary

This paper gives a review of the general method for determining the describing function in case of piecewise linear nonlinear characteristics having discontinuities of the first kind. This method establishes direct possibility for the solution of the inverse problem, that is for determining the nonlinearity on the basis of its describing function. The inverse characteristic is sought for in piecewise linear nonlinear form with discontinuities of the first kind. This nonlinear characteristic corresponds to the given values of the describing function in the preselected points.

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