THE GENERALIZATION OF THE THEOREM OF THREE MOMENTS

CLAPEYRON'S EQUATION

By

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Introduction

The aim of the present paper is to examine some problems of plane, straight and statically indeterminate beams.

A structure is called statically indeterminate if some of the internal and external forces cannot be determined from statical equations. Statical indeterminateness may have its origin e.g. in the statical indeterminateness of the supports of the structure, i.e. the reactive forces cannot be determined from statical equations. In such cases some of the internal forces cannot be determined either, consequently the structure is indeterminate both externally and internally. It may, however, occur that all the external forces are known or can be determined from statical equations, but the same is not valid for all the internal forces. In this case we may speak of internal indeterminateness. If the number of independent statical equations is k, the number of unknown forces n, then the degree of statical indeterminateness is the difference of the above two values, i.e.

$$n-k = H.$$

The other equations necessary for determining the unknown forces are to be set up on the basis of the correlations for the deflections of the structure. The so-called work theorems, such as those of Castigliano and of Betti, are of general validity and have a great significance on examining indeterminate structures, just on account of their general character. The Theorem of three moments (Clapeyron's equations), which has an important role in the theory of continuous straight beams, can also be derived from the above-mentioned work theorems.

1. The Theorem of three moments in its form, valid for statically determinate beam structures, expresses that the displacement of some point on the beam structure is in a determined direction, or the angular displacement of some beam cross section around a given axis is equal to the partial derivative of the deflection work with respect to the force acting at the examined point

in the direction of displacement, or with respect to the couple having a moment parallel to the axis of rotation, respectively. If we wish to employ the theorem for statically indeterminate structures, the constraints which are redundant from the aspect of supports, should be eliminated and in their place suitable forces and couples are to be made to act on the structure. Let $x_1, x_2, \ldots, x_i$ designate these forces and couples. Since the eliminated constraints were destined to prevent linear or angular displacements at certain points of the structure, we may write for these points, by force of the theorem, that

$$\frac{\partial U}{\partial X_1} = 0 , \quad \frac{\partial U}{\partial X_2} = 0 , \quad \frac{\partial U}{\partial X_i} = 0 .$$

As a final result, we may write in this way as many equations as the degree of statical indeterminateness.

2. According to Betti's theorem, if the suitably supported structure is loaded by two different systems of forces, then, depending on the sequence of loads in time, the work U_{12} performed by the first system during the deflection caused by the second system of forces is equal to the work U_{21} performed by the second system during the deflection caused by the first system.

Let e.g. f_K denote the deflection caused by the load at point K of a beam which has a statically determined and frictionless support. We shall regard the system of forces loading the beam as one of the force systems. The second force system consists of the unit force acting at point K in direction e, and of the pertaining reactive forces. In the absence of friction, the reactive forces do not perform any work during the deflection.

If the unit force, i.e. the second force system is applied to the beam first, and the actual load of the beam afterwards, then the unit force will perform the work

$$U_{21} = \bar{f}_k \bar{e}$$

while the actual load is being applied. The scalar projection of the required displacement $\bar{f}_k$ in the direction $\bar{e}$ can thus be easily calculated with the knowledge of U_{21}, which can be determined by the aid of Betti's theorem ($U_{21} = U_{12}$) on the one hand, and on the basis of the fact that the work of external forces is accumulated in the beam in the form of potential energy (spring energy), on the other hand.

The angular displacement of some cross section around a certain prescribed axis can be determined in a similar way.

If we want to employ the theorem for statically indeterminate structures, then we have to make first the structure statically determinate, as in the case of employing Castigliano's theorem. In the place of the removed constraints, linear and angular displacements have zero value. Accordingly the work of unit forces and couples employed here in the suitable direction will be zero:

$$U_{21} = \bar{f}_k \bar{e} = 0$$

or

$$U_{21} = \varphi_k 1 = 0 .$$

We may write just as many equations with the aid of Betti's theorem as the degree of indeterminateness.

3. The theorem of three moments (Clapeyron's equations). Clapeyron's equations serve to determine the bending moments at the vertical plane of the intermediate supports of a continuous beam. By each Clapeyron's equation the correlation between moments arising in the vertical planes of three subsequent supports is established. In the case of continuous beams one support is a joint, while the others are roller supports. The force arising at the joint is determined by two data, while those at the other supports by one value each, consequently $n+1$ equations are necessary for determining the reactive forces of an n-support beam. In the case of a beam with $n-1$ supports we may write $n-2$ Clapeyron's equations, thus these are sufficient, together with the three equilibrium equations, for the determination of the reactive forces of a multi-support beam.

4. Whichever of the described methods is being applied, we obtain a system of equations consisting of as many equations as the number of unknown forces. In the general case the solution of the system of equations requires, even for a relatively small number of unknown values, a tedious and lengthy work.

The solution will be simplified if we succeed to reduce the system of equations to smaller groups of equations independent of each other. The most favourable case is when each equation contains only one unknown value, that means that all the unknown quantities can be determined as the solution of a single equation, independently of the others.

Statically indeterminate beam structures can be made statically determinate by inserting joints, and the moments of couples, made to act at the place of the joints as a substitute for the undone material coherence, can be determined e.g. with the aid of the work theorem. By employing this method, joints used to be inserted above the supports of continuous beams. This method leads to the Theorem of three moments (Clapeyron's equations). If, however, the beam is transformed to a statically determinate Gerber's beam, with the aid of suitably arranged joints in the individual spans, then we obtain such a system of equations in which each equation contains only a single unknown quantity.

5. In the following we calculate for the case of a straight flexural member the value of the work $U_{21} = U_{12}$ which is necessary for the application of Betti's theorem.

In the case of bending

$$U = \int_0^L \frac{M_x^2}{21_x E} dz \tag{1}$$

is the deformation work (the x axis being the neutral axis of the cross section, and M_x the scalar projection in the x direction of the moment vector acting on the cross section).

By employing Betti's theorem, the beam is loaded by two systems of forces. Accordingly the moment function will also be the sum of the moment functions originating from the two loads. Let M_x denote the bending moment pertaining to the actual load of the beam, while m the second unit load. The total moment is

$$\overline{M}'_x = \overline{M}_x + \overline{m}_x \tag{2}$$

and

$$(\overline{M}_x + \overline{m}_x)^2 = M_x^2 + 2M_x m_x + m_x^2 .$$

Thus, if $\overline{M}$ and $\overline{m}$ are parallel,

$$U = \int_0^L \frac{M_x^2}{2I_x E} dz + \int_0^L \frac{M_x m_x}{I_x E} dz + \int_0^L \frac{m_x^2}{2I_x E} dz . \tag{3}$$

It is obvious that the first term is the work U_{11} performed by the first system of forces during the deflection caused by itself. The last term is U_{22} and thus

$$U_{21} = \int_0^L \frac{M_x m_x}{I_x E} dz = f_{yk} . \tag{4}$$

In the case of a beam with constant cross section the term $\frac{1}{I_x E}$ can be brought in front of the sign of integration.

On calculating the deformation work, the work originating from tension and shearing is generally neglected beside the work originating from bending.

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The aim of the present paper is to show that for an arbitrarily taken part with n supports on a continuous multi-support beam an equation, similar to Clapeyron's equation, can be written which establishes a correlation between the support point moments of the beam section, and which results in Clapeyron's equation in the case of $n = 3$. This equation may be named the generalization of Clapeyron's equation.

The generalization of Clapeyron's equations

A beam with n supports (Fig. 1) is $(n+1) - 3 = H = (n-2)$ fold indeterminate. To make it determinate we must remove a corresponding number of constraints, e.g. by removing $n-2$ roller supports and by simultaneously making the suitable forces act at the place of the removed supports.

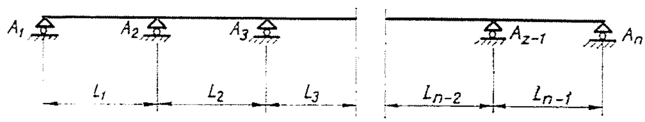

Fig. 1

Assuming a constant cross section, let us express the displacement at the place of one of the removed supports with the aid of Betti's theorem (Fig. 2).

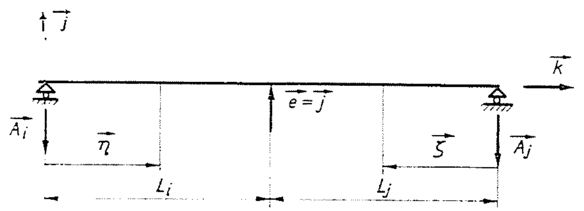

Fig. 2

At the place of the support the displacement is zero, therefore

$$f = \frac{1}{IE} \int_L \overline{M}\overline{m} ds = \frac{1}{IE} \left(\int_{L_i} \overline{M}\overline{m}_i ds + \int_{L_j} \overline{M}\overline{m}_j ds \right) = 0. \tag{5}$$

By writing in sections and by calculating with the vectors $\bar{\eta} = \eta\, \bar{k}$, $\bar{\zeta} = \zeta\, \bar{k}$,

$$\begin{aligned} \overline{m}_i &= \bar{A}_i x \bar{\eta} \quad (0 < \eta < L_i) \\ \overline{m}_j &= -\bar{A}_j x \bar{\zeta} \quad (-L_j < \bar{\zeta} < 0) \end{aligned} \tag{6}$$

$\overline{m} = \overline{m}\,(z)$ is the moment function pertaining to the unit load $\bar{e}$.

Upon substituting from (6) into equation (5),

$$\int \overline{m}\overline{M} ds = 0 = \bar{A}_i \int_{L_i} \bar{\eta} x \overline{M} ds - \bar{A}_j \int_{L_i} \bar{\zeta} x \overline{M} ds = 0. \tag{7}$$

The integrals represent the moments of the moment systems $\overline{M}$ pertaining to the individual sections, about points A_i and A_j respectively.

$$Q_i = \int_{L_j} \overline{M} ds = Q_i \bar{i} \qquad \text{and} \qquad Q_j = \int_{L_j} \overline{M} ds = Q_j \bar{i}$$

denote the resultant of the system of moment vectors on the sections L_i and L_j, respectively, $\bar{\eta}_{i0} = \eta_{i0}\bar{k}$ is the moment arm of Q_i, and $\bar{\zeta}_{j0} = \zeta_{j0j}$ That of Q_j, 3), then

$$\int_{L_i} \bar{\eta} x \overline{M} ds = \vec{\eta}_{i0} x \bar{Q}_i ,$$
$$\int_{L_j} \bar{\zeta} x \overline{M} ds = \bar{\zeta}_{j0} x \bar{Q}_j . \tag{8}$$

(In Fig. 3 η_{io} is positive, $\bar{\zeta}_{io}$ negative.) On the other hand

$$\bar{A}_i = -\frac{L_j}{L_i + L_j}\bar{j} ; \qquad \bar{A}_j = -\frac{L_i}{L_i + L_j}\dot{j} .$$

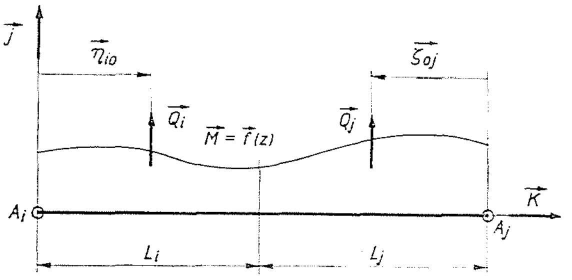

Fig. 3

Accordingly, upon substituting into (7),

$$-\frac{L_j}{L_i + L_j}\bar{j}\int \bar{\eta} x M ds + \frac{L_i}{L_i + L_j} j \int \bar{\zeta} x M ds = 0 ,$$

or

$$L_j \bar{j} \int_{L_i} \bar{\eta} x \overline{M} ds = L_i j \int_{L_j} \bar{\zeta} x \overline{M} ds .$$

From this, by considering (8),

$$\bar{j} \int_{L_i} \bar{\eta} x \overline{M} ds = \eta_{i0} \qquad Q_i j (\bar{k} x \bar{i}) = \eta_{i0} Q_i \tag{9}$$

and on the basis of

$$\bar{j} \int \bar{\zeta} x \overline{M} ds = \zeta_{j0} Q_j \tag{10}$$

we obtain

$$L_j \eta_{i0} Q_i = L_i \zeta_{j0} Q_j \tag{11}$$

or

$$\frac{\eta_{i0} Q_i}{\zeta_{j0} Q_j} = \frac{L_i}{L_j} . \tag{12}$$

By equation (12) the following is expressed: The moments of the moment vector systems on the sections L_i and L_j, — the beam sections to the left and right of the examined support, respectively, — about the supports A_i and A_j, respectively, are in proportion to the lengths of the corresponding sections. In the course of the deduction we have had no restrictions as regards the supports to be removed, with the exception of their number. This means that the remaining supports A_i and A_j may be any two of the original supports. The support, for the displacement of which Betti's theorem has been employed, had similarly been chosen arbitrarily. All this means that equations (11) and (12), respectively, are valid for any section on the beam between two, not adjacent, supports. Correlation (12) may be regarded as the generalized form of Clapeyron's equation. To verify this assertion let us apply correlation (11) for the arbitrarily taken section with n supports from the examined beam. The beam is transformed, with the aid of the joints inserted above the supports, to a statically determinate Gerber's beam, as shown in the figure. In the place of the individual supports, the material continuity is substituted by introducing suitable couples. These couples are just equal to the support point moments of the original beam.

The moment function of the beam, as obtained by superposition, can be written in the form

$$M = M_a + \Sigma M_i m_i$$

where M_a is the moment on the Gerber's beam originating from the load, $m_i = m_i(z)$ which denotes the moment function pertaining to the unit couple loading applied at the i-th joint. These are shown in Fig. 4. So as to apply equation (11), the beam is cut into the two sections as shown in the figure. The moment of the system of moment vectors should be calculated on the lines 1 and 2, in the left side section of length L_1 and the right side, respectively. With the notations of the figure,

$$Q_{21} = \frac{M_1 L_2}{2}, \qquad \eta_{21} = \frac{1}{3} L_2$$

$$Q_{12} = \frac{M_2 L_2}{2}, \qquad \eta_{12} = \frac{2}{3} L_2$$

$$Q_{22} = \frac{M_2 L_3}{2}, \qquad \eta_{22} = L_2 + \frac{1}{3} L_3$$

$$Q_{13} = \frac{M_3 L_3}{2}, \qquad \eta_{13} = L_2 + \frac{2}{3} L_3$$

$$Q_{23} = \frac{M_3 L_4}{2}, \qquad \eta_{23} = L_2 + L_3 + \frac{1}{3} L_4$$

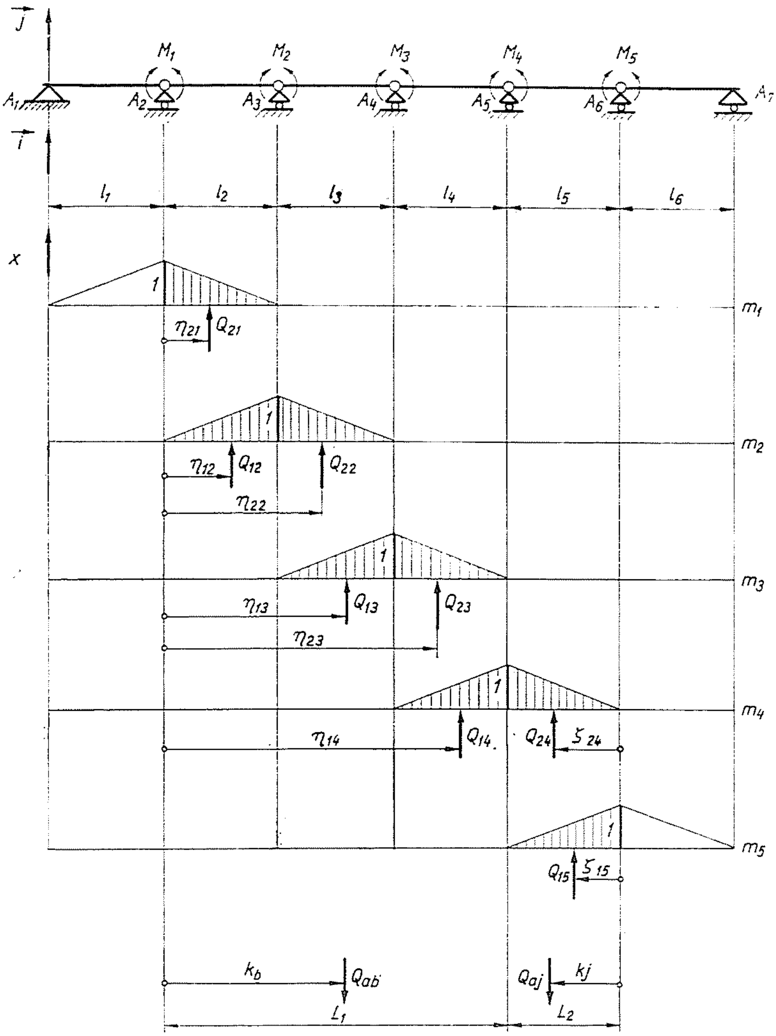

Fig. 4

$$Q_{14} = \frac{M_4 L_4}{2}, \qquad \eta_{14} = L_2 + L_3 + \frac{2}{3} L_4$$

$$Q_{24} = \frac{L_5 M_4}{2}, \qquad \zeta_{24} = -\frac{2}{3} L_5$$

$$Q_{25} = \frac{L_5 M_5}{2}, \qquad \zeta_{15} \quad -\frac{1}{3} L_5$$

Let Q_{ab} and Q_{aj} denote the resultant of the moments M_a, originating from the external load, on the two beam sections, respectively, while k_b and k_j the distance of these resultants from the lines 1 and 2, respectively. (In Fig. 4 Q_{ab} and Q_{aj} are negative, k_b is positive, and k_j negative.) With these values, equation (11) takes the following form:

$$L_2^* \left(Q_{ab} k_b + \sum_{i=1}^{3} Q_{2i}\, \eta_{2i} + \sum_{i=2}^{4} Q_{1i}\, \eta_{1i}\right) = L_1^* \left(Q_{aj} k_j + Q_{24} \zeta_{24} + Q_{15} \zeta_{15}\right).$$

Upon arranging the terms with unknown quantities on the left side, we find that

$$L_2^* \left(\sum_{i=1}^{3} Q_{21}\, \eta_{21} + \sum_{i=2}^{4} Q_{1i}\, \eta_{1i}\right) - L_1^* \left(Q_{24} \zeta_{24} + Q_{15} \zeta_{15}\right) = -L_2^*\, Q_{ab}\, k_b + L_1^*\, Q_{aj}\, k_j .$$

Or, with the former values of Q_{vi}, η_{vi} and ζ_{vi}, substitute the previously expressed Q, η and ζ values, as well as the L_1^* and L_2^* values, and multiply the equation by $\frac{6}{L_1^*\ L_2^*}$.

$$\frac{1}{L_2 + L_3 + L_4} \left[L_2^2 M_1 + (2\, L_2^2 + 3\, L_2\, L_3 + L_3^2)\, M_2\right] +$$

$$+ \frac{1}{L_2 + L_3 + L_4} \left\{\left[L_3 (2\, L_3 + L_2) + L_4 (L_4 + 3\, L_3 + 3\, L_2)\right] M_3\right\} +$$

$$+ \frac{1}{L_2 + L_3 + L_4} \left\{\left[L_4 (2\, L_4 + 3\, L_3 + 3\, L_2) + 2\, L_5 (L_2 + L_3 + L_4)\right] M_4\right\} +$$

$$+ \frac{1}{L_2 + L_3 + L_4} \left\{\left[L_5 (L_2 + L_3 + L_4)\right] M_5\right\} = - \frac{6 Q_{ab} k_b}{L_2 + L_3 + L_4} + \frac{6 Q_{aj} k_j}{L_5}.$$

By the equation a correlation between the support point moments of the examined beam section is expressed. Its character is obviously similar to that of the Theorem of three moments (Clapeyron's equation), which can be written for the three-support beam. If the equation is employed for a three-support beam section, in the manner described above, we obtain just Clapeyron's equation. Accordingly, equation (11) can actually be regarded as the generalization of the Theorem of three moments.

Summary

In the present paper the well-known Theorem of three moments (Clapeyron's equation), which relates to a beam section defined by three subsequent supports, is generalized for a beam section defined by arbitrarily chosen three supports and divided into two by a middle support. The character of the equation is evidently similar to that of Clapeyron's equation written for a three-support beam. If the equation described in the paper is employed for a three-support beam section, we obtain just Clapeyron's equation. Accordingly equation (11) can actually be regarded as the generalization of the Theorem of three moments.

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