SOME REMARKS ON THE CONSTRUCTION OF PHASE TRAJECTORIES

By

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1. Introduction

In control engineering, mechanics, electrical engineering, electronics for the construction of the phase trajectories of the differential equation

$$\ddot{x} + f(\dot{x}) + g(x) = 0$$

the PELL method, or for the simpler case when $g(x) = x$, the LIÉNARD method is used. A common disadvantage of both methods is the fact that for the construction of the phase trajectory only the tangent is precisely determined at each point, while an elementary section of the trajectory is fitted by an arc whose radius is only approximately true.

The purpose of the present study is to clarify, on the one hand, how much the accuracy of both methods is effected by the parameters of the differential equations, and, on the other hand, to find an adequate method of construction for the exact determination of the center and radius of curvature of the phase trajectory at each point.

2. The Pell method [1]

A differential equation to be solved is:

$$\ddot{x} + f(\dot{x}) + g(x) = 0 \tag{2—1}$$

Let it be

$$\dot{x} = v$$

then

$$\ddot{x} = \frac{dv}{dx} \cdot \frac{dx}{dt} = \frac{dv}{dx} \cdot \dot{x} = \frac{dv}{dx} \cdot v$$

Thus

$$\frac{dv}{dx} = -\frac{f(v) + g(x)}{v} \tag{2—2}$$

The PELL method [1] can be summarized as follows:

a) The curves $-x = f(v)$ and $-v = g(x)$ are drawn.

b) At the point F (Fig. 1) an angle of 45° is constructed. Thus the point

L is determined.

c) Drawing a parallel line to $\overline{HL}$ the point R is settled.

d) The direction of $\overline{RP}$ is the direction of the normal of the trajectory.

A small section of the curve beginning from the point P is approximated by an arc with centre R. Then from the new point the whole construction is repeated.

The beginning point P is fixed by the initial conditions $x(0)$ and $v(0)$.

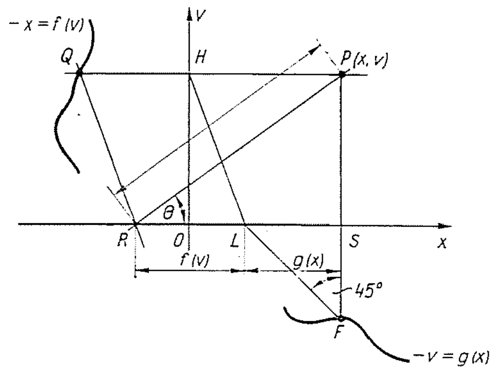

Fig. 1

3. Determination of the center of curvature

The accuracy of the PELL method can be greatly increased by the determination of the center of curvature. The course of the proposed construction can be depicted from Figs. 2 and 3.

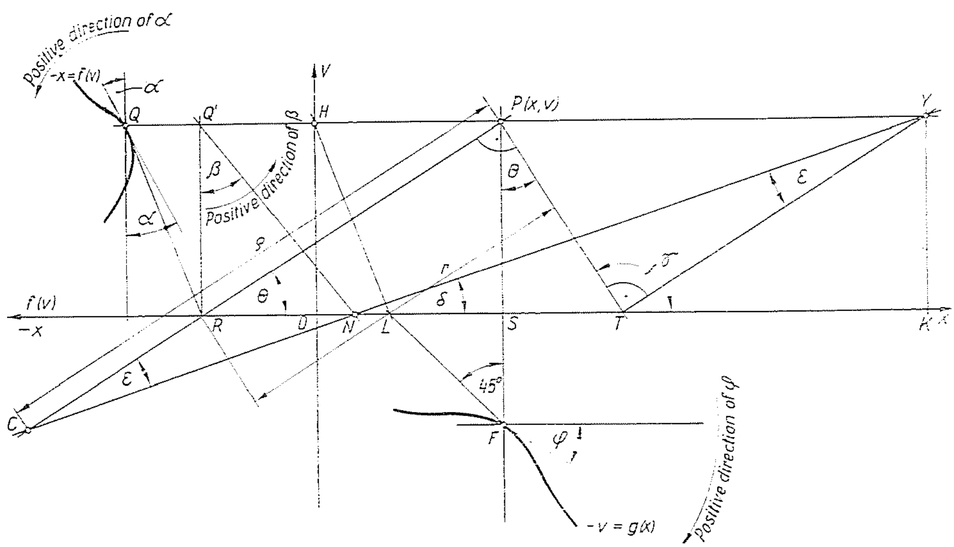

Fig. 2

The proposed method can be clarified by the following steps:

a) The PELL construction is performed according to clause 2.

b) Drawing in the point P a line at right angle to $\overline{PR}$, point T results.

c) The intersection of the continuation of $\overline{QP}$ and the parallel line from T to $\overline{PR}$ gives point Y.

d) Drawing a perpendicular line to the x-axis in point R yields point Q.

e) In point Q' the angle β is constructed in the shown manner. The side of the angle determines point N on the x-axis.

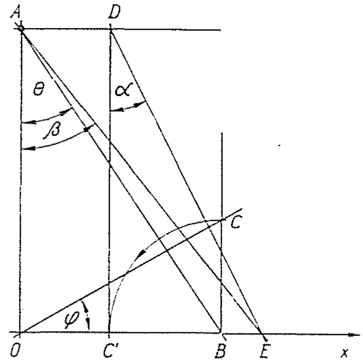

Fig. 3

f) The intersection of the continuation of $\overline{PR}$ and $\overline{NY}$ finally gives the center of curvature C. The radius of curvature ϱ is given by $\overline{CP}$.

An auxiliary construction is necessary for the determination of the angle β (Fig. 3). Let us introduce the following notations:

$$\frac{df(v)}{dv} = f'(v) = \tan \alpha \tag{3—1}$$

$$\frac{dv}{dx} = v'(x) = \tan \gamma \tag{3—2}$$

$$\frac{dg(x)}{dx} = g'(x) = \tan \varphi \tag{3—3}$$

The steps of the auxiliary construction are

a) At the point O angle φ is constructed.

b) The section $\overline{OA}$ is arbitrarily but adequately chosen.

c) At point A the angle Θ is constructed, the side of Θ determines point B.

d) A perpendicular line at point B gives the point C.

e) The latter point is turned around point B, thus, the point C' is obtained.

f) Constructing the angle α at point D the side yields point E.

h) The angle OAE is a final result for obtaining the angle β.

The four special cases for the construction of angle β in the function of α and φ are summarized in Fig. 4.

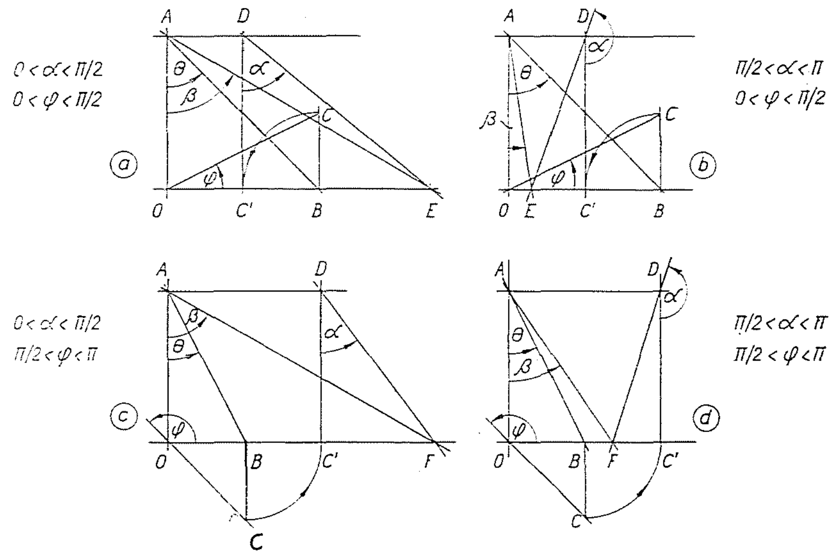

Fig. 4

4. The verification of the proposed construction

The solution of the differential equation

$$\ddot{x} + f(\dot{x}) + g(x) = 0$$

is assumed in the form $v = v(x)$, where $v = \dot{x}$. In other words, the expression of the phase trajectory is $v = v(x)$.

The radius of curvature is given by the formula:

$$\varrho = \frac{(1 + v'^2)^{3/2}}{v''} \tag{4—1}$$

Taking the relation

$$\frac{dv}{dx} = - \frac{f(v) + g(x)}{v} = v'$$

into consideration v'' is determined in the following way

$$v'' = -\frac{d}{dx}\,\frac{f(v)+g(x)}{v} = -\frac{\left[\frac{df(v)}{dv}v' + g'(x)\right]v - [f(v)+g(x)]\,v'}{v^2} =$$

$$= -\frac{[f'(v)\cdot v' + g'(x)] + \left[-\frac{f(v)+g(x)}{v}\,v'\right]}{v} =$$

$$= -\frac{[f'(v)v' + g'(x)] + v'^2}{v} \tag{4—2}$$

$$\text{where } g'(x) = \frac{d}{dx}g(x)$$

$$\text{and } f'(v) = \frac{d}{dv}f(v)$$

Substituting (4—2) into (4—1):

$$\varrho = -\frac{(1+v'^2)^{3/2}\cdot v}{f'(v)\cdot v' + g'(x) + v'^2} = -\frac{(1+v'^2)^{3/2}\cdot v}{f'(v)\cdot v' + g'(x) - 1 + (1+v'^2)}$$

Applying the notations of clause 3

$$\varrho = -\frac{\frac{1}{\cos^3\gamma}\cdot v}{\tan\alpha\cdot\tan\gamma + \tan\varphi - 1 + \frac{1}{\cos^2\gamma}}\ ;\ \text{as } 1 + v'^2 = \frac{1}{\cos^2\gamma}$$

After some algebraic manipulations and taking into account the relations

$$\cos\gamma = -\sin\Theta \text{ and } v/\sin\Theta = \overline{PR} = r$$

$$\sin\gamma = \cos\Theta$$

the following expression is obtained:

$$\varrho = \frac{r}{1 - \sin\Theta\cdot\cos\Theta\left[\tan\alpha - \tan\Theta\,(\tan\varphi - 1)\right]}$$

Let us introduce the relation

$$\tan\beta = \tan\alpha - \tan\Theta\ (\tan\varphi - 1) \tag{4—3}$$

As a final result, the formula of the radius of curvature is

$$\varrho = \frac{r}{1 - \sin\Theta\cdot\cos\Theta\cdot\tan\beta} \tag{4—4}$$

The validity of the construction can be verified on the basis of Fig. 2 by the following derivation:

From triangle MYT

$$\overline{PT} = r \cdot \tan \Theta$$

$$PCM \sphericalangle = MYT \sphericalangle = \varepsilon$$

Consequently

$$\overline{PT} = r \cdot \tan \varepsilon + \varrho \cdot \tan \varepsilon$$

Thus

$$r \cdot \tan \Theta = r \cdot \tan \varepsilon + \varrho \cdot \tan \varepsilon$$

Hence

$$\varrho = \frac{\tan \Theta - \tan \varepsilon}{\tan \varepsilon} r \tag{4—5}$$

Let us introduce the notation

$$\varepsilon = \Theta - \delta$$

thus

$$\tan \varepsilon = \frac{\tan \Theta - \tan \delta}{1 + \tan \Theta \cdot \tan \delta} \tag{4—6}$$

and as from triangle NKY

$$\tan \delta = \frac{v}{2 \cdot \dfrac{v}{\tan \Theta} + v \cdot \tan \Theta - v \cdot \tan \beta}$$

consequently

$$\tan \delta = \frac{\tan \Theta}{2 - \tan \Theta \cdot \tan \beta + \tan^2 \Theta} \tag{4—7}$$

After substituting Eqs. (4—6) and (4—7) into (4—5) and performing some algebraic manipulations the final result (4—4) is indeed obtained. Thus, the property of the construction is justified.

Now let us prove the truth of the auxiliary construction.

On the basis of Fig. 3

$$\overline{OB} = \overline{OA} \cdot \tan \Theta$$

$$\overline{BC} = \overline{OB} \cdot \tan \varphi = \overline{OA} \cdot \tan \Theta \cdot \tan \varphi$$

$$\overline{OC'} = \overline{OB} - \overline{C'B} = \overline{OB} - \overline{CB} = \overline{OA} (\tan \Theta - \tan \Theta \cdot \tan \varphi)$$

$$\overline{C'E} = \overline{OA} \cdot \tan \alpha$$

Thus

$$\overline{OE} = \overline{OA}\,[\tan \alpha - \tan \Theta\,(\tan \varphi - 1)]$$

that is

$$\tan \beta = \frac{\overline{OE}}{\overline{OA}} = \tan \alpha - \tan \Theta\,(\tan \varphi - 1) \qquad (4\text{—}8)$$

Eq. (4—8) is the very same as the definition (4—3), thus the auxiliary construction for obtaining the angle β is indeed true.

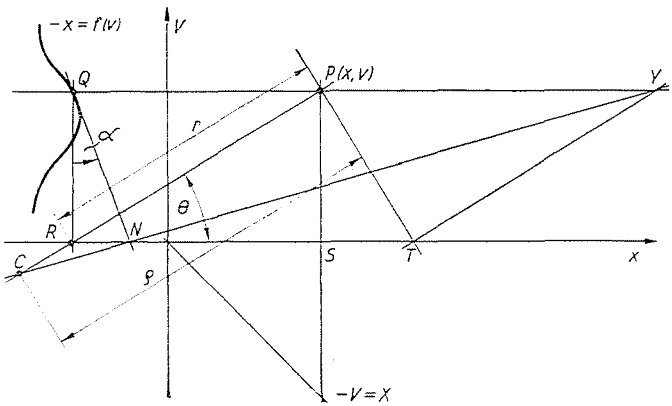

Fig. 5

5. Special cases

5.1. *Modification of the* LIÉNARD *method [2]*

The LIÉNARD method is nothing but a special case of the PELL method, when $g(x) = x$.

In the latter case our proposed method is also simplified significantly and we obtain the modified version of the LIÉNARD construction (see [2]).

Taking $\dot{x} = \tan \varphi = 1$ into consideration it can be seen from Eq. (4—3) that in this case $\tan \beta = \tan \alpha$, thus the auxiliary construction can be left out. The modified LIÉNARD construction is shown for this case in Fig. 5.

As

$$\varrho = \frac{r}{1 - \sin \Theta \cdot \cos \Theta \cdot f'(v)}$$

if $\sin \Theta \cdot \cos \Theta \cdot f'(v) \approx 0$ then $\varphi \approx r$, consequently the LIÉNARD method gives a good approximation. This arrives in the vicinity of $\Theta = 0°$, because of $\sin 0° = 0$, and in the vicinity of $\Theta = 90°$, because of $\cos 90° = 0$ and otherwise if $f'(v)$ is small enough.

5.2 *Second special case*

Let us now assume that

$$f(\dot{x}) = f(v) = v$$

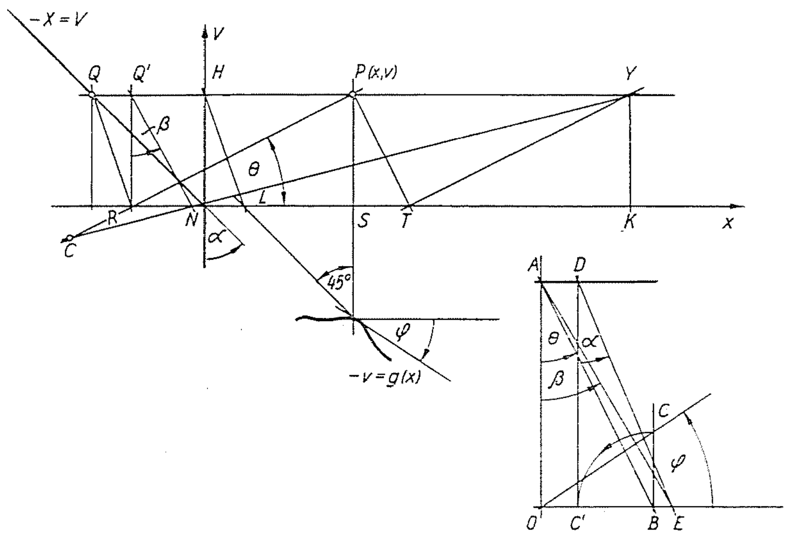

Fig. 6

Then the differential equation takes the following special form

$$\ddot{x} + \dot{x} + g(x) = 0$$

The construction for this case is shown in Fig. 6.

5.3 *Third special case*

If

$$f(\dot{x}) = f(v) = 2\,\zeta\,v$$

and at the same time

$$g(x) = x$$

then the nonlinear differential equation is reduced to a linear one:

$$\ddot{x} + 2\,\zeta\,\dot{x} + x = 0$$

The construction for the case of $2\,\zeta = 1$ is shown in Fig. 7.

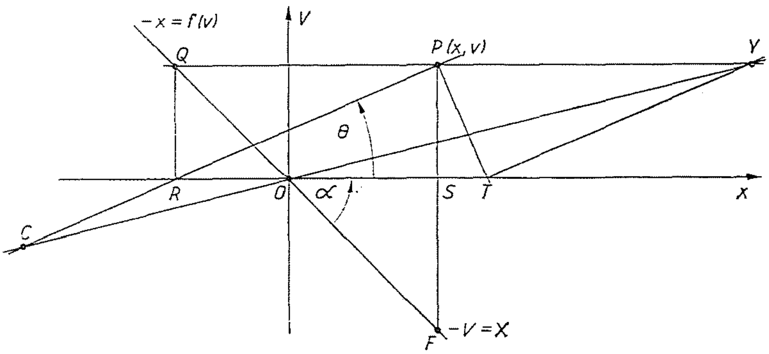

Fig. 7

6. Conclusions

In the previous treatment a construction was proposed for the determination of the centre and radius of curvature. The proposed method enables us to increase the accuracy of the drawing of the phase trajectories in the general enough case of Eq. (2—1). This method can be regarded as an essential development of the well-known PELL method.

The proposed method contains as a special case the modified LIÉNARD construction [2] in the same manner as the original LIÉNARD construction is a special case of the original PELL method.

Summary

A construction was proposed for obtaining the center and radius of curvature of the phase trajectories. The accuracy of drawing in the trajectories can thus be augmented. Some special cases are also given.

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