THE STATISTICAL ANALYSIS OF SAMPLED-DATA CONTROL SYSTEMS

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1. Mean-square errors

A current method of the analysis and synthesis of linear control systems is the *statistical method*. When employing this, the *input signal* ($r = s+n$) of the system is regarded as stationary and ergodic stochastic signal. In this case the *output signal* (c) itself is also stochastic. A *required* (ideal) output signal (i) is ordered to the *control input*. The *error signal* (e) is the difference of the ideal and of the actual output signal,

$$e(t) = i(t) - c(t). \tag{1}$$

One of the characteristics of the quality of the system is the mean of the square of the error signal, shortly the mean-square error.

Our task is the interpretation and calculation of the mean-square error in the case of a *linear sampled-data system*. It is assumed that the sampling period T is constant and the duration of sampling T_0 is much shorter, thus the sampled signals can correctly be approximated by Dirac impulses. The weighting function $w(t)$ of the closed system and the ideal weighting function $y(t)$ are regarded as given.

Two mean-square errors can be defined for sampled-data systems. The calculation is more simple, but the information is less in the case of the *discrete mean-square error*

$$\zeta^2 = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} e^2(nT) = \mathrm{M}_n\left\{e^2(nT)\right\}. \tag{2}$$

The calculation is more difficult, the information content, however, is greater in the case of the *continuous mean-square error*

$$\xi^2 = \lim_{T'\to\infty} \frac{1}{2T'} \int_{-T'}^{T'} e^2(t)\,dt = \mathrm{M}_n\left\{\frac{1}{T} \int_{nT-T}^{nT} e^2(t)\,dt\right\}. \tag{3}$$

It is known from the theory of continuous systems that the mean-square can be easily determined in the knowledge of the correlation function of the signal. On examining a sampled-data system, however, the difficulty is that the output signal $c(t)$ is not stationary, consequently its correlation function depends not only on the displacement time τ, but also on the examined moment, thus being a function of the latter only in the sense modulo T [8]. Accordingly the output signal is not ergodic, i.e. the correlation function can be interpreted only as an ensemble average, whereby its application and measurement are further made more difficult. This fact is often left out of consideration, thus faulty results are obtained [6, 9, 10].

2. Correlation sequences

The modified and simple discrete functions,

$$c\,[n, m] \equiv c\,(nT - T + m\,T),\; c\,[n] \equiv c\,[n+1, 0] \equiv c\,(nT + 0) \tag{4}$$

of the output signal $c(t)$ of the sampled-data system with stationary input signal, respectively, have the following characteristics: In the case of $c\,[n]$ and fixed $m\,(0 \leq m < 1)$, the sequence $c\,[n,m]$ is stationary. The signal $c\,[n,m]$ itself, however, is not stationary. A signal of this type will be denominated as *quasistationary*. We shall assume that the sequences are not only stationary in n, but also ergodic.

Let u and v designate quasistationary signals. The *simple correlation sequence* of these can be defined on the basis of the ensemble average as follows:

$$\psi_{uv}[k] = \lim_{M \to \infty} \sum_{i=1}^{M} u^{(i)}\,[n] \cdot v^{(i)}\,[n+k] \equiv \mathrm{E}_i \left\{ u^{(i)}\,[n] \cdot v^{(i)}\,[n+k] \right\} \cdot \tag{5}$$

On using the assumed ergodicity, we may also go over to the mean with respect to time:

$$\psi_{uv}\,[k] = \mathrm{M}_n \left\{ u\,[n] \cdot v\,[n+k] \right\}. \tag{6}$$

The *generalized correlation sequence* of the couple of signals is defined in thsis manner:

$$\psi_{uv}[k\,; m, h] \equiv \mathrm{E}_i \left\{ u^{(i)}\,[n, m]\; v^{(i)}\,[n+k, h] \right\}. \tag{7}$$

On taking the assumed ergodicity in n into consideration:

$$\psi_{uv}[k\,; m, h] = \mathrm{M}_n \left\{ u\,[n, m] \cdot v\,[n+k, h] \right\}. \tag{8}$$

Two special cases of this are

$$\psi_{uv}[k, m] \equiv \psi_{uv}[k\,; m, m]\,, \tag{9}$$

$$\psi_{uv}[k] \equiv \psi_{uv}[k\,; 0) \equiv \psi_{uv}[k\,; 0{,}0]. \tag{10}$$

The two-variable correlation function of the quasistationary couple of signals is

$$\varphi_{uv}(\tau, t) = \mathrm{E}_i\left\{u^{(i)}(t)\, v^{(i)}(t+\tau)\right\}. \tag{11}$$

This is periodic in t with respect to T. It can be easily realized that

$$\psi_{uv}[k] = \varphi_{uv}(kT, 0)\,, \tag{12}$$

$$\psi_{uv}[k\,; m, h] = \varphi_{uv}(kT + hT - mT, mT)\,. \tag{13}$$

If in turn $u = x$, $v = y$ are stationary, their correlation function is

$$\varphi_{xy}(\tau) = \mathrm{E}_i\left\{u^{(i)}(t)\;\; v^{(i)}(t+\tau)\right\}. \tag{14}$$

Let $t = nT$, and $t = nT - T + mT$, then it can easily be realized that

$$\psi_{xy}[k] = \varphi_{xy}[k]\,, \tag{15}$$

$$\psi_{xy}[k\,; m, h] = \begin{cases}\varphi_{xy}[k, h - m + 1]\,, & -1 \leqq h - m < 0, \\ \varphi_{xy}[k + 1\,, h - m]\,, & 0 \leqq h - m < 1\,,\end{cases} \tag{16}$$

$$\psi_{xy}[k\,; m] = \varphi_{xy}[k + 1, 0] = \varphi_{xy}[k]\,. \tag{17}$$

The last relation is conceivable: In the case of a stationary couple of signals ψ_{xy} depends only on the difference $m - m = 0$, not on m.

In the knowledge of the simple and generalized correlation sequence the *mean-square errors* can easily be expressed: On the basis of (2) and (3), on considering (6), (8) and (10), we obtain

$$\zeta^2 = \mathrm{M}_n\left\{e[n]\right\} = \psi_{ee}[0]\,, \tag{18}$$

$$\xi^2 = \int_0^1 \mathrm{M}_n\left\{e^2[n, m]\right\} dm = \int_0^1 \psi_{ee}[0\,; m]\, dm\,. \tag{19}$$

Both correlations are analogous with the relation $\overline{e^2(t)} = \varphi_{ee}(0)$ customary in the case of continuous systems.

3. Output signals

Our task is to express the discrete functions and correlation series of the **two** terms of the error signal $e = i - c$ with the aid of the corresponding functions of the input signal $r = s + n$ and of the weighting functions.

The output signal $c(t)$ is the result of the input signal $T_0 r^*(t)$, thus it follows from the definition of $w(t)$ that

$$c(t) = T_0 \sum_{n=-\infty}^{\infty} w(t - nT)\, r(nT)\,, \tag{20}$$

$$c[k, m] = T_0 \sum_{n=-\infty}^{\infty} w[k - n, m]\, r[n]\,, \tag{21}$$

$$c[k] = T_0 \sum_{n=-\infty}^{\infty} w[k - n]\, r[n]\,, \tag{22}$$

where, on account of the causality of the system,

$$w(t) = 0,\ t < 0;\ w[k, m] = 0, k \leqq 0;\ w[k] = 0, k < 0. \tag{23}$$

The ideal output signal $i(t)$ can be ordered in three ways to the control input $s(t)$:

Problem I. The ideal output signal $i(t)$ is ordered to the *sampled input signal* $T_0 s^*(t)$ with the aid of a *continuous weighting function*. Then $i(t)$ is *quasistationary*. The previously given correlations are valid in this case too, only (23) is not unconditionally satisfied:

$$i(t) = T_0 \sum_{n=-\infty}^{\infty} y(t - nT)\, s(nT)\,, \tag{24}$$

$$i[k, m] = T_0 \sum_{n=-\infty}^{\infty} y[k - n, m]\, s[n]\,, \tag{25}$$

$$i[k] = T_0 \sum_{n=-\infty}^{\infty} y[k - n]\, s[n]\,. \tag{26}$$

Problem II. The ideal output signal $i(t)$ is ordered to the *continuous input signal* $s(t)$ with the aid of the *continuous weighting function* $y(t)$. In this case we have a continuous system, thus $i(t)$ is *stationary*:

$$i(t) = \int_{-\infty}^{\infty} y(t - t')\ s(t')\, dt'\,. \tag{27}$$

By using this, $i[k,m]$ and $i[k]$ can be expressed, however this will not be necessary.

Problem III. The ideal output signal $i(t)$ is ordered to the *continuous signal* $s(t)$ with the aid of a discrete operation. Then $y(t)$ is a sampled function

$$y(t) = \sum_{p=-\infty}^{\infty} y_p \, \delta(t - pT) . \tag{28}$$

In this case $i(t)$ is stationary (the system is invariant) and

$$i(t) = \sum_{p=-\infty}^{\infty} y_p \, s(t - pT) , \tag{29}$$

$$i[k,m] = \sum_{p=-\infty}^{\infty} y_p \, s[k-p,m] = \sum_{n=-\infty}^{\infty} y_{k-n} \, s[n,m] , \tag{30}$$

$$i[k] = \sum_{n=-\infty}^{\infty} y_{k-n} \, s[n] . \tag{31}$$

Problem III is in fact a special case of Problem II, from the aspect of calculation technique, however, it is nearer to Problem I. The grounds for the separate discussion of this problem are given also by the fact that important problems, such as the follow-up system (ideal filter), for which $y(t) = \delta(t)$, the system advancing or retarding by a time pT, for which $y(t) = \delta\,(t \pm pT)$, etc., also belong to this group.

4. The correlation sequence of the output signals

On the basis of the relations in the preceding two sections the correlation sequence of the output signals c and i, and of some quasistationary signals u and v, respectively, can easily be determined. Thus e.g., on the basis of (6) and (22),

$$\begin{aligned} \psi_{uc}[k] &= \mathrm{M}_n\left\{u[n]\,c[n+k]\right\} = \\ &= \mathrm{M}_n\left\{u[n]\,T_0 \sum_{q=-\infty}^{\infty} w[n+k-q]\,r[q]\right\} = \\ &= T_0\,\mathrm{M}_n\left\{u[n] \sum_{p=-\infty}^{\infty} w[k-p]\,r[n+p]\right\} = \\ &= T_0 \sum_{p=-\infty}^{\infty} w[k-p]\,\mathrm{M}_n\left\{u[n]\,r[n+p]\right\} . \end{aligned} \tag{32}$$

As a final result we obtain from this (in a similar way) that

$$\psi_{uc}[k] = T_0 \sum_{p=-\infty}^{\infty} w[k-p]\,\psi_{ur}[p] , \tag{33}$$

$$\psi_{cv}[k] = T_0 \sum_{p=-\infty}^{\infty} w[p-k]\,\psi_{rv}[p], \tag{34}$$

$$\psi_{cc}[k] = T_0^2 \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} w[p-k]\,w[p-q]\,\psi_{rr}[q], \tag{35}$$

where $\psi_{rr}[q] = \varphi_{rr}[q]$, since r is stationary.

The relations are quite analogous to the index-changing rules known for the case of continuous systems.

The relations for the ideal signal in Problem I are similar, only the substitutions $c \to i$, $w \to y$, $r \to s$ should be carried out. In Problem II the simple correlation series will not be necessary, while the relations for Problem III are identical with those of Problem I by substituting $y[n] \to y_n$.

The generalized correlation sequence can be obtained in a similar way. E.g. on the basis of (8) and (21),

$$\begin{aligned}\psi_{uc}[k; m, h] &= \mathrm{M}_n\{u[n,m]\,c[n+k,h]\} = \\ &= \mathrm{M}_n\{u[n,m]\,T_0 \sum_{q=-\infty}^{\infty} w[n+k-q,h]\,r[q]\} = \\ &= T_0 \sum_{p=-\infty}^{\infty} w[k-p+1,h]\,\mathrm{M}_n\{u[n,m]\,r[n+p-1]\}.\end{aligned} \tag{36}$$

The final result is

$$\psi_{uc}[k; m, h] = T_0 \sum_{p=-\infty}^{\infty} w[k-p+1,h]\,\psi_{ur}[p; m, 0], \tag{37}$$

$$\psi_{cv}[k; m, h] = T_0 \sum_{p=-\infty}^{\infty} w[p-k+1,m]\,\psi_{rv}[p; 0, h], \tag{38}$$

$$\psi_{cc}[k; m, h] = T_0^2 \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} w[p-k+1,m]\,w[p-q+1,h]\cdot\psi_{rr}[q; 0{,}0], \tag{39}$$

where $\psi_{rr}[q; 0{,}0] = \psi_{rr}[q] = \varphi_{rr}[q]$. The formalism of the index-changing rule is quite evident.

The relations for the ideal signal are identical with the above in the case of Problem I, only the substitutions $c \to i$, $w \to y$, $r \to s$ should be carried out. The relations for Problem II will not be necessary. In Problem II, on the basis of (8) and (30), a transformation similar to that carried out in (36) can be performed, thus as a final result we have

$$\psi_{ui}[k; m, h] = \sum_{p=-\infty}^{\infty} y_{k-p}\,\psi_{us}[p; m, h], \tag{40}$$

$$\psi_{iv}[k\,;m,h] = \sum_{p=-\infty}^{\infty} y_{p-k}\,\psi_{sv}[p\,;m,h]\,, \tag{41}$$

$$\psi_{ii}[k\,;m,h] = \sum_{p=-\infty}^{\infty}\;\sum_{q=-\infty}^{\infty} y_{p-k}\,y_{p-q}\,\psi_{ss}[q\,;m,h]\,. \tag{42}$$

If u and v are stationary, then the correlation sequence on the right side of the relations can be expressed on the basis of formulae (15)—(17) in terms of the correlation functions. In formulae (40)—(41) this is feasible even on the left side since i is stationary here.

5. Transformed correlation sequences

The calculation of sampled-data systems is facilitated, as is well known, by the application of the discrete Laplace transformation. Let the variable of this be denominated by

$$Z \equiv z^{-1} \equiv e^{-sT} \tag{43}$$

where s is the variable of the Laplace transformation, while $z = e^{sT}$ is the variable generally used in the literature. Since at numerical calculations the expressions are generally to be ordered in terms of the powers of z^{-1}, it is more advantageous to employ the variable Z. The two-sided simple and modified discrete transforms ordered to the function $f(t)$ satisfying the respective mathematical conditions are

$$F(Z) = \mathfrak{Z} f(t) = \mathscr{D} f[k] = \mathfrak{Z}' F(s) = \sum_{k=-\infty}^{\infty} f[k]\,Z^k, \tag{44}$$

$$F(Z, m) = \mathfrak{Z}_m f(t) = \mathscr{D} f[k, m] = \mathfrak{Z}'_m F(s) = \sum_{k=-\infty}^{\infty} f[k, m]\,Z^k. \tag{45}$$

The transforms of the simple and modified correlation sequences are interpreted on this pattern:

$$\Psi_{uv}(Z) = \sum_{k=-\infty}^{\infty} \psi_{uv}[k]\,Z^k, \tag{46}$$

$$\Psi_{uv}(Z\,;m,h) = \sum_{k=-\infty}^{\infty} \psi_{uv}[k\,;m,h]\,Z^k, \tag{47}$$

where, in the sense of (9) and (10), respectively

$$\Psi_{uv}(Z\,;m) = \Psi_{uv}(Z\,;m,m)\,,\;\; \Psi_{uv}(Z) = \Psi_{uv}(Z\,;0{,}0)\,. \tag{48}$$

If $u = x$, $v = y$ are stationary, then by force of (15)—(17)

$$\Psi_{xy}(Z;m,h) = \begin{cases} \Phi_{xy}(Z, h-m+1), & -1 \leqq h-m < 0, \\ Z^{-1}\Phi_{xy}(Z, h-m), & 0 \leqq h-m < 1. \end{cases} \tag{49}$$

The special cases of this are

$$\Psi_{xy}(Z;m,0) = \Phi_{xy}(Z, 1-m) = \mathfrak{Z}'_{1-m}\Phi_{xy}(s), \tag{50}$$

$$\Psi_{xy}(Z;0,h) = Z^{-1}\Phi_{xy}(Z,h) = Z^{-1}\mathfrak{Z}'_h\Phi_{xy}(s), \tag{51}$$

$$\Psi_{xy}(Z;m) = \Psi_{xy}(Z) = \Phi_{xy}(Z) = \mathfrak{Z}'\Phi_{xy}(s). \tag{52}$$

Here attention is drawn again to the fact that $\Psi_{xy}(Z,m) = \Psi_{xy}(Z;m,m)$ does not depend on m, thus $\Phi_{xy}(s)$, or $\varphi_{xy}(\tau)$ are equal to the simple discrete transform.

On dimensional grounds, the discrete transfer functions are defined as follows:

$$W(Z,m) = T_0\mathfrak{Z}'_m W(s), \quad W(Z) = T_0\mathfrak{Z}' W(s), \tag{53}$$

and similar to it is the interpretation of $Y(Z,m)$ and $Y(Z)$ in Problem I, while in Problem III,

$$Y(Z) = \sum_{k=-\infty}^{\infty} y_k Z^k. \tag{54}$$

One of the important advantages of introducing the discrete transforms is that the convolution sums discussed in Section 4 go over to the product of the transforms. Thus, e.g., the transform of (33) is

$$\Psi_{uc}(Z) = \sum_{k=-\infty}^{\infty} T_0 \sum_{p=-\infty}^{\infty} w[k-p]\,\psi_{ur}[p]\,Z^k =$$

$$= \sum_{q=-\infty}^{\infty}\ \sum_{p=-\infty}^{\infty} T_0\, w[q]\,\psi_{ur}[p]\,Z^p Z^q. \tag{55}$$

As a final result

$$\Psi_{uc}(Z) = W(Z)\Psi_{ur}(Z), \quad \Psi_{cv}(Z) = W(Z^{-1})\Psi_{rv}(Z). \tag{56}$$

Analogous correlations are valid in Problems I and III as well:

$$\Psi_{ui}(Z) = Y(Z)\Psi_{us}(Z), \quad \Psi_{iv}(Z) = Y(Z^{-1})\Psi_{su}(Z). \tag{57}$$

In the case of Problem II these relationships are more complicated.

The transform of the generalized correlation sequences, on the basis of (37) is found to be

$$\Psi_{uc}(Z;m,h) = \sum_{q=-\infty}^{\infty} T_0 \sum_{p=-\infty}^{\infty} w[k-p+1,h]\, \psi_{ur}[p;m,0]\, Z^k =$$

$$= \sum_{q=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} T_0 w[q,h]\, \psi_{ur}[p;m,0]\, Z^p Z^q Z^{-1}, \tag{58}$$

as a final result

$$\Psi_{uc}(Z;m,h) = Z^{-1} W(Z,h)\, \Psi_{ur}(Z;m,0), \tag{59}$$

$$\Psi_{cv}(Z;m,h) = Z\, W(Z^{-1},m)\, \Psi_{rv}(Z;0,h), \tag{60}$$

$$\Psi_{cc}(Z;m,h) = W(Z^{-1},m)\, W(Z,h)\, \Psi_{rr}(Z;0,0), \tag{61}$$

where $\Psi_{rr}(Z;0,0) = \Psi_{rr}(Z) = \Phi_{rr}(Z)$. Analogous correlations are valid for Problem I, while in Problem III,

$$\Psi_{ui}(Z;m,h) = Y(Z)\, \Psi_{us}(Z;m,h), \tag{62}$$

$$\Psi_{iv}(Z;m,h) = Y(Z^{-1})\, \Psi_{sv}(Z;m,h). \tag{63}$$

From relations (56)—(63) the formalism of the index-changing rule is evident. If u and v are stationary, the transformed correlation sequences can be expressed by the transformed correlation functions.

6. Calculation of the mean-square errors

The discrete mean-square error can simply be expressed in the knowledge of the simple correlation sequence of the error signal. It follows from relations (18) and (46) that

$$\zeta^2 = \mathrm{M}_n\{e^2[n]\} = \frac{1}{2\pi j} \oint_\Gamma Z^{-1} \Psi_{ee}(Z)\, dZ\,; \quad \Gamma: |Z| = 1. \tag{64}$$

Since $e = i - c$, thus

$$\Psi_{ee}(Z) = \Psi_{ii}(Z) - \Psi_{ic}(Z) - \Psi_{ci}(Z) + \Psi_{cc}(Z). \tag{65}$$

The index c can be changed to index r on the ground of formulae (55)—(56).

$$\Psi_{ee}(Z) = \Psi_{ii}(Z) - W(Z)\, \Psi_{ir}(Z) - W(Z^{-1})\, \Psi_{ri}(Z) + + W(Z^{-1})\, W(Z)\, \Psi_{rr}(Z). \tag{66}$$

The problem is to eliminate index i with the aid of index s. This can be realized in different ways in the three problems. At the right side only the indices r

and s figure in this case. The correlation sequences of the stationary signals, in turn, can be expressed by their correlation functions, and with the aid of the Laplace transform (power spectrum) of these, respectively. Since $r = s + n$, we obtain with any variable,

$$\Phi_{sr} = \Phi_{ss} + \Phi_{sn}\,, \quad \Phi_{rs} = \Phi_{ss} + \Phi_{ns}\,,$$
$$\Phi_{rr} = \Phi_{ss} + \Phi_{sn} + \Phi_{ns} + \Phi_{nn}\,. \tag{67}$$

If the control input and the noise are uncorrelated, then $\Phi_{sn} = \Phi_{ns} = 0$.

In the case of Problems I and III, relation (57) can be employed. thus it is easily conceivable that

$$\Psi_{ee}(Z) = Y(Z^{-1})\,Y(Z)\,\Phi_{ss}(Z) - Y(Z^{-1})\,W(Z)\,\Phi_{sr}(Z) - W(Z^{-1})\,Y(Z)\,\Phi_{rs}(Z) + W(Z^{-1})\,W(Z)\,\Phi_{rr}(Z). \tag{68}$$

In Problem II signal i is stationary, thus at the right side of (67) Φ can be written in place of Ψ and the index-changing rule valid for continuous systems can be employed:

$$\Psi_{ee}(Z) = \mathfrak{Z}'\left[Y(-s)\,Y(s)\,\Phi_{ss}(s)\right] - W(Z)\,\mathfrak{Z}'\left[Y(-s)\,\Phi_{sr}(s)\right] - W(Z^{-1})\,\mathfrak{Z}'\left[Y(s)\,\Phi_{rs}(s)\right] + W(Z^{-1})\,W(Z)\,\Phi_{rr}(Z). \tag{69}$$

The continuous mean-square error can be calculated in a quite similar way. It follows from relations (19) and (47) that

$$\xi^2 = \mathrm{M}_n\left\{\int_0^1 e^2[n,m]\,dm\right\} = \frac{1}{2\pi j}\oint_\Gamma Z^{-1}\int_0^1 \Psi_{ee}(Z\,;m)\,dm\,dZ\,, \tag{70}$$

where

$$\Psi_{ee}(Z\,;m) = \Psi_{ii}(Z\,;m) - \Psi_{ic}(Z\,;m) - \Psi_{ci}(Z\,;m) + \Psi_{cc}(Z\,;m)\,. \tag{71}$$

The index c can be changed for the index r on the ground of (59)—(61).

$$\Psi_{ee}(Z\,;m) = \Psi_{ii}(Z\,;m) - Z^{-1}\,W(Z,m)\,\Psi_{ir}(Z\,;m,0) - Z\,W(Z^{-1},m)\,\Psi_{ri}(Z\,;0,m) + W(Z^{-1},m)\,W(Z,m)\,\Psi_{rr}(Z)\,. \tag{72}$$

The changing of index i to index s should be examined separately for all three problems.

In Problem I, by employing the sense of relations (59)—(61),

$$\Psi_{ee}(Z;m) = Y(Z^{-1},m)\;Y(Z,m)\,\Phi_{ss}(Z) - Y(Z^{-1},m)\;W(Z,m)\;\Phi_{sr}(Z) - W(Z^{-1},m)Y(Z,m)\;\Phi_{rs}(Z) + W(Z^{-1},m)\,W(Z,m)\;\Phi_{rr}(Z). \tag{73}$$

In Problem II, by force of relations (50)—(51) and of the index-changing rule relating to continuous systems,

$$\Psi_{ee}(Z;m) = \mathfrak{Z}'\left[Y(-s)\,Y(s)\,\Phi_{ss}(s)\right] - Z^{-1}\,W(Z,m)\,\mathfrak{Z}'_{1-m}\left[Y(-s)\,\Phi_{sr}(s)\right] - \\ - W(Z^{-1},m)\,\mathfrak{Z}'_m\left[Y(s)\,\Phi_{rs}(s)\right] + \\ + W(Z^{-1},m)\,W(Z,m)\,\Phi_{rr}(Z). \quad (74)$$

In the first term really $\mathfrak{Z}'$ (and not $\mathfrak{Z}'_m$) figures, since i is stationary, thus $\Psi_{ii}(Z;m) = \Phi_{ii}(Z)$.

In Problem III formulae (62)—(63) can be employed, then upon considering that i is stationary, the correlation functions can be deduced on the basis of (49):

$$\Psi_{ee}(Z;m) = Y(Z^{-1})\,Y(Z)\,\Phi_{ss}(Z) - Z^{-1}\,Y(Z^{-1})\,W(Z,m)\,\Phi_{sr}(Z,1-m) - \\ - W(Z^{-1},m)Y(Z)\,\Phi_{rs}(Z,m) + W(Z^{-1},m)W(Z,m)\,\Phi_{rr}(Z). \quad (75)$$

If in relationships (70)—(75) $m = 0$, then our results naturally go over into the relations (64)—(69).

The relationships given in this section are the solutions of the task of analysis: If the weighting functions (or the transfer functions) of the real and ideal systems, further the correlation functions (or the power spectra) of the input signal are known, then either the discrete, or the continuous mean-square error can be calculated on the ground of the given relationships.

7. Some critical remarks

As has been shown, there are theoretical and practical reasons for giving the expression of the mean-square error for six different cases separately. In the relevant literature the detailed discussion of all six problems cannot be found. ZYPKIN [5, 15] and KUZIN [7, 8] have interpreted the ideal transfer function in a too general way, thus their results are inconvenient in concrete cases. On the other hand, KUZIN has discussed only the calculation of the mean-square error in detail, while ZYPKIN has determined the mean of $e^2[n,m]$ with a fixed m value, of which ζ is obtained in the case of $m = 0$, while its integral with respect to m is intuitively identified with ξ^2. TOU [6, 9] has given the discrete mean-square error only for Problem II.

For characterizing quasistationary signals, the correlation sequences were employed, while in the literature the correlation functions are used. On calculating the discrete mean-square error (especially in the case of Problem I), formal analogies supply correct results, since both $c[n]$ and $i[n]$ are stationary. In connection with the examined continuous signals only KUZIN has correctly recognized the problems caused by the quasistationary character, therefore, he has employed a correlation function with two variables. His designations, however, are not consequent and not quite fortunate. The method of ZYPKIN is suitable for circumventing the problem, thus his results are faultless, but containing many arbitrary definitions and inconsistent designations; his results are difficult to employ in practice. CHANG [10], though he has recognized the quasistationary character of $c(t)$, but in spite of this he regards it as ergodic. TOU has produced the expression for Problem II on the basis of a formal analogy and his result is trivially incorrect. Thus e.g. in the first term of (72) the operation $\mathfrak{Z}'_m$ figures according to him, i.e. $\Phi_{ii}(Z,m)$. If we have e.g. a follow-up system, then $\Phi_{ii}(Z,m) = \Phi_{ss}(Z,m)$ and the integral of this with respect to m evidently does not supply the value of $s^2(t)$. It is very interesting that NISHIMURA [13] has obtained formally correct results without recognizing the essence of the problem, even by operating with a function which can evidently not be interpreted. His formalism is, however, complicated and unusual.

Further publications on the synthesis, such as [17, 18, 19] are generally based on the resuls of TOU, or criticize those. These are correct in respect to ζ^2, but as regards to ξ^2 these are incorrect both on the theoretical bases and the final result.

Summary

An important quality characteristic of sampled-data systems is the quadratic mean of the difference between the output signal arising in consequence of the stationary stochastic input signal and the ideal output signal ordered to the control input, i. e. of the actuating error. A discrete and a continuous mean-square error can be defined. The ideal signal can be ordered to the control input in three ways. Formulae for these six cases can suitably be given by introducing the correlation sequences and employing the discrete Laplace transformation. On deducing the expression for the continuous mean-square error the fact that the output signal is not stationary should be taken into consideration.

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