DETERMINATION OF DISTORTION CONDITIONS BY MEANS OF DIFFERENTIAL GEOMETRIC METHODS

By

J. SÁROSSY and V. SZÉKELY

Department of Theoretical Electricity and Department of Electron Tubes and Semiconductors, Polytechnical University, Budapest

(Received August 16, 1966)

Presented by Prof. Dr. I. P. VALKÓ

Many fields of physics and attached branches of science (such as optics, electron optics, geodesy, study of potential fields, etc.) work with geometric projection. Each of these special fields has developed a particular system of qualitative estimation of projection [7, 8, 9], however, a uniform method of description of geometric distortions is nevertheless lacking. This lack has moved us to introduce the distortion characteristics (ratio distortion, area distortion, etc.) by the present paper. We have discovered that these concepts are indeed applicable in several special fields.

Our method was first applied to characterize the distortion of television images. It occurred to us, in connection herewith, to try to describe distortions caused by any discretional technical means. Generalization succeeded in projections between discretionally curved surfaces, thus making it suitable among others, even for the computation of distortions of geodesic chart projection. The following step was the generalization to the more than two-dimensional "images"; no difficulties have been met with here either.

Experiments have as well been carried out to apply these methods of distortion computation for not merely geometric problems. The question of connection with the conform projection used for computing electrostatic fields, e. g., was tested. The results were surprising: our method of distortion computation is applicable even in the case of investigating potential fields, resp. the projection of potential fields. This kind of applicability exceeds the possibilities of conform projection, this latter being nothing but one special case of the method.

The present paper will be divided as follows: the first paragraph presents the concept of distortion tensor and defines the distortion characteristics. The second paragraph generalizes the derivation of the distortion tensor to the Riemannian spaces and derives the general form of the distortion characteristics. The third paragraph deals with the practical methods of determining the metric tensors indispensable for the computation of the distortion tensor, whereas the fourth paragraph is reserved for illustrating examples.

1. The concept of distortion tensor, ratio and area distortion

This paragraph is to introduce the concept of local distortion and distortion tensor by means of generalizing a simple example. The discussion should be as easy to understand as possible, thus it is limited to Cartesian coordinate systems and only the distortion of the neighbourhood of one single point will be examined. After having obtained the results, the entirely general solution of the problem will be given in para. 2.

The following definition concerning horizontal distortion of television image reproduction is generally accepted: in a fixed point the *relative deviation* describing distortion is given by the formula

$$R_{\%} = \left(\frac{v_p}{v_k} - 1\right) \cdot 100, \tag{1}$$

where v_p is the instantaneous velocity of electron-beam basepoint in the chosen point and v_k the mean velocity as referring to the whole length of the line [1].

If the horizontal size of a scanning element of the distorted picture is compared with the horizontal size of the corresponding scanning element of the original picture, no more happened there than that the above definition has been expressed in another, slightly more generalized form. Going now beyond this one-dimensional formulation, we should like to find a mode of describing the dimensional change of arbitrary direction of a scanning element. Additional difficulties arise by the fact that during distortion not only the length is changing, but also the direction of elementary arcs and even the angles between them. The problem which, at first sight, seems to be extremely difficult, becomes considerably simpler by expressing the distortion of a suitably small neighbourhood of a point in terms of one single quantity, finite number of components, a *tensor of second order*. (The second paragraph proves this statement for the general case.) Let us, therefore, substitute the elementary arcs taken in the n-dimensional neighbourhood of the point examined for the vectors as defined in the point examined; now we may study the changing effect of this tensor of second order on the multitude of these vectors.

Let the relation between the arbitrary vector u_j and the vector after the change $\tilde{u}_i$ be

$$\tilde{u}_i = H_{ij} u_j, \qquad \det |H_{ij}| \neq 0, \tag{2}$$

where the tensor H_{ij} causing u_j to change is called the *effective tensor*. Just as every tensor, H_{ij} may also be divided into the successive application of a symmetrical (S_{kj}) and an isometrical (I_{ik}) tensor [2]:

$$H_{ij} = I_{ik} S_{kj}. \tag{3}$$

According to their definition,

$$S_{kj} = S_{jk}, \tag{4}$$

and

$$I_{ik} I_{jk} = I_{ki} I_{kj} = \delta_{ij}, \tag{5}$$

where $\delta_{ij} = 1$, if $i = j$ and $\delta_{ij} = 0$, if $i \neq j$, i.e. δ_{ij} is the unit tensor.

The effect of the two factors of the effective tensor may be visualized as well: I_{ik} causing a *rotation* without any change in length and angle, and S_{kj} induces longitudinal and angular changes, i. e. *distortions*. The above property of I_{ik} is easily proved. Namely, data of length and angle are expressed by scalar products:

$$u = \sqrt{u_i u_i}, \quad \cos \varphi_{uv} = \frac{u_i v_i}{u v}; \tag{6}$$

whereas the isometric tensor itself leaves scalar products unchanged, since

$$(\tilde{u}, \tilde{v}) \equiv \tilde{u}_i \tilde{v}_i = I_{ij} u_j I_{ik} v_k = \delta_{jk} u_j v_k = u_j v_j. \tag{7}$$

(Here the relation (5) has been applied.)

The purpose of our study being the mathematical definition of *distortion conditions*, the rigid body-like rotation is ignored. Distortions, however, are characterized by longitudinal and angular changes, i.e. the modification of scalar products. Let us, therefore, study the scalar product changes:

$$(\tilde{u}, \tilde{v}) \equiv \tilde{u}_i \tilde{v}_i = H_{ij} H_{ik} u_j v_k = I_{ip} S_{pj} I_{iq} S_{qk} u_j v_k. \tag{8}$$

Making use again of relation (5),

$$(\tilde{u}, \tilde{v}) = S_{pj} S_{pk} u_j v_k = \gamma_{jk} u_j v_k. \tag{9}$$

where, as a definition,

$$\gamma_{jk} = S_{pj} S_{pk}. \tag{10}$$

γ_{jk} may now safely be called the *distortion tensor*, containing only tensor S_{kj} which characterizes distortions, but not the isometric factor, and giving direct information on scalar products composed by distorted vectors.

The distortion tensor γ_{ik} — although fully describing distortion — is rather difficult to be treated. Its components give no immediate information about the degree of distortion and are, in addition, dependent on the coordinate system applied. For instance, a direct answer is sought for to the question, which maximum change of length a vector can endure because of distortion. To express this more clearly: we look for the maximum change of length and the unit vector $\underset{1}{a_i}$, on which this is produced by the effective tensor [3].

According to (6) and (9) the distorted length square is:

$$\tilde{u}^2 = \gamma_{ik} u_i u_k. \tag{11}$$

The maximum of $\tilde{u}^2$ should be found assuming that $u_i u_i = 1$. Applying the method of Lagrange's multipliers, the absolute extreme value of the function

$$\tilde{u}^2(\lambda) = \gamma_{ik} u_i u_k - \lambda (u_i u_i - 1) \tag{12}$$

is to be found, so the vector $\underset{1}{a_i}$ sought for should satisfy the relations

$$\left.\frac{\partial \tilde{u}^2(\lambda)}{\partial u_j}\right|_{u_j = \underset{1}{a_j}} = 0 , \qquad \left.\frac{\partial \tilde{u}^2(\lambda)}{\partial \lambda}\right|_{u_j = \underset{1}{a_j}} = 0 . \tag{13}$$

After differentiating

$$\frac{\partial \tilde{u}^2(\lambda)}{\partial u_j} = \gamma_{jk} u_k + \gamma_{ij} u_i - 2\lambda u_j = 0 , \tag{14}$$

$$u_i u_i = 1.$$

Since γ_{jk} is symmetric (see [10]),

$$\gamma_{jk} u_k - \lambda u_j = 0 , \qquad u_i u_i = 1 , \tag{15}$$

where the first equation is the *eigenvalue-equation* of the tensor. A non-trivial solution is obtained only if

$$\det | \gamma_{jk} - \lambda \delta_{jk} | = 0 , \tag{16}$$

as known in the theory of homogeneous equation systems. Equation (16) has n real solutions, the *eigenvalues*: let them be $\underset{1}{\lambda}, \underset{2}{\lambda}, \ldots \underset{n}{\lambda}$.

In the following, let us assume a sequence of eigenvalues always obtainable by appropriate interchanges of subscripts:

$$\underset{1}{\lambda} \leq \underset{2}{\lambda} \leq \ldots \leq \underset{n}{\lambda} . \tag{17}$$

By substituting in order the eigenvalues in equation (15), the sequence of *eigenvectors* is obtained: $\underset{1}{a_i}, \underset{2}{a_i}, \ldots \underset{n}{a_i}$; where

$$\underset{r}{a_i} \underset{r}{a_i} = 1 . \qquad \text{(not summed for } r) \tag{18}$$

Considering the well-known theorem according to which the eigenvectors of symmetrical tensors are orthogonal (or, at least, orthogonizable),

$$\underset{r}{a_i} \underset{s}{a_i} = \underset{rs}{\delta} . \tag{19}$$

If a function defined on a compact region is continuous, moreover continuously differentiable, then it takes on its maximum and minimum as well within this closed region, in particular at some of the zeros of the differential quotient. For function (12) these assumptions are fulfilled on the closed unit sphere $u_i u_i = 1$, consequently the maximum may only be $\underset{n}{\lambda}$, the minimum $\underset{1}{\lambda}$.

It can easily be proved that a symmetrical tensor transformed to the coordinate system spanned by its eigenvectors becomes:

$$\bar{\gamma}_{ij} = \underset{i}{\lambda}\,\delta_{ij} \quad \text{(not summed for } i\text{)} \tag{20}$$

i. e.

$$\bar{\gamma}_{ij} = \begin{bmatrix} \underset{1}{\lambda} & 0 \ldots 0 \\ 0 & \underset{2}{\lambda} \ldots 0 \\ \vdots & \vdots \quad\; \vdots \\ 0 & 0 \ldots \underset{n}{\lambda} \end{bmatrix}. \tag{20a}$$

In this coordinate system $\bar{x}_i$ determined by the isometric transformation $x_i = \underset{j}{a_i}\bar{x}_j$ (11) becomes:

$$\tilde{u}^2 = \underset{1}{\lambda}\,\bar{u}_1^2 + \underset{2}{\lambda}\,\bar{u}_2^2 + \ldots + \underset{n}{\lambda}\,\bar{u}_n^2\,, \tag{21}$$

whereby

$$\bar{u}_1^2 + \bar{u}_2^2 + \ldots + \bar{u}_n^2 = 1\,. \tag{21a}$$

These two equations determine because of the positive eigenvalues (which is explained by [10]) an ellipsoid, the so-called indicatric ellipsoid the axes of which point in the direction of the coordinate vectors, i. e. the eigenvectors and with the lengths as follows: $\sqrt{\underset{1}{\lambda}}, \sqrt{\underset{2}{\lambda}}, \ldots \sqrt{\underset{n}{\lambda}}$.

Let us try to illustrate the change of neighbourhood of a predetermined point on the basis of the aforesaid. Recent results show that — while vector u_i is running at the surface of the unit sphere surrounding the chosen point — its distorted equivalent ($\tilde{u}_i$) runs along the surface of the ellipsoid. This means that a circle becomes distorted in two dimensions to an ellipse, the square to a rectangle, rhombus, rhomboid (see Figs 1 and 2), whereas in three dimensions an ellipsoid is obtained instead of a sphere, a general parallelepiped instead of a cube. (The above illustration refers as, of course, all considerations discussed so far, to a *small* neighbourhood of the point.)

According to equations (2) and (3), the change of the scanning element occurs in two steps. First the neighbourhood of the point suffers pure distortion, followed by a rigid body-like rotation (Fig. 2). It may be seen that the rigid body-like rotation does not essentially alter the estimation of distortion, consequently it is really ignorable. (This problem will be referred to again at the end of this paragraph, where a deeper reason is given for our procedure.)

It is worth observing that the informations obtainable from the distortion of the scanning element may be ranged in two groups. The *absolute* characteristics (the minor, resp. major axis of the ellipse, for example, its eccentricity, its area, etc.) belong to the first group and are the invariants of the dis-

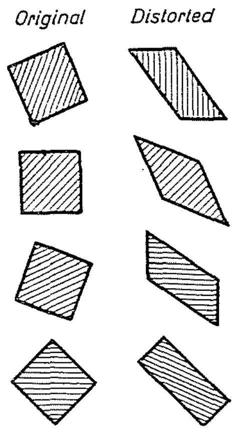

Fig. 1

tortion tensor. The second group is formed by the *relative* characteristics (such as the direction of maximum, resp. minimum distortion, etc.) which are in connection with the eigenvectors of the distortion tensor and thus dependent on the coordinate system (on the view point). Although both groups of data are necessary to completely describe the distortion, the absolute characteristics are more expressive because of their independency from the view point.

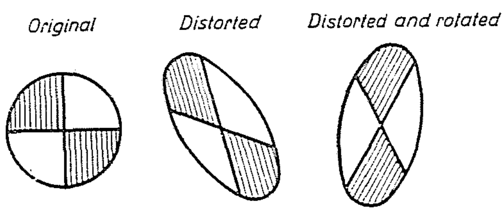

Fig. 2

Now let us consider the absolute characteristics, the invariants of the distortion tensor. Such characteristics are, for example, the eigenvalues. It is not difficult to prove that their totality forms an independent and, at the same time, complete system. (It is sufficient to consider the form the distortion tensor is taking after the transformation on the main axes.) It follows from this fact that every invariant is unambiguously determinable by the eigenvalues.

The eigenvalues have an expressive content as well, giving — following from equations (21) and (21a) — just the square of the axes of the distortion ellipsoid, whereas a conception of the nature and degree of distortion may be formed with the knowledge of the length of the axes. There is, however, a possibility to form even more practical invariants. The introduction of *volume distortion* (T_n) is very useful; this invariant is the ratio between the volume of the distortion ellipsoid and the volume of the unit sphere. Its connection with eigenvalues is evident:

$$T_n^2 = \prod_{k=1}^{n} \lambda_k . \tag{22}$$

A similar definition may be given on the distortion of volume element of an arbitrary l-dimensional subspace, depending, of course, on the orientation of the subspace as well. Its maximum and minimum may be given as invariants. Relations with eigenvalues are again simple:

$$T_{l\,\max}^2 = \prod_{k=n-l+1}^{n} \lambda_k , \tag{23}$$

$$T_{l\,\min}^2 = \prod_{k=1}^{l} \lambda_k . \tag{23a}$$

Evidently

$$T_{n\,\max}^2 = T_{n\,\min}^2 = T_n^2 . \tag{23b}$$

Another, similarly effective and expressive sequence of invariants is the system of *ratio distortions* (A_l). The l-dimensional ratio distortion is the quotient of maximum and minimum volume distortion of an l-dimensional volume element:

$$A_l^2 = \frac{\prod\limits_{k=n-l+1}^{n} \lambda_k}{\prod\limits_{k=1}^{l} \lambda_k} , \; (l < n). \tag{24}$$

It follows from equation (24) that

$$A_l = A_{n-l} . \tag{24a}$$

Let us examine how an independent and, at the same time, complete system may be chosen from among the above derived invariants. First of all, $\{T_{l\max}\}$, $(1 \leqq l \leqq n)$ as well as $\{T_{l\min}\}$, $(1 \leqq l \leqq n)$ form an independent and complete system. (For the sake of simplicity, demonstration is neglected.) Accordingly, there is an unambiguous relation between the two systems:

$$T_{l\,\max} \cdot T_{n-l\,\min} = T_n , \tag{25}$$

i.e. the elements of the two systems may be conversed to each other. Because of relation (24a), however, only $\{A_l\}$, $\left(1 \leq l \leq \left[\frac{n}{2}\right]\right)$ forms an independent, but visibly not complete system. They may be completed to an independent and complete system in the following manner: $\{A_l\}$ $\{T_{k\,\max}\}$, $\left(1 \leq l \leq \left[\frac{n}{2}\right],\right.$ $\left.\left[\frac{n}{2}\right] < k \leq n\right)$. In this case

$$\left.\begin{array}{lll} T_{k\,\min} = \dfrac{T_{k\,\max}}{A_{n-k}} & & \left(\left[\dfrac{n}{2}\right] < k < n\right), \\ T_{l\,\max} = \dfrac{T_n}{T_{n-l\,\min}}, & T_{l\,\min} = \dfrac{T_n}{T_{n-l\,\max}} & \left(1 \leq l \leq \left[\dfrac{n}{2}\right]\right) \end{array}\right\}, \tag{26}$$

respectively if n is even:

$$T_{n/2\,\max} = \sqrt{A_{n/2}\,T_n}\,, \quad T_{n/2\,\min} = \sqrt{\frac{T_n}{A_{n/2}}}\,. \tag{26a}$$

There are, of course, still numerous ways to produce invariants, but in our opinion it is exactly the above invariant systems which are, in the majority of applications, the most practical, and, from the point of view of computations, at the same time, the most simple ones. This approach seems to be particularly advisable for general descriptions of picture reproduction and transmission.

After having generally discussed the distortion invariants, let us consider which data are the most expressive to describe the distortion of the most frequent two-dimensional images.

There are two eigenvalues of the distortion tensor in two dimensions, hence two independent invariants may be chosen. It seems to be advisable to mark them as the two-dimensional volume distortion of the picture — the *area distortion* — and as the ratio of the major and minor axis of the deformation ellipse — the *ratio distortion*:

$$T_2 = \sqrt{\lambda_1 \cdot \lambda_2}\,, \quad (22a) \qquad\qquad A = \sqrt{\frac{\lambda_2}{\lambda_1}}\,. \tag{24b}$$

These two characteristics are well applicable for the qualification of distorted pictures, since — by specifying the limits of the ratio distortion, resp. the ratio between local and general area distortion — there are two data giving reliable information on the quality of the distorted image. (By all means more reliable information than, e.g., the present index system used for the qualification of television picture, where the image that becomes unenjoyable because of jitters caused by errors in synchronism or by network disturbance does not show any irregularity in its data of vertical-horizontal linearity and pincushion distortion.)

Let us study how the area and ratio distortion may be determined when knowing the distortion tensor γ_{ij}. The characteristical equation (16) may be expressed in the following well-known form:

$$\lambda^2 - \lambda \Gamma_1 + \Gamma_2 = 0 . \tag{16a}$$

Γ_2 is the *determinant* of the distortion tensor, Γ_1 its *trace*; these two invariants may be computed in the Cartesian system as follows:

$$\left.\begin{aligned} \Gamma_1 &= \gamma_{ii} = \gamma_{11} + \gamma_{22} \\ \Gamma_2 &= \gamma_{11}\gamma_{22} - \gamma_{12}\gamma_{21} \end{aligned}\right\} . \tag{27}$$

The two solutions of equation (16a) are the two eigenvalues. According to the Vieta-formulas of the quadratic equation $\Gamma_2 = \lambda_1 \cdot \lambda_2$, i.e.

$$T_2 = \sqrt{\Gamma_2} . \tag{28}$$

In order to express the ratio distortion, equation (16a) should be solved.

$$A = \sqrt{\frac{\lambda_2}{\lambda_1}} = \frac{\lambda_2}{\sqrt{\lambda_1 . \lambda_2}} = \frac{1}{\sqrt{\Gamma_2}} \frac{\Gamma_1 + \sqrt{\Gamma_1^2 - 4\Gamma_2}}{2} = \frac{\Gamma_1}{2\sqrt{\Gamma_2}} + \sqrt{\left(\frac{\Gamma_1}{2\sqrt{\Gamma_2}}\right)^2 - 1} ,$$

$$\ln A = \operatorname{arch} \frac{\Gamma_1}{2\sqrt{\Gamma_2}} , \tag{29}$$

where relation $arch\, x = ln\,(x \pm \sqrt{x^2 - 1})$ is used and where we limited ourselves to the positive branch of the *arch* function. (The other branch would give the reciprocal of A, $\sqrt{\lambda_1/\lambda_2}$.)

Computation of ratio distortion is even simpler in the special, but frequently occurring case of $\gamma_{12} = \gamma_{21} = 0$. Here

$$A = \begin{cases} \sqrt{\gamma_{11}/\gamma_{22}} & \text{when} \quad \gamma_{11} \geq \gamma_{22} \\ \sqrt{\gamma_{22}/\gamma_{11}} & \text{when} \quad \gamma_{11} \leq \gamma_{22} \end{cases} \tag{29a}$$

resulting from (20a) and (24b).

It is noteworthy that there is an unambiguous relation between ratio distortion and angle distortion. The maximum distorted equivalent of a rectangle, $\tilde{\varphi}$ is computable as follows:

$$\operatorname{tg} \tilde{\varphi} = \frac{2A}{A^2 - 1} . \tag{30}$$

The relations of ratio and area distortion derived hereabove will be especially useful in the following chapter since, with their help, it is possible to obtain from the distortion tensor given in the general coordinates, distortion data easy to be dealt with.

Limited still to a two-dimensional image and Cartesian coordinates, let us study a view-point dependent, relative distortion feature: the *setting angle* of the distortion ellipse axes as referred to the direction of the coordinate axis

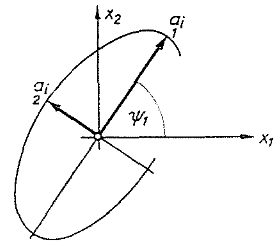

Fig. 3

x_1. The direction of the axes of the ellipse is determined by the eigenvectors a_i of the distortion tensor, the setting angles being thus expressed by their components (Fig. 3):

$$\operatorname{tg} \Psi = \frac{a_2}{a_1}.$$

Let us proceed from the first line of the equation system (15) serving to determine the eigenvectors:

$$\gamma_{11} a_1 + \gamma_{12} a_2 - \lambda a_1 = 0, \tag{15a}$$

$$\frac{a_2}{a_1} = \frac{\lambda - \gamma_{11}}{\gamma_{12}} = \frac{\Gamma_1 \pm \sqrt{\Gamma_1^2 - 4\Gamma_2} - 2\gamma_{11}}{2\gamma_{12}},$$

$$\frac{a_2}{a_1} = \ldots = \frac{\gamma_{22} - \gamma_{11}}{2\gamma_{12}} \pm \sqrt{\left(\frac{\gamma_{22} - \gamma_{11}}{2\gamma_{12}}\right)^2 + \frac{\gamma_{21}}{\gamma_{12}}}. \tag{31}$$

Considering now the symmetry ($\gamma_{12} = \gamma_{21}$) of the matrix representing the tensor γ, and using the relation *arsh* $x = \ln(x \pm \sqrt{x^2 + 1})$, we obtain:

$$\ln \operatorname{tg} \Psi = \operatorname{arsh} \frac{\gamma_{22} - \gamma_{11}}{2\gamma_{12}}. \tag{31a}$$

It should as well be considered that the *arsh* function is multivalued: if $y =$ *arsh* x, then $j\pi - y =$ *arsh* x is valid, too. For this reason, relation (31a) gives two data: the setting angle of the major and the minor axis — showing, of course, a deviation of 90°.

Comparing distorted and undistorted picture elements to be found in different spaces the rigid body-like rotation naturally cannot be defined. Let us consider the problem in that case when the distorted and undistorted pictures are in the same space. In this case the condition for the definition of the rigid body-like rotation is that there should be an unambiguous procedure for making two picture elements to cover each other; herewith the isometric tensor can be derived according to the above discussed description. The problem of the above mentioned procedure is dealt with in detail in the literature: the conditions of unambiguous *parallel displacement* or with another name the *absolute parallelism* are derived in detail e.g. in [5] and [6]. We quote here the final result: unambiguous parallel displacement exists only in euclidean space.

An obvious conclusion of the above result is that *rigid body-like rotation can be defined only in euclidean space.* But there are further restrictions for the I_{ik} isometric tensor defined by (3) and (5). If we accept the criterium of the continuous deformation then the continuous curves of the undistorted picture should be mapped also into continuous curves in the course of distortion. On the other hand, the relation of the arc elements of such curves is determined by H_{ij}:

$$d\tilde{x}_i = H_{ij}(x_p)\, dx_j \,. \tag{32}$$

Consider the following simple case: $H_{ij}(x_p) = I_{ij}(x_p)$ i.e. $S_{kj}(x_p) = \delta_{kj}$. For the integrability of (32) there are well-known criteria in the theory of differential equations, according to which e.g. in case of the above determined H_{ij} only the $H_{ij}(x_p) = \underset{\circ}{I}_{ij}$ choice can be made ($\underset{\circ}{I}_{ij}$ is constant). There are similar restrictions, too, if $S_{kj} \neq \delta_{kj}$.

After the above explanation — dispensing with the detailed proof — it can be stated that if the isometric tensor is known at one point then the integrability conditions determine it at every other point. In other words: knowing the S_{kj} symmetric component pointwise the rigid body-like rotation can also be determined point by point to the extent of a constant rotation valid for the whole picture. As a result of the above discussion giving the distortion conditions for the whole picture we determine in fact the rigid body-like rotation, too; for this reason this latter is ignored in the forthcoming part.

2. General introduction of the distortion tensor

The question arises how the concepts discussed in the above chapter may be derived generally, without the unnecessary restrictions mentioned there. First of all, distortion conditions of the entire image space should be discussed instead of studying the neighbourhood of one single point. Consequently, it should not be any longer assumed that the undistorted, resp. the corresponding distorted picture elements may be found at the same place, i.e. the distortion tensor forms a connection between spaces more or less independent of each other. Secondly: in order to keep the discussion general, it is necessary to use general coordinates. It is an essential condition that the ***object space*** as well as the ***image space*** (in a wider sense) should be metric (this being, of course, the sine qua non of mentioning distortion at all). This condition means, in our case, nothing but that contravariant and covariant vectors may be defined at every point of the studied region, and there is a symmetric metric tensor (g_{ij}) forming a connection between these vectors. In the course of derivations, the differentiability of coordinate functions is assumed everywhere ([5], [6]).

Our task is therefore to submit the geometric (i.e. geometrically characterizable) relation between *object* and *image* to a general study. Part of this relation, the deviation between object and picture, is the *difference*; the quantitative description of one element of this difference — the distortion — is the proper purpose of our study. The other part of the relation is an *identity* between object and image. The character of identity limits all further discussion; the topologically homeomorphic property of object and image is assumed. This means that the points of object and image correspond conversely unambiguously and conversely continuously to each other, which simultaneously results the equidimensionality of the transformation.

For the sake of analytical discussion, the space expanded by the object, resp. by the image is to be embedded in a coordinate system. Hereafter, we restrict ourselves to embeddings where the correspondence between the points and coordinates of the image, resp. of the object is always unambiguous and conversely continuous [5] [6]. The two embeddings may occur independently of each other as well, but in this case a special formula should be sought for in order to describe the correspondence of object and image points. (As a result of constraints made in embedding, this functional relation characterizing the distortion is able to reflect the homeomorphic property of the relation between object and image, because in this case the functional relation is always unambiguous and conversely continuous, too.) For practical purposes let us choose the embedding so as to express the correspondence of the object space and image space by the relation

$$\tilde{x}^i(\tilde{P}) = x^i(P)\,. \tag{1}$$

(P is the point of object space, $\tilde{P}$ the corresponding point of the picture space, x^i, resp. $\tilde{x}^i$ the coordinate-n-tuples belonging to the points.)

Assumption (1) does not affect generality, since — if in case of an independent embedding — the formula describing the correspondence of the coordinate-n-tuples was

$$F\left[\tilde{x}^i(\tilde{P});\, x^j(P)\right] = 0\,, \tag{2}$$

then, introducing new $\bar{x}^i$ coordinates in the object space with the transformation relation

$$F\left[\bar{x}^i(P); x^j(P)\right] = 0, \tag{2a}$$

the relation of the coordinates of the corresponding points is described by the equation

$$\tilde{x}^i(\tilde{P}) = \bar{x}^i(P). \tag{1a}$$

The small neighbourhood (more precisely: the infinitely small neighbourhood) of the arbitrary point P of the object space is transformed into the similarly small (infinitely small) neighbourhood of point $\tilde{P}$ of the image space. In order to be able to discuss this correspondence in a simple way, such quantities defined in point P, resp. $\tilde{P}$ should be sought for, which are at least able to describe the immediate neighbourhood of the selected points. Quantities like that may be constructed as follows. Let us draw a set of curves consisting of smooth curves passing through point P and densely covering the small neighbourhood in question of point P. Let the arc length measured from point P be the parameter variable for each curve. (Those singular curves where this cannot be done are excluded from our study.) From a curve as defined above

$$x^i = x^i(s) \tag{3}$$

the quantity

$$u^i(P) = \frac{d\,x^i(s)}{ds} \tag{4}$$

may be formed, for each curve separately. It is clear that $u^i(P)$ is a contravariant vector, its absolute value being *1*.

The equivalent of the above set of curves covers the neighbourhood of point $\tilde{P}$ following from the homeomorphism. Since the chosen curve (3) is mapped — according to relation (1) — to the curve

$$\tilde{x}^i \equiv \tilde{x}^i(s) = x^i(s), \tag{5}$$

it still holds that

$$\tilde{u}^i(\tilde{P}) = \frac{d\tilde{x}^i(s)}{ds} = \frac{dx^i(s)}{ds} = u^i(P). \tag{6}$$

We must consider that in this latter case, s is no longer an arc length parameter, hence $\tilde{u}^i(\tilde{P})$ is not a unit vector. Relation (6) obtained for the contravariant components is only the result of (1) and does not hold for the covariant components.

In an arbitrary coordinate system the scalar product of two vectors is produced by the bilinear form obtained by the metric tensor and the components of the two vectors. Thus, for instance, the scalar product in the coordinate system x^i of the object space is

$$(u, v) = g_{ij}(P)\, u^i(P)\, v^j(P)\,, \tag{7}$$

where u^i and v^j are two arbitrary vectors, determined by equation (4). Accordingly, in the image space

$$(\tilde{u}, \tilde{v}) = \tilde{g}_{ij}(\tilde{P})\, \tilde{u}^i(\tilde{P})\, \tilde{v}^j(\tilde{P})\,. \tag{8}$$

Let us form the scalar product $(\tilde{u}, \tilde{v})$ in the image space by means of vectors u^i, resp. v^j in the object space. Considering (6), this is easily done:

$$(\tilde{u}, \tilde{v}) = \tilde{g}_{ij}(\tilde{P})\, u^i(P)\, v^j(P)\,. \tag{9}$$

Equation (9) is a *tensor equation* in the object space, since u^i and v^j are vectors at point P and $(\tilde{u}, \tilde{v})$ is an invariant scalar ordered to point P. Equation (9) is therefore defining $\tilde{g}_{ij}(\tilde{P})$ as a tensor at point P of the object space.

Our equation forms a connection between the scalar product of the unit vectors of the object space and their equivalent in the image space. Let us therefore regard in such a tensor relation of

$$(\tilde{u}, \tilde{v}) = \gamma_{ij}(P)\, u^i(P)\, v^j(P) \tag{10}$$

the tensor γ_{ij} as the distortion characteristic, i.e. as the *distortion tensor*. When comparing (9) and (10), this distortion tensor is defined more clearly:

$$\gamma_{ij}(P) = \tilde{g}_{ij}(\tilde{P})\,. \tag{11}$$

(It is easy to see that even in case of more general postulates only the distortion tensor γ_{ij} is required for describing the geometric deviations i. e. distortion in the small neighbourhood of the corresponding points of the two objects. That is to say, the whole system of geometric differences is known if the equivalent in the image space of all geometric invariants definable in the point in question of the object space can be given. For this purpose it is only necessary to know, in the known way by differential geometry [4] [6], the metric tensor and its derivates in the point of image space; in our case, it is equivalent to know the metric tensor at *all* points of the given spaces, whilst considering the conditions concerning differentiability.)

The metric tensor being in any case symmetric, so the distortion tensor is symmetric as well.

Let us return to equation (10) and compare it with the relation of the preceding part (1.9). It is clear that (10) is a generalization of (1.9), hence considerations following the eigenvalue equation are applicable by corresponding generalization in this case, too. The eigenvalue equation now takes the following form [4]:

$$(\gamma_{ij} - \lambda g_{ij})\, u^j = 0\,, \tag{12}$$

$$\det \left| \gamma_{ij} - \lambda g_{ij} \right| = 0\,. \tag{12a}$$

From the eigenvalues (whether real or complex) obtained by the equation, the distortion invariants introduced in the first part may further on also be formed, on the basis of relations (1.22)—(1.24).

Instead of the determinant of the matrix of equation (12a) the characteristic equation may read by means of the invariant determinant differing from it only by the constant, as follows [6]:

$$0 = \frac{1}{n!}\, \varepsilon^{r_1 r_2 \ldots r_n} \varepsilon^{s_1 s_2 \ldots s_n} (\gamma_{r_1 s_1} - \lambda g_{r_1 s_1})(\gamma_{r_2 s_2} - \lambda g_{r_2 s_2}) \ldots (\gamma_{r_n s_n} - \lambda g_{r_n s_n}). \tag{13}$$

Here $\varepsilon^{r_1 r_2 \ldots r_n}$ is the Levi-Civita tensor the definition of which is, in a right-handed system, the following:

$$\varepsilon^{r_1 r_2 \ldots r_n} = g^{-1/2} \cdot \underset{r_1 r_2 \ldots r_n}{e}\,; \quad \underset{r_1 r_2 \ldots r_n}{e} = \begin{cases} 0, & \text{if at least two indices equal} \\ 1, & \text{if the indices form an even permutation} \\ -1, & \text{if the indices form an odd permutation} \end{cases}$$

where $\underset{r_1 r_2 \ldots r_n}{e}$ is the antisymmetric unit matrix and g the matrix determinant of the covariant metric tensor.

Using the binomial theorem on (13), it results in the following form:

$$0 = \sum_{l=0}^{n} (-1)^{n-l} \lambda^{n-l} \Gamma_l\,, \tag{14}$$

where

$$\Gamma_l = \frac{1}{(n-l)!\, l!}\, \varepsilon^{r_1 r_2 \ldots r_n} \varepsilon^{s_1 s_2 \ldots s_n} \gamma_{r_1 s_1} \ldots \gamma_{r_l s_l}\, g_{r_{l+1} s_{l+1}} \ldots g_{r_n s_n}\,. \tag{15}$$

Since there are only tensors at the right side of (15), the Γ_l quantities are invariants. $\Gamma_1, \ldots \Gamma_n$ are the invariants of tensor γ_{ij}; $\Gamma_0 = 1$. The expression of invariants is considerably simplified if the distortion tensor γ_{ij} is used in the compound form

$$\gamma^k_{\cdot j}(P) = g^{ki}(P)\, \gamma_{ij}(P) = g^{ki}(P)\, \tilde{g}_{ij}(\tilde{P})\,. \tag{16}$$

In this case, one of the ε-tensors has to be written in the covariant form in (15) and the compound metric tensor representable by the unit matrix should be substituted into the equation. The following simple relation is obtained:

$$\Gamma_l = \frac{1}{(n-l)!\,l!} \underset{r_1 \ldots r_l r_{l+1} \ldots r_n}{e} \; \underset{s_1 \ldots s_l r_{l+1} \ldots r_n}{e} \; \gamma^{r_1}_{\cdot s_1} \ldots \gamma^{r_l}_{\cdot s_l} . \tag{15a}$$

Subsequently let us sum up the general method of computation of the distortion characteristics. The first step is to embed object and image in a suitably chosen coordinate system, noting condition (1). In both coordinate systems the metric tensor is determined as the function of the point. By means of relation (16), the distortion tensor is obtained as a function of the coordinates of point P. After having determined the invariants Γ_l (15a), the eigenvalues can be computed from (14). They give the different distortion characteristics as a function of the point. These latter two steps may be combined to one single step by previously directly expressing the distortion characteristics with the invariants Γ_l, as, e.g., in the preceding part in the two-dimensional ratio and space distortion relation.

Finally the so obtained general results are studied concerning two-dimensional object and image and, at the same time, it is shown that for practical computations the compound form of the distortion tensor is the most advisable one. Forming the two invariants of two-dimensional $\gamma^{i}_{\cdot j}$ according to (15a),

$$\left.\begin{aligned} \Gamma_1 &= \underset{r_1 r_2}{e}\; \underset{s_1 r_2}{e}\; \gamma^{r_1}_{\cdot s_1} = \gamma^{1}_{\cdot 1} + \gamma^{2}_{\cdot 2} \\ \Gamma_2 &= \frac{1}{2} \underset{r_1 r_2}{e}\; \underset{s_1 s_2}{e}\; \gamma^{r_1}_{\cdot s_1} \gamma^{r_2}_{\cdot s_2} = \gamma^{1}_{\cdot 1} \gamma^{2}_{\cdot 2} - \gamma^{1}_{\cdot 2} \gamma^{2}_{\cdot 1} \end{aligned}\right\} \tag{17}$$

the relation equivalent to the equation of the preceding chapter (1.27) has been obtained. Whilst, therefore, the invariants are obtained by the covariant, resp. contravariant form of the distortion tensor only by simultaneous application of the metric tensor (see equation (15)), the matrix representing the distortion tensor in a compound form gives them *in itself* — actually according to rules entirely corresponding with the invariant formation in the Cartesian system.

Ratio and area distortion may thus be formed further by means of (1.28), (1.29) and (17). The question of set-up of the distortion indicatric figure must be studied, on the other hand, in fuller details. The matrix representing $\gamma^{i}_{\cdot j}$ in a compound form is, namely, not bound to be symmetric; there is nothing known but that its two possible forms are mirror images of one another as referred to the main diagonal:

$$\gamma^{i}_{\cdot j} = \gamma_{j}^{\cdot i} = \gamma^{i}_{j} .$$

Transferring the eigenvalue equation (12) to the compound form,

$$(\gamma_j^i - \lambda \, \delta_j^i) \, u^j = 0 \tag{12b}$$

and taking out its "first line"

$$\gamma_1^1 u^1 + \gamma_2^1 u^2 - \lambda u^1 = 0$$

a relation of identical form to (1.15a) is obtained. The ratio of contravariant components of the eigenvectors may thus be given in accordance with (1.31):

$$\frac{a^2}{a^1} = \frac{\gamma_2^2 - \gamma_1^1}{2 \, \gamma_2^1} \pm \sqrt{\left(\frac{\gamma_2^2 - \gamma_1^1}{2 \, \gamma_2^1} \right)^2 + \frac{\gamma_1^2}{\gamma_2^1}} \tag{18}$$

or

$$\ln \frac{a^2}{a^1} = 0{,}5 \ln \frac{\gamma_1^2}{\gamma_2^1} + \operatorname{arsh} \frac{\gamma_2^2 - \gamma_1^1}{2 \sqrt{\gamma_2^1 \gamma_1^2}} \, . \tag{18a}$$

With this knowledge, the direction of eigenvectors, i.e. the orientation of the distortion indicatrix, the *set-up*, is determined.

3. Determination of the metric tensor on general surfaces

The preceding part specified the method of general description of the distortion between object and image space by introducing the distortion tensor. The invariants of the distortion tensor help to give a direct picture about the character and degree of distortion. As to the computation of the distortion tensor γ_j^i embracing all further distortion informations, however, entirely general and, therefore, not too expressive directions for special cases were given. Relation

$$\gamma_j^k (P) = g^{ki} (P) \, \tilde{g}_{ij} (\tilde{P})$$

of (2.16) is worth something only if the metric tensors are known in the one-to-one points of object and image space, which is a not negligible requirement in a general object and image coordinate system.

This chapter has just the purpose to facilitate the practical computation by giving a general method for the determination of the metric tensor in the two-dimensional surface points with an arbitrary coordinate network, embedded in the three-dimensional Euclidean space. The results will be useful in a great part of our computations, our distorted images frequently appearing at a two-dimensional curved surface from the dome fresco to the cinerama screen.

As a start, the surface studied is embedded in some trivial three-dimensional coordinate system as defined in the Euclidean space. First of all the rectangular, cylindrical and spherical coordinates are taken into account; their metric tensors being well known (Fig. 4). If the surface where the metric tensor is sought for, coincides with one level surface of one of the coordinates (further on always called the third coordinate, for the sake of simplicity), there are no special troubles. In this case, the restriction to the surface, the "staying at the surface" is synonymous with the statement that the third component of contravariant vectors drawn at any point of the surface is zero. In the relations containing the metric tensor, as for instance, in the expression

$$(a, b) = g_{ij}\, a^i b^i$$

there is no place for the third line and column of the covariant metric tensor. By canceling these, the two-dimensional surface metric tensor is obtained.

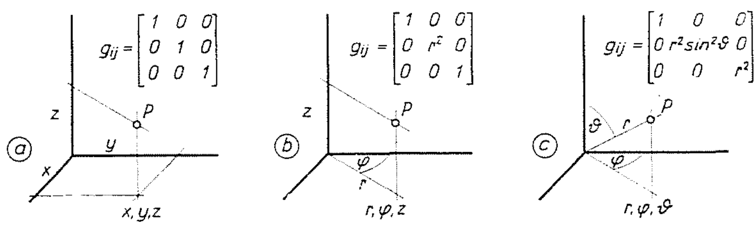

Fig. 4

If there does not exist the fortunate case of coincidence of the surface studied and one level surface, this state may be produced by a suitable coordinate transformation. A general solution is given therefore by the following set of operations:

a) Starting from the already mentioned trivial embedding: $\bar{x}^\alpha$. (The Greek superscripts mean here and further on running integers from 1 to 3.)

b) The next step is the second embedding of the surface: x^α. In this new coordinate system the surface studied must absolutely be a level surface — e.g. $x^3 = 0$. x^1 and x^2 may be arbitrary (it should be taken into account, however, that they have to satisfy a relation of the (2.1) kind in the course of further computations). If, in the system $\bar{x}^\alpha$, the surface is represented by

$$\bar{x}^\alpha = f_\alpha (x^1 ; x^2), \tag{1}$$

where x^1 and x^2, the new surface coordinates required, are parameters then the requirements as given in *b)* are satisfied, for instance, by the system x^α defined by the following relation:

$$\bar{x}^\alpha = f_\alpha (x^1 ; x^2) + x^3 \cdot \bar{m}^\alpha , \tag{2}$$

where vector $\bar{m}^\alpha$ varies along the surface so that its component perpendicular to the surface does not disappear anywhere and is differentiable along the surface.

The coordinate derivatives along the studied surface $x^3 = 0$ are as follows:

$$d_{\alpha\beta} = \frac{\partial \bar{x}^\alpha}{\partial x^\beta} = \begin{bmatrix} \frac{\partial f_1}{\partial x^1} & \frac{\partial f_1}{\partial x^2} & \bar{m}^1 \\ \frac{\partial f_2}{\partial x^1} & \frac{\partial f_2}{\partial x^2} & \bar{m}^2 \\ \frac{\partial f_3}{\partial x^1} & \frac{\partial f_3}{\partial x^2} & \bar{m}^3 \end{bmatrix}. \tag{3}$$

c) After this, the known metric tensor $\bar{g}_{\alpha\beta}$ of the trivial embedding $\bar{x}^\alpha$ is transformed by means of (3) into the system x^α: $g_{\alpha\beta}$. Since our surface in x^α is a level surface, the surface metric tensor is obtained by deleting the third line and column of $g_{\alpha\beta}$. It is not even worthwhile to compute these useless components; it is sufficient to form only the two-dimensional surface metric tensor:

$$g_{ij} = \frac{\partial \bar{x}^\alpha}{\partial x^i} \frac{\partial \bar{x}^\beta}{\partial x^j} \bar{g}_{\alpha\beta}. \tag{4}$$

It may be observed that in this relation the last column of (3) has not even been used, the transformation being thus carried out but by the 3×2 matrix of the coordinate derivatives (actually, by the derivatives of (1)):

$$\frac{\partial \bar{x}^\alpha}{\partial x^i} = \begin{bmatrix} \frac{\partial \bar{x}^1}{\partial x^1} & \frac{\partial \bar{x}^1}{\partial x^2} \\ \frac{\partial \bar{x}^2}{\partial x^1} & \frac{\partial \bar{x}^2}{\partial x^2} \\ \frac{\partial \bar{x}^3}{\partial x^1} & \frac{\partial \bar{x}^3}{\partial x^2} \end{bmatrix}. \tag{5}$$

The above sequence of thoughts proved the transformation rule concerning coordinate transformation inherent in dimension reduction; further, the final results (4) and (5) just obtained will be used.

Application of this method is shown by an example. The metric tensor of the spherical surface with radius R is sought for, together with the surface coordinates φ and ϑ as defined in accordance with Fig. 4c. (The task is so simple in order for it to be controllable.) Nothing but the metric tensor of Cartesian coordinates may be considered as known, thus

a) the trivial embedding takes place in the rectangular coordinates x, y, z; $g_{\alpha\beta} = \delta_{\alpha\beta}$. Our relations corresponding to (1):

$$\begin{aligned} x &= \mathrm{R} \cdot \sin\vartheta \cdot \cos\varphi \\ y &= \mathrm{R} \cdot \sin\vartheta \cdot \sin\varphi \qquad && x, y, z \Rightarrow \bar{x}^\alpha \\ z &= \mathrm{R} \cdot \cos\vartheta && \varphi, \vartheta \Rightarrow x^i. \end{aligned}$$

As to relation (2) concerning embedding *b)*, it is not necessary to note this, but it may be sufficient to form the coordinate derivatives (5):

$$\frac{\partial \bar{x}^\alpha}{\partial x^i} = \begin{bmatrix} -R \sin \vartheta \sin \varphi & R \cos \vartheta \cos \varphi \\ R \sin \vartheta \cos \varphi & R \cos \vartheta \sin \varphi \\ 0 & -R \sin \vartheta \end{bmatrix}$$

c) By means of (4), transformation is carried out:

$$g_{ij} = \frac{\partial \bar{x}^\alpha}{\partial x^i} \frac{\partial \bar{x}^\alpha}{\partial x^j} = \begin{bmatrix} R^2 \sin^2 \vartheta & 0 \\ 0 & R^2 \end{bmatrix}.$$

Resulting from this method, the metric tensor of the spherical surface is obtained with respect to the surface coordinates φ, ϑ. This result in fact corresponds with the metric tensor visible on Fig. 4c if the first line and column of the same are deleted and $r = R$.

So far, the *general* method of the determination of the metric tensor has been indicated and illustrated by way of example. Further on, procedures derivable from the general method will be dealt with.

The method of local embedding is identical with the general procedure included in items *a)—c)*, with the restriction that the new surface coordinates x^i are equal to the two coordinates of the original $\bar{x}^\alpha$ embedding. (As a matter of course, $\bar{x}^1$ and $\bar{x}^2$ should meet with the general requirements concerning coordinates embedding.) In this case, (1) becomes:

$$\begin{aligned} \bar{x}^1 &= x^1 \\ \bar{x}^2 &= x^2 \\ \bar{x}^3 &= f_3(x^1; x^2), \end{aligned} \tag{1a}$$

and the coordinate derivatives:

$$d_{\alpha i} = \frac{\partial \bar{x}^\alpha}{\partial x^i} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \dfrac{\partial f_3}{\partial x^1} & \dfrac{\partial f_3}{\partial x^2} \end{bmatrix}. \tag{5a}$$

The surface metric tensor may be, henceforward, computed by means of (4).

It is worthwhile considering which data of the surface were used for the determination of the metric tensor: as a matter of fact, nothing but the two differential coefficients in the last line of (5a) have been necessary. This fact might be illustrated by the following considerations. The last line of (2)

$$\bar{m}^3 x^3 = \bar{x}^3 - f_3(x^1; x^2) = \bar{x}^3 - f_3(\bar{x}^1; \bar{x}^2)$$

is regarded as a scalar-vector function in the coordinate system $\bar{x}^\alpha$. The gradient vector hereof is:

$$\frac{\partial}{\partial \bar{x}^i}(\bar{m}^3 x^3) = \left(-\frac{\partial f_3}{\partial \bar{x}^1}; -\frac{\partial f_3}{\partial \bar{x}^2}; 1\right) = \left(-\frac{\partial f_3}{\partial x^1}; -\frac{\partial f_3}{\partial x^2}; 1\right) \tag{6}$$

perpendicular to the surfaces $\bar{m}^3 x^3 = const$, thus to $x^3 = 0$ as well. Accordingly, the first two components of the normal vector reduced to the third component stand in the last line of (5a):

$$\frac{\partial \bar{x}^\alpha}{\partial x^i} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -\frac{n_1}{n_3} & -\frac{n_2}{n_3} \end{bmatrix}. \tag{7}$$

The components n_α are to be understood in the $\bar{x}^\alpha$ system and are — taking into account (6) and the definition-like covariant character of the gradient vector — covariants.

Our recent results led to interesting statements. For the determination of g_{ij} it is not necessary to explicitly know the equation of the surface in question; it is sufficient to know nothing but the surface normal at the points studied. This possibility highly facilitates the solution of some tasks. The application is shown in example 6.

If the tangent-plane would be formed point by point by means of the surface normal vector, and if, as step *b)* of our general method, a local coordinate system would be drawn where the *tangent-plane* is the level surface, it might be possible to determine the surface metric tensor point by point, *locally*. This consideration would lead to the same result as (7); the above described procedure is called, for this reason, "the method of local embedding".

Method of common embedding. The methods used so far are suitable for the computation of both $g_{ij}(P)$ and $\tilde{g}_{ij}(\tilde{P})$. In case of object and image space determined in the highest degree independently from each other, there is nothing to do but to determine separately the two metric tensors in corresponding points by means of effective embeddings. In the very frequent case, however, when object and image allow themselves to be embedded in a *common* coordinate system, there is still another way to compute the metric tensors, at least in any of the object and image spaces.

Let the coordinate 3-tuple $x^\alpha(\tilde{P})$ of the image points in the coordinate system embedding the object and depending on the coordinates $x^i(P)$ of the object points be given, for instance, by the following equations:

$$x^\alpha(\tilde{P}) = f_\alpha\left[x^1; x^2(P)\right]. \tag{8}$$

A new coordinate system regarding the image and meeting with the requirement (2.1) may be introduced by the following equations:

$$x^\alpha(\tilde{P}) = f_\alpha\left[\tilde{x}^1; \tilde{x}^2(\tilde{P})\right]. \tag{1b}$$

This relation is similar to the initial equation (1) of the general metric tensor computation method by means of which the transformation similar to that met with there may be carried out. The results are particularly easy to be dealt with, if $x^i(P) = x^i(\tilde{P})$ — which, in turn, is valid in many cases. (E.g in case of every projection task to the coordinate φ and ϑ of the spherical coordinate system fixed in the centre of projection.) According to the foregoing,

$$\left.\begin{array}{l} x^1(\tilde{P}) = x^1(P) = \tilde{x}^1(\tilde{P}) \\ x^2(\tilde{P}) = x^2(P) = \tilde{x}^2(\tilde{P}) \\ x^3(\tilde{P}) = f_3\left[\tilde{x}^1; \tilde{x}^2(\tilde{P})\right] \end{array}\right\}. \tag{2b}$$

The coordinate derivatives are:

$$d_{\alpha l} = \frac{\partial x^\alpha}{\partial \tilde{x}^i} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \dfrac{\partial x^3(\tilde{P})}{\partial \tilde{x}^1(\tilde{P})} & \dfrac{\partial x^3(\tilde{P})}{\partial \tilde{x}^2(\tilde{P})} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \dfrac{\partial x^3(\tilde{P})}{\partial x^1(P)} & \dfrac{\partial x^3(\tilde{P})}{\partial x^2(P)} \end{bmatrix}. \tag{3b}$$

The new metric tensor of the image could be expressed with the aid of (4):

$$\tilde{g}_{ij}(\tilde{P}) = g_{ij}(\tilde{P}) + g_{i3}(\tilde{P})\frac{\partial x^3(\tilde{P})}{\partial x^j(P)} + g_{3j}(\tilde{P})\frac{\partial x^3(\tilde{P})}{\partial x^i(P)} + g_{33}(\tilde{P})\frac{\partial x^3(\tilde{P})}{\partial x^i(P)}\frac{\partial x^3(\tilde{P})}{\partial x^j(P)}. \tag{9}$$

The method is substantially the following: the well-known "trivial" embedding serving as the starting point for the computation of the metric tensor of the image surface is the coordinate system embedding the object. In this way, the metric tensor of the image is obtained by a relatively convenient transformation, simultaneously satisfying (2.1).

Example 5 applies the method of "common embedding".

The method of fitting surfaces is applicable in case of the object and image surface not bending outwards from each other, but joining each other. Here the third dimension is not required, computations may be carried out in two dimensions.

Let the surface coordinates be x^i and the known metric tensor g_{ij}. Let the one-to-one mapping of the object points P and the image points $\tilde{P}$ be as follows:

$$x^i(\tilde{P}) = f_i\left[x^1; x^2(P)\right]. \tag{10}$$

A new coordinate system may be introduced in the image space meeting with (2.1):

$$x^i(\tilde{P}) = f_i\left[\tilde{x}^1; \tilde{x}^2(\tilde{P})\right]. \tag{1c}$$

The metric tensor of the image coordinate system may be computed from its object space:

$$\tilde{g}_{ij}(\tilde{P}) = \frac{\partial x^p(\tilde{P})}{\partial \tilde{x}^i(\tilde{P})} \cdot \frac{\partial x^q(\tilde{P})}{\partial \tilde{x}^j(\tilde{P})} g_{pq}(\tilde{P}). \tag{11}$$

Accordingly, the distortion tensor is

$$\gamma_j^k = g^{ki}(P)\tilde{g}_{ij}(\tilde{P}) = g^{ki}(P)\frac{\partial x^p(\tilde{P})}{\partial \tilde{x}^i(\tilde{P})} \frac{\partial x^q(\tilde{P})}{\partial \tilde{x}^j(\tilde{P})} g_{pq}(\tilde{P}),$$

taking the very convenient form of

$$\gamma_j^k = \frac{\partial x^p(\tilde{P})}{\partial \tilde{x}^k(\tilde{P})} \cdot \frac{\partial x^p(\tilde{P})}{\partial \tilde{x}^j(\tilde{P})} = \frac{\partial x^p(\tilde{P})}{\partial x^k(P)} \cdot \frac{\partial x^p(\tilde{P})}{\partial x^j(P)} \tag{12}$$

in the Cartesian system x_i.

This method will be used in examples 2 and 3.

4. Examples for application

Entire generalization has been strived for so far in derivations and results, and it is for this reason that the individual possibilities of application have not yet been discussed. This paragraph is designed to present the applicability of our methods of distortion computation by way of some examples. These examples should illustrate the wide applicability of the processes of the computations mentioned.

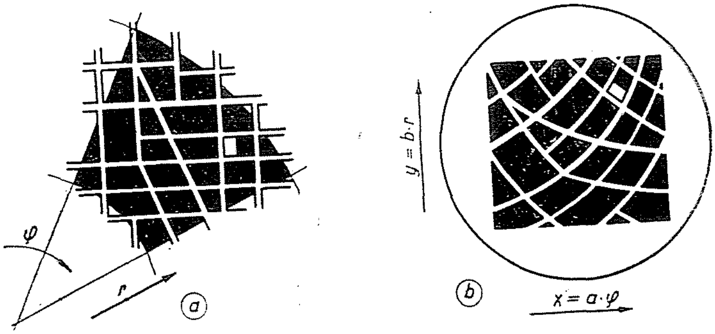

Fig. 5

1st example. The distortion of picture B of radiolocators is studied. In this mode of operation some sector is scanned by the locator, whereby the individual points of the scanned surface (Fig. 5a, black space) are determined by distance r and the azimuth φ. Potentials, at the same time, proportional with the coordinates φ, r, control the deviation x—y of the locator picture tube and this draws it in an orthogonal system:

$$x = a \cdot \varphi \,,$$
$$y = b \cdot r \,. \tag{1}$$

The picture appearing on the screen will apparently become distorted (which is clearly visible in Fig. 5b).

Our task is to determine the distortion tensor, the ratio and are a distortion as well as the setting, with reference to each point of the picture. In the first step, the object and the image are embedded in a suitable coordinate system, whilst complying with (2.1). In our case, the system φ, r seems to be advisable. Further on, the metric tensor concerning the coordinates φ, r is to be determined on the object as well as on the image. φ and r on the object mean polar coordinates, the metric tensor being of the well-known form

$$g_{ij} = \begin{bmatrix} r^2 & 0 \\ 0 & 1 \end{bmatrix} \Longrightarrow g^{ij} = \begin{bmatrix} r^{-2} & 0 \\ 0 & 1 \end{bmatrix}.$$

The metric tensor of the image is known only in the Cartesian system x, y $(\tilde{x}^i) : \tilde{g}_{ij} = \delta_{ij}$. With its transformation, the metric tensor of the picture is obtained in the system φ, r $(\bar{\tilde{x}}^i)$:

$$\bar{\tilde{g}}_{ij} = \frac{\partial \tilde{x}^p}{\partial \bar{\tilde{x}}^i} \frac{\partial \tilde{x}^q}{\partial \bar{\tilde{x}}^j} \tilde{g}_{pq} \,.$$

The coordinate derivatives from (1):

$$\frac{\partial \tilde{x}^i}{\partial \bar{\tilde{x}}^j} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}.$$

Thus, in the system $\bar{\tilde{x}}^i$ complying with the relation $\bar{\tilde{x}}^i(\tilde{P}) = x^i(P)$

$$\bar{\tilde{g}}_{ij} = \begin{bmatrix} a^2 & 0 \\ 0 & b^2 \end{bmatrix}.$$

In system φ, r of the object, the distortion tensor is obtained from (2.16) as follows:

$$\gamma^i_k = g^{ij} \bar{\tilde{g}}_{jk} = \begin{bmatrix} a^2/r^2 & 0 \\ 0 & b^2 \end{bmatrix}.$$

Area distortion by (1.28):

$$T = \sqrt{\Gamma_2} = \frac{ab}{r} \,.$$

Ratio distortion according to (1.29a):

$$A = \max \left\{ \frac{rb}{a} ; \frac{a}{rb} \right\}.$$

It is clear that the ratio distortion as well as the area distortion are both depending on the coordinate r. $A = 1$ (i.e. there is no ratio distortion) at the places $r = \frac{a}{b}$; here a b-fold magnification occurs, i.e. $T = b^2$. The direction of coordinate lines φ, r is obtained for the direction of the eigenvectors, thus for the setting of the distortion, on the basis of (2.18).

The distortion of the picture of C, resp. E type locators could be similarly discussed. It is easy to carry out tests concerning the *common* distortion resulting "officially" from the mode of representation, respectively, from the errors of the picture tube deflecting system.

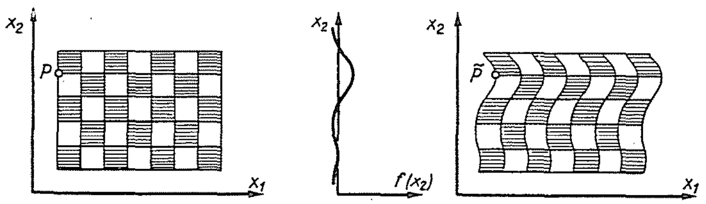

Fig. 6

2nd example. This image distortion is a frequently occurring phenomenon in television receivers. If a disturbance superimposes the horizontal deflexion signal (for instance hum or control swing in the indirect synchronous circuit), some lines of the picture are slipping as compared to each other; an unpleasant distortion is observed. Fig. 6 represents the distortion of checkerboard pattern television picture, function $f(x_2)$ indicating the degree of slip of the single lines.

When solving the example, the original and the distorted picture are considered as lying in a plane, and the method of "fitting surfaces" of para. 3 is applied. Points $\widetilde{P}$ of the distorted image are given, according to (3.10), by the Cartesian coordinates of the undistorted image as follows:

$$\begin{aligned} x_1(\widetilde{P}) &= x_1(P) + f[x_2(P)] \\ x_2(\widetilde{P}) &= x_2(P) \end{aligned} \quad \Longrightarrow \quad \frac{\partial x_i(\widetilde{P})}{\partial x_j(P)} = \begin{bmatrix} 1 & \dfrac{df}{dx_2} \\ 0 & 1 \end{bmatrix}.$$

Regarding the Cartesian origin of system x_i, the distortion tensor is formed by (3.12):

$$\gamma_{jk} = \frac{\partial x_p(\tilde{P})}{\partial x_j(P)} \frac{\partial x_p(\tilde{P})}{\partial x_k(P)} = \begin{bmatrix} 1 & \dfrac{df}{dx_2} \\ \dfrac{df}{dx_2} & 1+\left(\dfrac{df}{dx_2}\right)^2 \end{bmatrix}.$$

The invariants of the distortion tensor are:

$$\Gamma_2 = 1 + \left(\frac{df}{dx_2}\right)^2 - \left(\frac{df}{dx_2}\right)^2 = 1,$$

$$\Gamma_1 = 2 + \left(\frac{df}{dx_2}\right)^2,$$

the area distortion therefore being (1.28): $T = \sqrt{\Gamma_2} = 1$, which was to be expected considering the figure. The ratio distortion (1.29):

$$A = \exp\left\{\text{arch}\,\frac{\Gamma_1}{2\sqrt{\Gamma_2}}\right\} = \exp\left\{\text{arch}\left[1 + \frac{1}{2}\left(\frac{df}{dx_2}\right)^2\right]\right\}.$$

In the practically interesting case $\dfrac{df}{dx_2} < 1$

$$A \approx \exp\frac{df}{dx_2}.$$

From (1.31a), the setting angle Ψ is

$$\ln \text{tg}\,\Psi = \text{arsh}\,\frac{\left(\dfrac{df}{dx_2}\right)^2}{2\dfrac{df}{dx_2}} = \text{arsh}\,\frac{1}{2}\frac{df}{dx_2}.$$

Using the relations concerning the hyperbolic functions:

$$\text{tg}\,\Psi = A^{\pm\frac{1}{2}}.$$

In the course of television picture transmission various forms of distortion may occur, only one of which was picked out here. The trapezoidal distortion of the iconoscope, the pincushion distortion of picture tubes with great angles of deflexion, the distortions resulting from the inadequate wave form of the horizontal-vertical deflexion current, the distortion arising from the curvature of the display screen — may without exception easily be described by means of the distortion tensor.

3rd example. The distortion to be tested in this case occurs with slit-shutter cameras. Should moving objects be photographed and the speed of the camera projection of the object (w_i) is comparable with the slit-shutter speed (v) of the apparatus, the image of the object shows a strong distortion by the photograph. Our method of computation is now applied for this phenomenon.

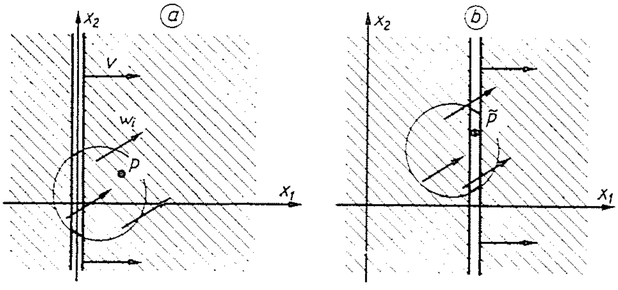

Fig. 7

The investigation is carried out with the following assumptions:
— the slit of the shutter is considered as infinitely small
— the "ideal snapshot" of the object is considered as undistorted
— the rigid body-like movement of the projection of the object is assumed
— we content ourselves with the case of $w_1 < v$.

The investigation is effected in the Cartesian coordinates x_i fixed to the photographic plate.

Let the slit moving from left to right be exactly at $x_1 = 0$ at the moment $t = 0$. This instant is represented in Fig. 7a. The undistorted projection of the object is e.g. a circular plane, its point P being separately indicated. The image of this point only later gets, at the instant of Fig. 7b, in point $\widetilde{P}$ onto the photographic plate.

The time passed between Fig. 7a, respectively, Fig. 7b may be described in two ways:

$$\frac{x_1(\widetilde{P})}{v} = \frac{x_1(\widetilde{P}) - x_1(P)}{w_1}.$$

Using this relation, the connection between coordinates of points P and $\widetilde{P}$ becomes as follows:

$$\begin{aligned} x_1(\widetilde{P}) &= x_1(P) \cdot \frac{v}{v - w_1} \\ x_2(\widetilde{P}) &= x_1(P) \cdot \frac{w_2}{v - w_1} + x_2(P) \end{aligned} \quad \Longrightarrow \quad \frac{\partial x_i(\widetilde{P})}{\partial x_j(P)} = \begin{bmatrix} \dfrac{v}{v - w_1} & 0 \\ \dfrac{w_2}{v - w_1} & 1 \end{bmatrix}.$$

Again applying the method of fitting surfaces as well as (3.12):

$$\gamma_{ij} = \begin{bmatrix} \dfrac{v^2 + w_2^2}{(v - w_1)^2} & \dfrac{w_2}{v - w_1} \\ \dfrac{w_2}{v - w_1} & 1 \end{bmatrix}.$$

The invariants of the distortion tensor are:

$$\Gamma_2 = \ldots = \frac{v^2}{(v - w_1)^2}$$

$$\Gamma_1 = \frac{v^2 + w_2^2 + (v - w_1)^2}{(v - w_1)^2}.$$

Area and ratio distortion according to (1.28) and (1.29):

$$T = \frac{v}{v - w_1}; \qquad \ln A = \operatorname{arch}\left[1 + \frac{1}{2}\,\frac{w_1^2 + w_2^2}{v\,(v - w_1)}\right].$$

The results obtained are valid even in case of an object of varying speed; the distortion of the neighbourhood of each point is characterized by the speed of the object projection taken at the moment of passing the slit.

So far, object as well as image have been, in our examples, in a plane surface. Examples concerning picture distortion between unfitting surfaces "bending" from each other are started by the

4th example. What kind of distortion dependent on the place should be used for a dome fresco on a hemispherical dome of radius R, if the observer standing below the dome-zenith at a depth of $R+H$ should have the impression that the fresco is arranged on a cylindrical surface with vertical axis, the base circle of which is identical with that of a hemisphere (Fig. 8)?

The task may be formulated in other words as follows: The single points of the cylinder as object surface are projected to the spherical surface by straight lines passing through the view-point. What kind of distortion has resulted?

When solving this task, the coordinate system on both surfaces is set up so that the coordinates of related points should comply with (2.1), and the metric tensors are separately determined.

Let the coordinates concerning both surfaces be ϑ and φ. The metric tensor of the spherical surface has already been mentioned above:

$$\tilde{g}_{ij} = \begin{bmatrix} \mathrm{R}^2 & 0 \\ 0 & \mathrm{R}^2 \cdot \sin^2 \vartheta \end{bmatrix}.$$

The metric tensor of the cylindrical surface should be noted to the coordinates z, φ according to Fig. 4:

$$\bar{g}_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & R^2 \end{bmatrix}.$$

Proceeding to system ϑ, φ, we must know the relation between the values z and ϑ belonging to one point. According to Fig. 8,

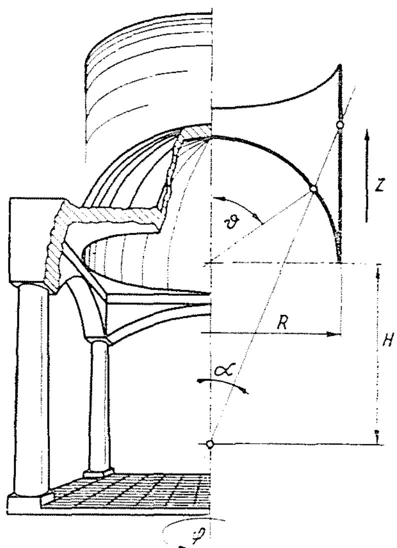

Fig. 8

$$\operatorname{ctg} \alpha = \frac{z - z_0}{R} = \frac{H + R\cos\vartheta}{R\sin\vartheta} = \frac{K + \cos\vartheta}{\sin\vartheta},$$

where $K = \dfrac{H}{R}$.

The new coordinates ϑ, φ' of the cylindrical surface may be introduced by the relations of transformation:

$$z = z_0 + R\,\frac{K + \cos\vartheta}{\sin\vartheta} \qquad \vartheta,\ \varphi \Longrightarrow x^i$$

$$\varphi = \varphi' \qquad z,\ \varphi' \Longrightarrow \bar{x}^i$$

The coordinate-derivatives:

$$\frac{\partial \bar{x}^i}{\partial x^j} = \ldots = \begin{bmatrix} -R\,\dfrac{K\cos\vartheta + 1}{\sin^2\vartheta} & 0 \\ 0 & 1 \end{bmatrix}.$$

Transformation of the metric tensor is carried out as follows:

$$g_{ij} = \frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial \bar{x}^q}{\partial x^j} \bar{g}_{pq} = \begin{bmatrix} R^2 \frac{(K \cos \vartheta + 1)^2}{\sin^4 \vartheta} & 0 \\ 0 & R^2 \end{bmatrix},$$

whereas the *contravariant* metric tensor concerning ϑ, φ of the cylindrical surface becomes

$$g^{ij} = g_{ij}^{-1} = \frac{1}{R^2} \begin{bmatrix} \frac{\sin^4 \vartheta}{(K \cos \vartheta + 1)^2} & 0 \\ 0 & 1 \end{bmatrix}.$$

The distortion tensor is composed according to (2.16):

$$\gamma_j^k = g^{ki} \tilde{g}_{ij} = \begin{bmatrix} \frac{\sin^4 \vartheta}{(K \cos \vartheta + 1)^2} & 0 \\ 0 & \sin^2 \vartheta \end{bmatrix}.$$

Applying (1.27—28—29a—31):

$$T = \sqrt{\Gamma_2} = \frac{\sin^3 \vartheta}{1 + K \cos \vartheta}; \qquad A^{-1} = \frac{\sin \vartheta}{1 + K \cos \vartheta};$$

the setting becomes coordinate line-directed.

By way of an example, the following tasks may be similarly solved:

— which are the distortion conditions between the complex-number plane and the Riemannian sphere?

— what kind of distortion is caused on the map obtained by the projection of the surface of the Earth onto a plane (cylinder, cone) by means of straight lines passing through the centre of the Earth?

— how does an illumination produced by a spherical radiant light-source change along a plane surface? (In this case, a sphere-plane projection is to be tested where the point of projection is the spherical centre; the illumination is proportional to the inverse of the area distortion.)

The *5th example* is, just as the above one, of a projection character. Considering the fact that calculations are tiresome because of the complex nature of this problem, nothing but the main steps for the solution are discussed as well as the final results.

The screen of the cinerama is part of a circular cylinder surface with the radius R, the projector standing at the axis of a cylindrical surface. The picture projected in this set-up is regarded as undistorted. How great will the distortion be if the projector is displaced by a distance l towards (or in any direction from) the screen centre? (This question would always arise if several stationary projectors are operating for one screen.)

In the course of computation, the position of the projector is regarded as fixed, and the screen is displaced at a distance l as compared to the projector (Fig. 9a). Our results will be valid for the entire circular cylindrical screen. Let us imagine the screen centre to be at the place $\varphi = 0$ (just as on the figure), the solution thus concerns the displacement towards the centre; but if, for instance, the centre is at the point $\varphi = 90°$, the result concerns distortion conditions of the displacement perpendicular to the direction of the projector-screen centre (case of projectors standing beside each other).

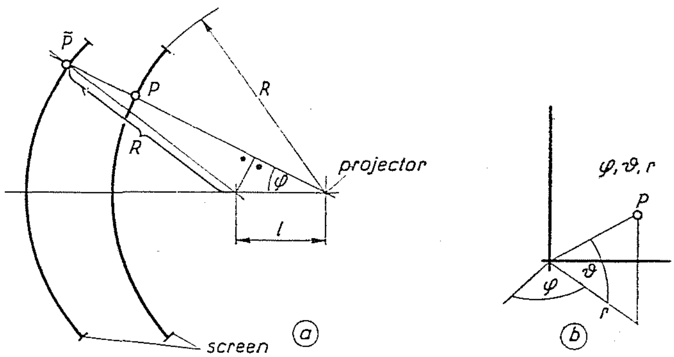

Fig. 9

Considering the cylindrical screen, the introduction of cylindrical coordinates φ, z, r should be obvious. Nevertheless, it seems to be advisable to use the angle of elevation coordinate ϑ instead of z, computations thus being facilitated by the method of "common embedding" of para. 3. The metric tensor of the obtained non-orthogonal cylindrical coordinates $\varphi, \vartheta, r(x^\alpha)$ is thus

$$g_{\alpha\beta} = \begin{bmatrix} r^2 & 0 & 0 \\ 0 & r^2/\cos^4 \vartheta & r \cdot \mathrm{tg}\, \vartheta/\cos^2 \vartheta \\ 0 & r \cdot \mathrm{tg}\, \vartheta/\cos^2 \vartheta & 1/\cos^2 \vartheta \end{bmatrix}. \tag{2}$$

The screen being the level surface $r = R$ of the cylindrical coordinates in the undistorted case, the surface metric tensor concerning coordinates φ, ϑ is

$$g_{ij} = \begin{bmatrix} R^2 & 0 \\ 0 & R^2/\cos^4 \vartheta \end{bmatrix}; \qquad g^{ij} = \begin{bmatrix} 1/R^2 & 0 \\ 0 & \cos^4 \vartheta/R^2 \end{bmatrix}. \tag{3}$$

The x^α coordinates of the image points $\widetilde{P}$ are expressed according to (3.8) by the coordinates of the object points P:

$$\varphi(\widetilde{P}) = \varphi(P)$$
$$\vartheta(\widetilde{P}) = \vartheta(P)$$
$$r(\widetilde{P}) = l \cdot \cos\varphi + \sqrt{R^2 - l^2 \sin^2\varphi}\,.$$

(The third line may be written according to the two right triangles of Fig. 9a.) Hence,

$$\frac{\partial x^3(\widetilde{P})}{\partial x^1(P)} = \frac{\partial r(\widetilde{P})}{\partial\varphi} = -l \cdot \sin\varphi \left(1 + \frac{l \cdot \cos\varphi}{\sqrt{R^2 - l^2 \sin^2\varphi}}\right),$$
$$\frac{\partial x^3(\widetilde{P})}{\partial x^2(P)} = \frac{\partial r(\widetilde{P})}{\partial\vartheta} = 0\,. \tag{4}$$

(3.9), (2) and (4) are used to compute $\widetilde{g}_{ij}$. Summing up the four tensor terms of (3.9),

$$\widetilde{g}_{ij}(\widetilde{P}) = \ldots = r^2 \begin{bmatrix} \dfrac{R^2 + l^2 \sin^2\varphi \,\mathrm{tg}^2\vartheta}{R^2 - l^2 \sin^2\varphi} & -\dfrac{\mathrm{tg}\,\vartheta}{\cos^2\vartheta}\,\dfrac{l \cdot \sin\varphi}{\sqrt{R^2 - l^2 \sin^2\varphi}} \\ -\dfrac{\mathrm{tg}\,\vartheta}{\cos^2\vartheta}\,\dfrac{l \cdot \sin\varphi}{\sqrt{R^2 - l^2 \sin^2\varphi}} & \dfrac{1}{\cos^4\vartheta} \end{bmatrix}.$$

From this and from (3), the distortion tensor is formed by (2.16):

$$\gamma^i_k = r^2 \begin{bmatrix} \dfrac{1 + \left(\dfrac{l}{R}\right)^2 \sin^2\varphi \,\mathrm{tg}^2\vartheta}{R^2 - l^2 \sin^2\varphi} & -\mathrm{tg}\,\vartheta \cos^2\vartheta \,\dfrac{l \cdot \sin\varphi}{R^2\sqrt{R^2 - l^2 \sin^2\varphi}} \\ -\dfrac{\mathrm{tg}\,\vartheta}{\cos^2\vartheta}\,\dfrac{l \cdot \sin\varphi}{R^2\sqrt{R^2 - l^2 \sin^2\varphi}} & \dfrac{1}{R^2} \end{bmatrix}.$$

Introducing the designation $K = \dfrac{l}{R}\sin\varphi$, the area distortion becomes

$$T = \sqrt{\Gamma_2} = \ldots = \frac{r^2}{R^2\sqrt{1 - K^2}},$$

the ratio distortion

$$\ln A = \ldots = \mathrm{arch}\,\frac{1 + \dfrac{K^2}{2}(\mathrm{tg}^2\vartheta - 1)}{\sqrt{1 - K^2}},$$

and the setting

$$\ln\frac{a^2}{a^1} = \ldots = 2 \cdot \ln\cos\vartheta - \mathrm{arsh}\,\frac{K}{\sin 2\vartheta\sqrt{1 - K^2}}\,.$$

6th example. Distortion caused by the plane-parallel refractive medium is studied. Above the original picture, but in close contact with the same, there is a medium of b thickness and n refraction index. (Such a system is produced, for instance, by a picture developing on the phosphoric layer of a flat-faced cathode-ray tube and by the plane-parallel front-plate of several mm thickness of the bulb.) The picture is observed from a point lying at a height of a above the refractive medium by means of some devices with an infinitely small aperture (Fig. 10). This device is looking at a virtual picture arranged at a surface of revolution. How great is the distortion between the real picture and this virtual picture?

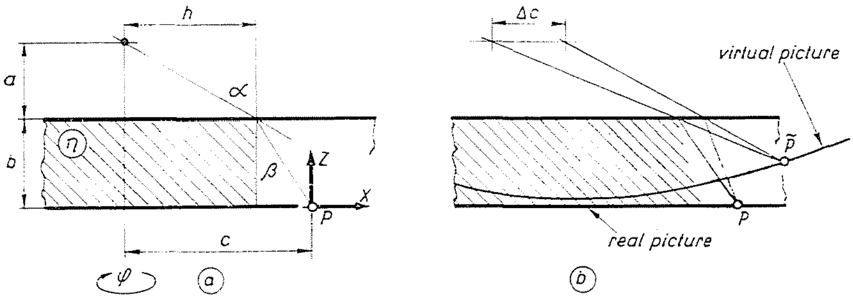

Fig. 10

This task wishes to give an example for the "local embedding method" of metric tensor determination; further steps of rather lengthy computations will only be outlined.

The real as well as the virtual picture are described by the coordinates h, φ according to Fig. 10a.

First step: determination of the position of the virtual picture. Setting the coordinate system x, z to some point P of the real picture, let us write the equation of the ray-straight line running from P into the view-point:

$$z = a + b - \frac{a}{h\,(c)}(x + c)\,.$$

(The meaning of c is shown in Fig. 10.) Point $\widetilde{P}$ of the virtual picture will be arranged on this straight line. By a small displacement Δc of the view-point, another straight line is obtained (Fig. 10b), and the device with its very small aperture is "seeing" point $\widetilde{P}$ of the virtual picture at the point of intersection concerning $\Delta c \to 0$ of these two straight lines. $\widetilde{P}$ will thus be in the very point where $\dfrac{dz}{dc} = 0$.

$$\frac{dz}{dc} = \frac{a}{h^2}\frac{dh}{dc}(x + c) - \frac{a}{h} = 0 \quad \Longrightarrow \quad x + c = h\frac{dc}{dh}\,.$$

The surface of the virtual picture

$$r = h \frac{dc}{dh}$$

$$z = a + b - a \frac{dc}{dh}$$

is a surface of revolution about an axis, perpendicular from the view-point to the plane-parallel plate, the distance measured from this axis being $r = c + x$.

Second task of detail: determination of the metric tensor of the real and the virtual picture.

The metric tensor of the real picture in the polar coordinates c, φ is

$$\bar{g}_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & c^2 \end{bmatrix},$$

which may easily be transformed into the coordinates h, φ:

$$g_{ij} = \begin{bmatrix} \left(\frac{dc}{dh}\right)^2 & 0 \\ 0 & c^2 \end{bmatrix}; \qquad g^{ij} = \begin{bmatrix} \left(\frac{dh}{dc}\right)^2 & 0 \\ 0 & \frac{1}{c^2} \end{bmatrix}. \tag{5}$$

When determining the metric tensor of the virtual picture surface, let us start at the cylindrical coordinates r, φ, z with their metric tensor known in Fig. 4. The virtual picture being a surface of revolution, its normal vector will not have a component of direction φ, thus $n_2/n_3 = 0$. Regarding the fact that the $r-z$ section of the cylindrical coordinates is orthonormal, the relation between the other two normal vector components reads:

$$\frac{n_1}{n_3} = -\frac{dz}{dr} = -\frac{dz}{dh}\frac{dh}{dr} = a\frac{d^2c}{dh^2}\frac{dh}{dr}.$$

By means of the local embedding method (3.4), (3.7), respectively, by using the metric tensor $\bar{g}_{\alpha\beta}$ of the system r, φ, z the metric tensor of the virtual picture surface concerning r, φ is obtained:

$$\bar{\bar{g}}_{ij} = \begin{bmatrix} 1 + a^2\left(\frac{d^2c}{dh^2}\frac{dh}{dr}\right)^2 & 0 \\ 0 & r^2 \end{bmatrix}.$$

Transformed into system h, φ:

$$\tilde{g}_{ij} = \ldots = \begin{bmatrix} \left(\frac{dc}{dh} + h\frac{d^2c}{dh^2}\right)^2 + a^2\left(\frac{d^2c}{dh^2}\right)^2 & 0 \\ 0 & h^2\left(\frac{dc}{dh}\right)^2 \end{bmatrix}.$$

With the so obtained metric tensor of the virtual picture and the metric tensor of the real picture (5), the distortion tensor is described as follows:

$$\gamma_k^i = g^{ij}\tilde{g}_{jk} = \begin{bmatrix} \left(1 + h\frac{dh}{dc}\frac{d^2c}{dh^2}\right)^2 + \left(a\frac{dh}{dc}\frac{d^2c}{dh^2}\right)^2 & 0 \\ 0 & \left(\frac{h}{c}\frac{dc}{dh}\right)^2 \end{bmatrix}.$$

In the last step of the solution, function $c = c(h)$ should be determined. Applyng the rule of refraction

$$n = \frac{\sin \alpha}{\sin \beta}$$

the results from Fig. 10a are:

$$c = h + b \cdot \operatorname{tg} \beta = \ldots = h + \frac{b}{\sqrt{n^2 + \frac{n^2 a^2}{h^2} - 1}}.$$

Knowing this relation and the above defined distortion tensor, the problem may be considered as solved.

7th example. Let us determine the electrostatic field of an infinite plane of potential U_1 and the conducting line pushing the same perpendicularly!

This example somewhat differs from those solved so far and of purely geometrical concern, it seems, however, advisable to mention it in this place. Since the distortion tensor helps to establish methods for the solution of electrostatic problems, it is considered necessary to give one example at least for such a possibility.

First of all, let us follow here the general train of thoughts by means of which the distortion tensor facility may be rendered suitable for electrostatic investigations. It will be proved that this train of thoughts also leads to the method of solution known as the *conformal mapping* of plane problems; the method will then be developed in all its details in connection with another group of examples, the *spherical problems.*

In the course of electrostatical investigations, a subject field (x^i) and an image field $(\tilde{x}^i)$ are assumed as well whose points are harmonized to relation (2.1):

$$x^i(P) = \tilde{x}^i(\tilde{P}). \tag{6}$$

The potential field of the subject space is mapped into the image field so that the potentials of the corresponding points are assumed to be identical:

$$U(x^i) = \widetilde{U}(\widetilde{x}^i) . \tag{7}$$

Question: by which connection between subject and image field — i.e. with which *distortion tensor* — is it true that in the subject field an arbitrary potential function fulfilling $\Delta U = 0$ corresponds to a potential function in the image space by fulfilling $\Delta \widetilde{U} = 0$, too? By solving this question (perhaps with restrictions), such *permissible* subject-image space relations may be found which map the arbitrary potential function (perhaps satisfying certain conditions) again into a potential function!

Let us study, therefore, which kind of distortion does not influence the validity of equation $\Delta U = 0$?

For the sake of simplicity, let us be content with the orthogonal system in the subject as well as in the image space:

$$g_{ij} = g_i^2 \delta_{ij} \quad \text{(not summed for } i\text{)}$$
$$\widetilde{g}_{ij} = \widetilde{g}_i^2 \delta_{ij} \quad \text{(not summed for } i\text{)}$$

the distortion tensor is orthogonal as well:

$$\gamma_j^i = g^{ik} \widetilde{g}_{kj} = \left(\frac{\widetilde{g}_i}{g_i}\right)^2 \delta_{ij} = \gamma_i^2 \delta_{ij} \text{ (not summed for } i\text{)}, \tag{8}$$

The assumption $\Delta U = 0$ for the potential function of the subject space may be expressed as follows:

$$\Delta U = \frac{1}{g_1 g_2 g_3} \sum_{k=1}^{3} \frac{\partial}{\partial x^k} \frac{g_{k+1} \cdot g_{k+2}}{g_k} \frac{\partial U}{\partial x^k} = 0 \tag{9}$$

(whereby $g_l = g_{l-3}$, for $l > 3$).

In the image space the same condition is met with:

$$\Delta \widetilde{U} = \frac{1}{\widetilde{g}_1 \widetilde{g}_2 \widetilde{g}_3} \sum_{k=1}^{3} \frac{\partial}{\partial \widetilde{x}^k} \frac{\widetilde{g}_{k+1} \widetilde{g}_{k+2}}{\widetilde{g}_k} \frac{\partial \widetilde{U}}{\partial \widetilde{x}^k} = 0 ,$$

which may be expressed by using (6), (7), (8):

$$\sum_{k=1}^{3} \frac{\partial}{\partial x^k} \frac{\gamma_{k+1} \gamma_{k+2}}{\gamma_k} \frac{g_{k+1} g_{k+2}}{g_k} \frac{\partial U}{\partial x^k} = 0 . \tag{10}$$

Now it is desirable to find a distortion tensor assuring that if (9) holds true, (10) is fulfilled as well. As long as there is no specification as to $U(x^i)$, this remains only in case if three members of (9) and (10) differ in pairs only by a multiplier $K(x^i)$. This condition is expressed for the two first members in the following way:

$$K(x^i)\cdot\frac{\partial}{\partial x^1}\frac{g_2 g_3}{g_1}\frac{\partial U}{\partial x^1}=\frac{\gamma_2\gamma_3}{\gamma_1}\frac{\partial}{\partial x^1}\frac{g_2 g_3}{g_1}\frac{\partial U}{\partial x^1}+\frac{g_2 g_3}{g_1}\frac{\partial U}{\partial x^1}\cdot\frac{\partial}{\partial x^1}\frac{\gamma_2\gamma_3}{\gamma_1}, \tag{11}$$

$$\left[K(x^i)-\frac{\gamma_2\gamma_3}{\gamma_1}\right]\frac{\partial}{\partial x^1}\left[\frac{g_2 g_3}{g_1}\frac{\partial U}{\partial x^1}\right]=\left[\frac{g_2 g_3}{g_1}\frac{\partial U}{\partial x^1}\right]\cdot\frac{\partial}{\partial x^1}\frac{\gamma_2\gamma_3}{\gamma_1},$$

which holds on base of an arbitrary $U(x^i)$ if

$$K(x^i)=\frac{\gamma_2\gamma_3}{\gamma_1}$$

and

$$\frac{\partial}{\partial x^1}\frac{\gamma_2\gamma_3}{\gamma_1}=0\,.$$

(9) and (10) produce two additional conditions still, similar to (11); from these

$$\left.\begin{array}{lll} & K=\dfrac{\gamma_2\gamma_3}{\gamma_1}=\dfrac{\gamma_1\gamma_3}{\gamma_2}=\dfrac{\gamma_1\gamma_2}{\gamma_3} & \Longrightarrow \gamma_1=\gamma_2=\gamma_3=\gamma \\ \text{and} & & \\ & \dfrac{\partial\gamma}{\partial x^i}=0 & \Longrightarrow \gamma=\text{const}\end{array}\right\}. \tag{12}$$

The only permissible distortion tensor therefore is

$$\gamma_j^i=\gamma^2\cdot\delta_{ij}\,;$$

i.e. the well-known result is obtained according to which the spaces passing into each other by magnification, rotation, reflection independent from place are correctly transforming the potential function.

More useful results may be achieved by introducing certain restrictions for $U(x^i)$.

Let, for instance, be $\dfrac{\partial U}{\partial x^3}=0$! Consequently, the third member in (9) as well as in (10) will be zero; from the requirements under (12) no more is left than:

$$\frac{\gamma_2\gamma_3}{\gamma_1}=\frac{\gamma_1\gamma_3}{\gamma_2}\qquad\Longrightarrow (\gamma_1)^2=(\gamma_2)^2$$

$$\frac{\partial \gamma_3}{\partial x^1} = 0 \qquad \frac{\partial \gamma_3}{\partial x^2} = 0 .$$

The mappings whose distortion tensor has the form of

$$\gamma_j^i = \begin{bmatrix} f(x^i) & 0 & 0 \\ 0 & f(x^i) & 0 \\ 0 & 0 & g(x^3) \end{bmatrix} \tag{13}$$

met with the above conditions, are therefore mapping the potential functions with the property of $\frac{\partial U}{\partial x^3} = 0$ into potential functions again.

By means of the "permissible" distortion tensor (13) it is desired to achieve, first of all, the method of conformal representation. The space is covered with the Cartesian coordinates x_i, the coordinates after projection of the single points being expressed in the same way:

$$x_1(\widetilde{P}) = f_1\left[x_1 ; x_2(P)\right]$$

$$x_2(\widetilde{P}) = f_2\left[x_1 ; x_2(P)\right]$$

$$x_3(\widetilde{P}) = x_3(P) ,$$

and with the restriction of $\frac{\partial U}{\partial x_3} = 0$ we limit ourselves to the plane problems.

The coordinate derivatives are

$$\frac{\partial x_i(\widetilde{P})}{\partial x_j(P)} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & 0 \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & 0 \\ 0 & 0 & 1 \end{bmatrix} .$$

The distortion tensor is computed with the method of "fitting surfaces" applicable even in three dimensions and with (3.12):

$$\gamma_k^j = \frac{\partial x_p(\widetilde{P})}{\partial x_j(P)} \frac{\partial x_p(\widetilde{P})}{\partial x_k(P)} = \begin{bmatrix} \left(\frac{\partial f_1}{\partial x_1}\right)^2 + \left(\frac{\partial f_2}{\partial x_1}\right)^2 & \frac{\partial f_1}{\partial x_1}\frac{\partial f_1}{\partial x_2} + \frac{\partial f_2}{\partial x_1}\frac{\partial f_2}{\partial x_2} & 0 \\ \frac{\partial f_1}{\partial x_1}\frac{\partial f_1}{\partial x_2} + \frac{\partial f_2}{\partial x_1}\frac{\partial f_2}{\partial x_2} & \left(\frac{\partial f_1}{\partial x_2}\right)^2 + \left(\frac{\partial f_2}{\partial x_2}\right)^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} .$$

γ_k^j becomes a permissible distortion tensor in the form of (13), if the following two conditions are fulfilled:

$$\frac{\partial f_1}{\partial x_1}\frac{\partial f_1}{\partial x_2} + \frac{\partial f_2}{\partial x_1}\frac{\partial f_2}{\partial x_2} = 0, \tag{14}$$

$$\left(\frac{\partial f_1}{\partial x_1}\right)^2 + \left(\frac{\partial f_2}{\partial x_1}\right)^2 = \left(\frac{\partial f_1}{\partial x_2}\right)^2 + \left(\frac{\partial f_2}{\partial x_2}\right)^2. \tag{15}$$

(14) and (15) result — by way of algebraic transformations — in equivalent conditions:

$$\frac{\partial f_1}{\partial x_1} = \pm \frac{\partial f_2}{\partial x_2}, \tag{16}$$

$$\frac{\partial f_1}{\partial x_2} = \mp \frac{\partial f_2}{\partial x_1}. \tag{17}$$

The two obtained relations agree, in case of signs written above, with the Cauchy—Riemann differential condition of the complex function $x_1 + jx_2 - u_1 + ju_2$ $[u_i = f_i(x_1;\ x_2)]$; hence, applying our method for plane problems, the same result is obtained as with the conformal mapping by differentiable complex functions. (The lower signs of (16) and (17) indicate that the reflection of the potential field is permitted in addition to the mapping produced by the differentiable complex functions.)

Now let us return to our original example; to the case of the infinite plane and the conducting line pushing the same perpendicularly.

If the origin of a spherical coordinate system r, φ, ϑ is fixed at the point of intersection, the condition $\frac{\partial U}{\partial r} = 0$ is met with (except the nearest neighbourhood of the point of intersection — which may, however, be left out of account, just as the edge effects in case of plane problems). For carrying out the mapping, a space should be found whose relation with the spherical coordinates is described by the distortion tensor (13). This is found in the cylindrical coordinates z, φ, t (the interpretation of the single coordinates is visible on Fig. 11). After the scale transformation $z = ln\ r$ of coordinate z, the contravariant metric tensor of the cylindrical coordinates is

$$g^{ij} = \begin{bmatrix} \frac{1}{r^2} & 0 & 0 \\ 0 & \frac{1}{t^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let us carry out a still pending scale transformation on the coordinate ϑ of the spherical coordinates r, φ, ϑ:

$$\begin{aligned} r &= r \\ \varphi &= \varphi \\ \vartheta &= f(t) \end{aligned} \quad \Longrightarrow \quad \frac{\partial \tilde{x}^i}{\partial \bar{\tilde{x}}^j} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \dfrac{d\vartheta}{dt} \end{bmatrix}.$$

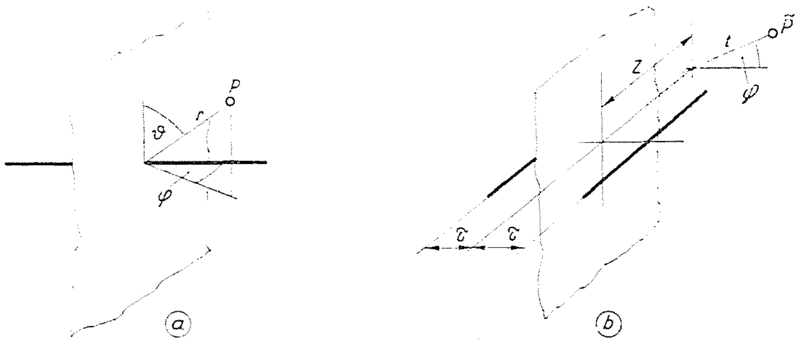

Fig. 11

Applying here the metric tensor $\tilde{g}_{pq}$ of the spherical coordinates r, φ, ϑ given in Fig. 4, the results are that

$$g_{ij} = \frac{\partial \tilde{x}^p}{\partial \bar{\tilde{x}}^i} \frac{\partial \tilde{x}^q}{\partial \bar{\tilde{x}}^j} \tilde{g}_{pq} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 \sin^2\vartheta & 0 \\ 0 & 0 & r^2 \left(\dfrac{d\vartheta}{dt} \right)^2 \end{bmatrix}.$$

The distortion tensor

$$\gamma^i_k = g^{ij} \bar{\tilde{g}}_{jk} = \begin{bmatrix} \dfrac{1}{r^2} & 0 & 0 \\ 0 & \dfrac{r^2 \sin^2\vartheta}{t^2} & 0 \\ 0 & 0 & r^2 \left(\dfrac{d\vartheta}{dt} \right)^2 \end{bmatrix}$$

will have the identical form with (13), if

$$\frac{r^2 \sin^2\vartheta}{t^2} = r^2 \left(\frac{d\vartheta}{dt} \right)^2$$

$$\frac{\sin \vartheta}{t} = \frac{d\vartheta}{dt} .$$

The solution of this differential equation is $t = \tau \cdot tg\, \vartheta/2$. Thus introducing in ϑ the new scale, relation (7) produces a permissible potential function transformation between the two spaces. In the cylindrical coordinates r, φ, t our example appears to be a plane problem, considering the fact that $\frac{\partial U}{\partial r} = 0$.

Let the plane with the potential U_1 in the spherical coordinates be

$$\varphi = \pm 90^\circ ; \quad \text{whilst} \quad 0 < r < \infty$$
$$0 < \vartheta < 180^\circ .$$

After mapping into the cylindrical coordinates, this becomes passing into the plane

$$\varphi = \pm 90^\circ ; \qquad - \infty < z < \infty$$
$$0 < t < \infty .$$

The conducting line perpendicularly intersecting plane U_1 in the spherica coordinates:

$$\varphi = 0^\circ,\ 180^\circ ; \qquad 0 < r < \infty$$
$$\vartheta = \quad 90^\circ ;$$

after projection to the cylindrical coordinates

$$\varphi = 0^\circ,\ 180^\circ ; \qquad - \infty < z < \infty$$
$$t = \tau ;$$

passing on both sides of the plane with the potential U_1, into parallel straight lines (Figs 11a—b). The potential field of this latter arrangement being known, it may be expressed in the coordinates φ, ϑ as well. Neglecting detailed computations, it holds for the half field on one side of the plane that

$$U = U_1 + \text{const} \cdot \ln \frac{\text{ch} \ln \text{tg}\, \vartheta/2 + \cos \varphi}{\text{ch} \ln \text{tg}\, \vartheta/2 - \cos \varphi} . \tag{18}$$

With the here introduced mapping from spherical coordinates to cylindrical coordinates many other problems may be solved, e.g., the field of two straight line-conductors crossing each other nonperpendicularly, the field of conical surfaces with common vertex, etc.

*

To sum up it should be mentioned that when applying the basic relations of the Riemannian geometry playing an important part in differential geometry, the concept of the distortion tensor and distortion invariants presented itself quite obviously. Using these concepts and the concerning methods of computation, the comparison between geometric objects (e.g. pictures) standing in some relation of mapping with each other, as well as the evaluation of the mapping process may be solved with good results.

Practically it may be important in picture transmission, for instance, particularly in television engineering, to investigate the picture distortion effects of the apparatus applied (e.g. distortions resulting from the geometric structure of a television picture tube). At the same time, the concepts and definitions serving as a basis for mathematical examinations seem to become useful as starting points for the preparation of unambiguous measuring methods.

As to mathematics, the introduction of distortion characteristics of higher order, the interpretation of distortions as defined in indefinite metric spaces, the extension to non-commutative tensor algebra as well as the generalization to the Hilbert-space may all be the direction for further steps. It is hoped that the application of this method in other fields will bring further interesting results.

Summary

Authors present a general method for computation of the distortion of geometric projection. This method is based on Riemannian geometry and tensor calculus. The concepts of ratio distortion and volume distortion are introduced which are useful even in case of arbitrary dimensional values. The distortion tensor introduced in this paper is the basis of the discussion. The further part of the paper helps to carry out practical computations, whilst the conclusion gives a number of examples in order to illustrate the method.

References

1. IEC Publications No. 107, 1960.
2. DUSCHEK, A.—HOCHRAINER, A.: Grundzüge der Tensorrechnung in analytischer Darstellung. Springer Verlag. 1960.
3. PACH, ZS. P.—FREY, T.: Vektor- és tenzoranalízis. Műszaki Kiadó, Budapest, 1960.
4. PETROW, A. S.: Einstein-Räume. Akademie-Verlag, Berlin, 1964.
5. RASCHEWSKI, P. K.: Riemannsche Geometrie und Tensoranalysis. Deutsch. Vrlg. der Wiss., Berlin, 1959.
6. DUSCHEK, A.—MAYER, W.: Lehrbuch der Differentialgeometrie. Teubner Vrlg., Leipzig, 1930 II. Band.
7. LINFOOT, E. H.: Qualitätsbewertung optischer Bilder. F. Vieweg & Sohn, Braunschweig, 1960.
8. KELMAN, V. B.—JAVOR, S. J.: Elektronoptika. Akadémiai Kiadó, Budapest, 1965.
9. HAZAY, I.: Földi vetületek. Akadémiai Kiadó, Budapest, 1954.

József SÁROSSY
Vladimir SZÉKELY } Budapest XI., Sztoczek u. 2. Hungary