THE SYNTHESIS OF MULTIVARIABLE SAMPLED-DATA CONTROL SYSTEMS

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I. The theoretical basis

Introduction

Such systems are designated as multivariable systems which have more than one controlled variable (N pieces) and an identical number of reference inputs. The number of the inputs of the final controlled plant (M) may be identical ($M = N$, symmetrical system), it may be higher ($M > N$, excess systems) or lower ($M < N$, deficiency systems).

The system is denominated as noninteracting if the controlled variables depend only on their own reference input, and are independent — in a certain sense — of "extraneous" reference inputs. The functioning of the noninteracting system is evidently better, its designing simpler.

Only linear systems having a purely discrete operation will be examined. The sampling period T is assumed to be constant and the samplers synchronized.

We assume that the impulses can be substituted by Dirac impulses. The variable and the parameter of the discrete Laplace transformation, respectively, are $Z = e^{-sT} = z^{-1}$;

$$\lambda = \frac{t - nT}{T}, \quad nT \leq t < (n+1)\,T, \quad 0 \leq \lambda < 1,$$

where s denotes the variable of the Laplace transformation.

The task of the designer is to determine the transfer matrix of the impulse compensator.

1. *The description of the system*

The controlled variable y_i of the controlled system is a linear function of the input f_j of the controlled element:

$$Y_i(s) = \sum_{j=1}^{M} G_{Sij}(s)\, F_j(s), \quad i = 1, 2, \ldots, N. \tag{1.1}$$

The transfer functions $G_{Sij}(s)$ of the controlled element are represented by a transfer matrix $\mathbf{G_S}$ consisting of N rows and M columns, while the controlled variables and the inputs of the controlled element by the column matrices $\mathbf{Y}(s)$ and $\mathbf{F}(s)$ having N and M rows, respectively.

In front of the controlled system, a zero order hold is inserted on most occasions, the transfer function of which is given by

$$G_T(s) = K_T \frac{1 - e^{-sT}}{s} . \tag{1.2}$$

Let us introduce the joint transfer matrix of the controlled system and of the hold element:

$$\mathbf{G}(s) = G_T(s) \ \mathbf{G}_S(s) . \tag{1.3}$$

The corresponding discrete and modified transfer matrices are designated by $\mathbf{G}(Z)$ and $\mathbf{G}(Z, \lambda)$, respectively.

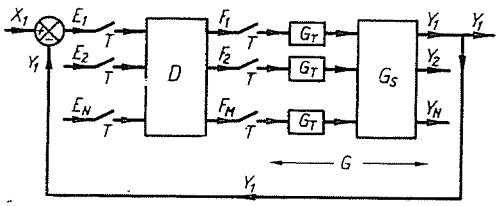

Fig. 1

According to the block diagram in Fig. 1, we can write, for the discrete and the modified discrete matrices of the controlled variables, that

$$\mathbf{Y}(Z) = \mathbf{G}(Z) \ \mathbf{F}(Z) , \tag{1.4}$$

$$\mathbf{Y}(Z, \lambda) = \mathbf{G}(Z, \lambda) \ \mathbf{F}(Z) . \tag{1.5}$$

The series $f_j\,(nT)$ of the inputs of the controlled element is a linear function of the series $e_k(mT)$ of actuating signals:

$$F_j(Z) = \sum_{k=1}^{N} D_{jk}(Z) \, E_k(Z), \quad j = 1,2, \ldots, M. \tag{1.6}$$

By summing the transfer functions $D_{jk}(Z)$ of the impulse compensator to a discrete transfer matrix $\mathbf{D}(Z)$ consisting of M rows and N columns,

$$\mathbf{F}(Z) = \mathbf{D}(Z) \, \mathbf{E}(Z) . \tag{1.7}$$

In the case of the rigid feedback according to Fig. 1,

$$\mathbf{E}(Z) = \mathbf{Y}(Z) - \mathbf{X}(Z) , \tag{1.8}$$

where $\mathbf{X}\,(Z)$ denotes the N-row column matrix of the reference inputs.

By expressing the controlled variables in terms of the actuating signals,

$$\mathbf{Y}(Z) = \mathbf{G}(Z)\,\mathbf{F}(Z) = \mathbf{G}(Z)\,\mathbf{D}(Z)\,\mathbf{E}(Z)\,. \tag{1.9}$$

Upon substituting this into (1.8),

$$\mathbf{E}(Z) = \mathbf{G}(Z)\,\mathbf{D}(Z)\,\mathbf{E}(Z) - \mathbf{X}(Z)\,. \tag{1.10}$$

Let us introduce the quadratic and diagonal unit matrix **I** (where $I_{ij} = 0$, if $i \neq j$, and $I_{ii} = 1$). On rearrangement we obtain the column matrix of the actuating signals:

$$\mathbf{E}(Z) = [\mathbf{I} + \mathbf{G}(Z)\,\mathbf{D}(Z)]^{-1}\,\mathbf{X}(Z)\,, \tag{1.11}$$

where the exponent (-1) denotes the inverse of the matrix.

Upon substituting expression (1.11) into (1.4) and (1.5) and introducing

$$\mathbf{L}(Z) = \mathbf{G}(Z)\,\mathbf{D}(Z) \tag{1.12}$$

the quadratic discrete transfer matrix of the open-loop system, the expression for the controlled variables will be

$$\mathbf{Y}(Z) = \mathbf{L}(Z)\,[\mathbf{I} + \mathbf{L}(Z)]^{-1}\,\mathbf{X}(Z) = \mathbf{W}(Z)\,\mathbf{X}(Z)\,, \tag{1.13}$$

$$\mathbf{Y}(Z, \lambda) = \mathbf{G}(Z, \lambda)\,\mathbf{D}(Z)\,[I + \mathbf{L}(Z)]^{-1}\,\mathbf{X}(Z) = \mathbf{W}(Z, \lambda)\,\mathbf{X}(Z)\,. \tag{1.14}$$

The discrete and the modified discrete matrices of the closed system are (N-row, N-column matrices),

$$\mathbf{W}(Z) = \mathbf{L}(Z)\,[\mathbf{I} + \mathbf{L}(Z)]^{-1} = [\mathbf{I} + \mathbf{L}(Z)]^{-1}\,\mathbf{L}(Z)\,, \tag{1.15}$$

$$\mathbf{W}(Z, \lambda) = \mathbf{G}(Z, \lambda)\,\mathbf{D}(Z)\,[\mathbf{I} + \mathbf{L}(Z)]^{-1} = \\ = \mathbf{G}(Z, \lambda)\,\mathbf{D}(Z)\,[\mathbf{G}(Z)\,\mathbf{D}(Z)]^{-1}\,\mathbf{W}(Z)\,. \tag{1.16}$$

The second form of the relationships can be easily verified. In the case of a symmetrical system $[\mathbf{G}\,\mathbf{D}]^{-1} = \mathbf{D}^{-1}\mathbf{G}^{-1}$, thus

$$\mathbf{W}(Z, \lambda) = \mathbf{G}(Z, \lambda)\,\mathbf{G}^{-1}(Z)\,\mathbf{W}(Z)\,, \quad M = N\,, \tag{1.17}$$

which is analogous to the relationship valid for systems with a single variable.

2. *The noninteracting system*

All the controlled variables of a closed control system depend on all the reference inputs. This is naturally disadvantageous from the aspect of the operation of the system, since thus an intentional or accidental change of any of the reference inputs causes a change in all the controlled variables. On the other hand, couplings make the analysis of the system more difficult and designing more complicated.

It is a self-evident requirement to eliminate or to reduce these harmful couplings as far as possible. Let $y_{ik}(t)$ designate the component of the controlled variable $y_i(t)$, which is produced by the reference input $x_k(t)$, if the other reference inputs are equal to zero. For controlled variables of the character $y_{ii}(t)$, the requirements usual in the case of systems with a single variable are valid, which will not be discussed here. With respect to mixed index components $(i = k)$, we insist in any case on having a value of zero in the steady state. In other words, in the steady state all the controlled variables may depend only on their respective reference inputs, since otherwise we cannot speak of control any longer. The ideal case would be if all the coupled components (with mixed index) were identically zero.

In sampled-data systems several types of noninteracting can be distinguished. These are, in the order of stringency, the following:

1. *Steady-state noninteracting:* The mixed index components asymptotically tend to zero with the increasing of time:

$$\lim_{t \to \infty} y_{ik}(t) = 0, \qquad i \neq k. \tag{2.1}$$

2. *Finite settling time noninteracting:* The value of the mixed index components in the sampling instants is zero after the elapse of the settling time:

$$y_{ik}(nT) = 0, \quad nT > T_s\,; \quad \lim_{t \to \infty} y_{ik}(t) = 0, \qquad i \neq k\,. \tag{2.2}$$

3. *Ripple-free, finite settling time noninteracting:* The value of the mixed index components is zero after the elapse of the settling time:

$$y_{ik}(t) = 0, \quad t > T_s\,, \qquad i \neq k\,. \tag{2.3}$$

4. *Sampled noninteracting:* The value of the mixed index components is zero at the sampling instants:

$$y_{ik}(nT) = 0; \quad \lim_{t \to \infty} y_{ik}(t) = 0, \qquad i \neq k\,. \tag{2.4}$$

5. *Continuous (complete) noninteracting:* The value of the mixed index components is zero:

$$y_{ik}(t) = 0, \qquad i \neq k\,. \tag{2.5}$$

As we have already mentioned, requirement 1 should be satisfied in any case, but this is generally not sufficient. In connection with requirements 2 and 3 it should be noted that those can be satisfied only with respect to certain types of reference inputs. This follows from the concept of the finite settling time. We generally demand that the settling time should be finite if the reference input is of m or of a lower order, where the m-order reference input is

$$x(t) = 1(t) \frac{t^m}{(m-1)!}, \quad m = 1, 2, 3, \tag{2.6}$$

i.e. the unit step ($m = 1$), the unit ramp ($m = 2$), and the acceleration step ($m = 3$). Requirement 5, i.e. the continuous noninteracting, cannot be satisfied on most occasions in sampled-data systems. The system is left to itself in the sampling instants. The controlled variables are changing between the sampling instants in accordance with the time constants and natural frequencies of the controlled system. The amplitudes of the individual components can be influenced by controlling the amplitudes of the inputs of the controlled element, acting at the sampling instants. The superposition of components with different time constants and natural frequencies however cannot be equal to zero. It follows from this that continuous noninteracting with respect to one or several controlled variables can be ensured only if the transfer matrix of the controlled system has a special structure concerning the time constants and natural frequencies. Expressing this more simply: Some of the time constants and the natural frequencies should be identical. We shall later revert more concretely to this question.

3. *The mathematical conditions of noninteracting*

Let us formulate the various requirements of noninteracting in a mathematical form. This means the conditions imposed on the transfer functions $W_{ik}(Z)$ and $W_{ik}(Z, \lambda)$ with a mixed index $i \neq k$. For the transfer functions with identical indices $W_{ii}(Z)$ or $W_{ii}(Z, \lambda)$ essentially the same requirements are valid as in the case of a single variable.

The discrete transform of the m-order reference input is

$$X(Z) = \frac{\Phi_m(Z)}{(1-Z)^m}, \quad m = 1, 2, 3, \tag{3.1}$$

where $\Phi_m(Z)$ is an $(m-1)$-degree polynomial. The transforms of the corresponding controlled variable are

$$Y_{ik}(Z) = W_{ik}(Z) \frac{\Phi_m(Z)}{(1-Z)^m}, \tag{3.2}$$

$$Y_{ik}(Z, \lambda) = W_{ik}(Z, \lambda) \frac{\Phi_m(Z)}{(1-Z)^m}. \tag{3.3}$$

From this, the condition of steady-state noninteracting is evidently

1. $$W_{ik}(Z, \lambda) = (1-Z)^m R_{ik}(Z, \lambda), \quad i \neq k \tag{3.4}$$

where $R_{ik}(Z, \lambda)$ denotes a rational fractional function having all its poles outside of the unit circle, but otherwise arbitrary.

The condition of the finite settling time and of the ripple-free finite settling time noninteraction is that $Y_{ik}(Z)$ and $Y_{ik}(Z, \lambda)$, respectively, should be polynomials. Accordingly the requirements are:

2. $$W_{ik}(Z) = (1-Z)^m P_{ik}(Z), \qquad i \neq k \tag{3.5}$$

3. $$W_{ik}(Z, \lambda) = (1-Z)^m P_{ik}(Z, \lambda), \quad i \neq k \tag{3.6}$$

where $P_{ik}(Z)$ and $P_{ik}(Z, \lambda)$ are arbitrary polynomials in terms of the variable Z.

The conditions of the sampled and continuous noninteraction are independent of the reference input, evidently it is

4. $$W_{ik}(Z) = 0, \qquad i \neq k, \tag{3.7}$$

5. $$W_{ik}(Z, \lambda) = 0, \qquad i \neq k. \tag{3.8}$$

Now we have also mathematically demonstrated that the requirements are more and more stringent in the order of enumeration.

Requirements 1, 2, and 3 are closely connected with the type of the reference input. In the case of a more general reference input, the system does not show decoupledness, but every controlled variable depends on all the reference inputs. Since reference inputs do not change during the real operation as step signals do, the controlled variables are actually changing under the influence of any of the reference inputs. Afterwards this change tends to zero asymptotically or with a finite settling time. Regarding the problem from the aspect of the designer, the application of the statistical designing methods is practically hopeless. In principle there is nothing to prevent the formation of the quadratic mean errors $\overline{y_{ik}^2(t)}$, or $\overline{y_{ik}^2(nT)}$, and afterwards to minimize these. Since, however, the number of these quadratic mean error values is N^2 and they are all very complicated functions of the free parameters of the impulse compensator, the actual execution of this process is not very promising. Circumstances are considerably more favourable in the case of sampled noninteracting systems. The system with several variables can be substituted with respect to the sampling instants by N pieces of single-variable subsystems, which can be designed according to the usual method, independent of each other. The quadratic mean error $\overline{y_{ik}^2(nT)}$ is zero $(i \neq k)$, while the mean error $\overline{y_{ii}^2(nT)}$ can be minimized.

II. Designing methods

4. *Finite settling time noninteracting system*

a) The basic correlations

The designing method of systems becoming noninteracting after a finite settling time was elaborated by Nishida and Imai [6]. The procedure was given for symmetrical systems ($M = N$), but the generalization can be performed without difficulties, as we shall later see.

The calculation is based on the examination of the equivalent open system, containing a fictive impulse compensator with the transfer matrix $\mathbf{C}(Z)$ (Fig. 2).

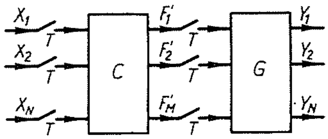

Fig. 2

The matrix has M rows and N columns. The designing consists of two steps: First the matrix $\mathbf{C}(Z)$ is determined, afterwards, with the knowledge of this, the transfer matrix $\mathbf{D}(Z)$ is calculated.

The transfer matrix of the closed system is, in view of Fig. 2,

$$\mathbf{W}(Z, \lambda) = \mathbf{G}(Z, \lambda)\,\mathbf{C}(Z)\,. \tag{4.1}$$

One of the conditions of finite settling-time noninteracting is, according to formulae (3.5) and (3.6), that all $W_{ik}(Z, \lambda)$ functions should be polynomials. Let $N_j(Z)$ denote the least common multiple of the elements $G_{ij}(Z)$ figuring in the j-th column of the matrix $\mathbf{G}(Z)$, then

$$C_{jk}(Z) = N_j(Z)\,C'_{jk}(Z); \quad j = 1, 2, \ldots, M; \quad k=1,2, \ldots, N \tag{4.2}$$

should be chosen, where $C'_{jk}(Z)$ is a polynomial to be determined later.

It is evident, if the finite settling time is stipulated not only for the coupled components, but also for the own components. By restricting our considerations to follow-up systems (generalization can be performed without difficulty), the other condition system of the finite settling time is

$$\lim_{Z \to 1} \mathbf{G}(Z)\;\mathbf{C}(Z) = \mathbf{I}\,, \tag{4.3}$$

$$\lim_{Z \to 1} \frac{\mathrm{d}^{\mu}}{\mathrm{d}Z^{\mu}} \mathbf{G}(Z)\,\mathbf{C}(Z) = \mathbf{0}, \;\mu = 1, 2, \ldots, (m-1), \tag{4.4}$$

where m denotes the highest order of the still compensated reference inputs.

Let us regard the index k as being fixed. In this case the number of condition equations given by (4.3) and (4.4) is mN. The matrix elements $C_{jk}(Z)$ should contain altogether at least as many free parameters. Let p_{jk} designate the degree of the polynomial $C'_{jk}(Z)$, then the condition equation

$$\sum_{j=1}^{M} (p_{jk} + 1) = mN \tag{4.5}$$

should be satisfied for all the indices k. If in a special case all the $C'_{jk}(Z)$ have an identical degree by a given k, then

$$p_k \geqq m \frac{N}{M} - 1. \tag{4.6}$$

In a symmetrical system this is reduced to the form $p_k \geqq m-1$ [6]. The quality characteristics of the system can later be influenced by the choice of the free parameters.

It is easily conceivable that the settling time is determined by the maximum degree of the functions $C_{jk}(Z)$. Let q_j designate the degree of the functions $N_j(Z)$, then

$$\frac{T_s}{T} = \max (q_j + p_{jk}). \tag{4.7}$$

The discrete transfer matrix of the closed system, in view of formulae (4.1) and (1.15) is given by

$$\mathbf{W}(Z) = \mathbf{G}(Z)\ \mathbf{C}(Z) = \mathbf{G}(Z)\ \mathbf{D}(Z)\ [\mathbf{I} + \mathbf{G}(Z)\ \mathbf{D}(Z)]^{-1} =$$

$$= [\mathbf{I} + \mathbf{G}(Z)\ \mathbf{D}(Z)]^{-1}\ \mathbf{G}(Z)\ \mathbf{D}(Z)\,. \tag{4.8}$$

Let us examine the second and third terms in the series of equations. We obtain after matrix algebraical rearrangements,

$$\mathbf{G}(Z)\Big[[\mathbf{I} - \mathbf{C}(Z)\ \mathbf{G}(Z)]\ \mathbf{D}(Z) - \mathbf{C}(Z)\Big] = 0\,. \tag{4.9}$$

If the multiplier of the matrix $\mathbf{G}$ is a zero matrix, the equation is satisfied. In the case of a symmetrical system, this is the only solution (since in this case $\mathbf{G}^{-1}$ does exist). In the general case, other solutions are also possible, but it is not easy to find one. By accepting this solution, the transfer matrix of the impulse compensator is

$$\mathbf{D}(Z) = [\mathbf{I} - \mathbf{C}(Z)\ \mathbf{G}(Z)]^{-1}\ \mathbf{C}(Z)\,. \tag{4.10}$$

Let us now regard the second and fourth terms in the series of equations (4.8). One of the solutions is obtained in a similar way:

$$\mathbf{D}(Z) = \mathbf{C}(Z)\ [\mathbf{I} - \mathbf{G}(Z)\ \mathbf{C}(Z)]^{-1}. \tag{4.11}$$

The two expressions are theoretically equivalent.

The difference in practice is that a matrix of dimension M should be inverted for the first form and one of dimension N for the second form.

b) The statistical error

We shall examine the operation of the system in that case of stochastic reference inputs. Only stationary processes are examined, by accepting the ergodic hypothesis.

Let the reference inputs consist of two parts: the useful signal $m(t)$ and the noise $n(t)$.

$$x_k(t) = m_k(t) + n_k(t), \quad k = 1, 2, \ldots, N. \tag{4.12}$$

We shall restrict our considerations to the examination of the values occurring at the sampling instants, thus we regard the series

$$x_k(rT) = m_k(rT) + n_k(rT), \quad k = 1, 2, \ldots, N, \tag{4.13}$$

or the corresponding autocorrelation and cross-correlation series as given.

Let $y_{0i}(rT)$ designate the required series of the i-th controlled variable. This is generally some linear function of the useful signal value $m_i(pT)$ and can be expressed by the ideal transfer function $W_{0i}(Z)$, that is assumed to be a general power series.

The error signal series of the i-th controlled variable is the difference of the actual and the desired series of signals,

$$\psi_i(rT) = y_i(rT) - y_{0i}(rT) = \sum_{k=1}^{N} y_{ik}(rT) - y_{0i}(rT). \tag{4.14}$$

Let the statistical error be, by definition, the mean square value of the series $\psi_i(rT)$:

$$\zeta_i^2 = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{r=N}^{N} \psi_i^2(rT). \tag{4.15}$$

The statistical error can be expressed by the autocorrelation function of the error signal and by the two-sided discrete transform $\Phi_{\psi_i\psi_i}(Z)$ thereof, respectively:

$$\zeta_i^2 = \frac{1}{2\pi j} \oint_{|Z|=1} \Phi_{\psi_i\psi_i}(Z) \frac{dZ}{Z} = \sum_{|Z_j|<1} \operatorname*{Res}_{Z=Z_j} \frac{1}{Z} \Phi_{\psi_i\psi_i}(Z), \tag{4.16}$$

where by force of (4.14) and (4.15)

$$\Phi_{\psi_i\psi_i}(Z) = \sum_{k=1}^{N}\sum_{j=1}^{N} \Phi_{y_{ij}y_{ik}}(Z) - \sum_{k=1}^{N} \Phi_{y_{ik}y_{0i}}(Z) - \sum_{k=1}^{N} \Phi_{y_{0i}y_{ik}}(Z) + \Phi_{y_{0i}y_{0i}}(Z). \tag{4.17}$$

Express the transformed correlation functions occurring here with the aid of the rule of index changes. Upon introducing the designation $\overset{*}{W}(Z) =$

$\equiv W(Z^{-1})$, omitting the argument Z, we obtain on the basis of (4.12) the following:

$$\Phi_{y_{ij}y_{ik}} = \hat{W}_{ij} W_{ik} \Phi_{x_j x_k} = \hat{W}_{ij} \hat{W}_{ik} [\Phi_{m_j m_k} + \Phi_{m_j n_k} + \Phi_{n_j m_k} + \Phi_{n_j n_k}], \quad (4.18)$$

$$\Phi_{y_{ik}y_{oi}} = \hat{W}_{ik} W_{oi} \Phi_{x_k m_i} = \hat{W}_{ik} W_{oi} [\Phi_{m_k m_i} + \Phi_{n_k m_i}], \quad (4.19)$$

$$\Phi_{y_{oi}y_{ik}} = \hat{W}_{oi} W_{ik} \Phi_{m_i x_k} = \hat{W}_{oi} W_{ik} [\Phi_{m_i m_k} + \Phi_{m_i n_k}], \quad (4.20)$$

$$\Phi_{y_{oi}y_{oi}} = \hat{W}_{oi} W_{oi} \Phi_{m_i m_i}. \quad (4.21)$$

On most occasions it is justified to assume that the useful signals and the noises are uncorrelated by pairs. Then the expressions are simplified, but even in this case the function $\Phi_{\psi_i\psi_i}(Z)$ is complicated. If all the reference inputs and noises too are uncorrelated by pairs, then the situation is relatively simple, since in this case the system is noninteracting in the stochastical sense. Naturally in this case this is not a characteristic of the system, but a consequence of the uncorrelatedness of the reference inputs by pairs.

It should be noted that since the individual functions $W_{ik}(Z)$ are polynomials, thus the poles of the function $\Phi_{\psi_i\psi_i}(Z)$ are essentially identical with the poles of the transformed correlation functions. By the transfer functions the place $Z = 0$ is introduced, at the most, as a new pole. In spite of this, the evaluation of the statistical error is very lengthy, since (4.17) consists of a very great number of terms. In the general case the residues of $(2N + 1)^2$ functions should be calculated to a single variable. Even if the useful signals and the noises are uncorrelated, the number of functions is $2N(N + 1) + 1$ even in this condition, i.e. in the case of $N = 2$, the number is 13, while in the case of $N = 3$, it is already 25.

c) *Evaluation of the method*

With the aid of the given relationships the controlled variables and the statistical error can be checked. With the suitable choice of the free parameters the quality characteristics can be improved. If necessary, the degree p_{jk} of the polynomials $C'_{jk}(Z)$ is chosen higher. It is useful, by considering expression (4.7) for the settling time, to increase, above all, those degrees p_{jk} which do not increase the settling time. If the performance of the system is already suitable, then the transfer matrix of the impulse compensator is determined according to (4.10) or (4.11).

The advantage of the process is that it can be employed both in the case of $M \geq N$ and $M < N$. One of the drawbacks of the method is that the system is actually not noninteracting, thus its operation is not completely satisfactory, as has already been mentioned, in spite of the formally reassuring quality characteristics. A drawback in the calculation technique is that first the elements of the matrix $\mathbf{C}(Z)$ are to be determined and the effect of these is clarified only during further calculations. The calculation of the statistical error is very lengthy. The calculation of the impulse compensator requiring a matrix inversion is similarly cumbersome.

5. *Sampled noninteracting excess systems*

a) General correlations

Systems with several variables are designated as sampled noninteracting systems, if the controlled variables depend on the sampling instants only at their reference inputs, not on the other reference inputs. The condition of sampled noninteracting is, according to (4.7), that $W_{ik}(Z) = 0$, $i \neq k$, with other words, the transfer matrix of $\mathbf{W}(Z)$ should be diagonal.

The expression of the transfer matrix of the closed system shown in Fig. 1, according to (1.15) is

$$\mathbf{W}(Z) = \mathbf{L}(Z)\,[\mathbf{I} + \mathbf{L}(Z)]^{-1}. \tag{5.1}$$

The matrix $\mathbf{W}$ is diagonal only in the case if the matrix $\mathbf{L} = \mathbf{GD}$ is also diagonal. Accordingly the condition of sampled noninteracting is

$$L_{ik}(Z) = \sum_{j=1}^{M} G_{ij}(Z)\,D_{jk}(Z) = 0, \qquad k = 1, 2, \ldots, N,$$
$$i = 1, 2, \ldots, N,\ i \neq k \tag{5.2}$$

If in the above equation the index k is assumed to be fixed, the number of the condition equations is $(N - 1)$. The functions $G_{ij}(Z)$ are given, while the number of the unknown functions $D_{jk}(Z)$ is M. One of these must be regarded as given (e.g. the function $D_{1k}(Z)$), since otherwise only the trivial solution $D_{jk}(Z) = 0$ exists, that naturally cannot be used. Thus the number of the functions $D_{jk}(Z)$ which can be chosen at will is $(M - 1)$. It is evident that $(N - 1)$ equations with $(M - 1)$ unknown terms can be satisfied only if $M \geqq N$, i.e. if this is an excess system (there are more inputs of the controlled element than controlled variables). For designing symmetrical systems, $(M = N)$, the method of Tou [10] can be utilized, while the designing of excess systems can be performed by way of the generalization of this method. The essence of the generalization is that the $N(N - 1)$ pieces of functions $D_{jk}(Z)$ are expressed in terms of the others, which will be determined later.

Let us regard the functions $D_{kk}(Z)$ $(k = 1, 2, \ldots, N)$, figuring in the main diagonal of the upper quadratic block of the matrix $\mathbf{D}(Z)$, as given. The other elements are expressed with the help of the coupling matrix $\mathbf{J}(Z)$:

$$D_{jk}(Z) = J_{jk}(Z)\,D_{kk}(Z), \qquad j = 1, 2, \ldots M,$$
$$k = 1, 2, \ldots N, \tag{5.3}$$

where naturally

$$J_{kk}(Z) = 1, \qquad k = 1, 2, \ldots, N. \tag{5.4}$$

Accordingly one element (D_{kk}) in the k-th column of the matrix $\mathbf{D}(Z)$ is given, $(N-1)$ pieces of elements can be determined with the aid of relationship (5.2) and $(M-N)$ elements can be entered arbitrarily.

Let us regard the upper quadratic block of the matrix $\mathbf{D}(Z)$ (with the exception of the main diagonal) as being determined, while the lower block is chosen. It is advisable to express the elements of the latter similarly by the elements $D_{kk}(Z)$, this means that

$$D_{jk}(Z) = K_{jk}(Z)\, D_{kk}(Z), \qquad j = N+1, N+2, \ldots, M, \quad k = 1, 2, \ldots, N, \tag{5.5}$$

where the functions $K_{jk}(Z)$ are arbitrary rational functions having no pole at $Z = 0$. (The condition of the realizability of the impulse compensator is that $Z = 0$ may not be a pole of the functions $D_{jk}(Z)$).

Divide the equation (5.5) by $D_{kk}(Z)$:

$$\sum_{j=1}^{M} G_{ij}(Z)\, J_{jk}(Z) = 0, \qquad i \neq k, \qquad i, k = 1, 2, \ldots, N. \tag{5.6}$$

Upon arranging the quantities regarded as known on the right side, and performing on the basis of formulae (5.3) and (5.5) the substitution

$$J_{jk}(Z) = K_{jk}(Z), \quad j = N+1, \ N+2, \ldots, M, \quad k = 1, 2, \ldots N, \tag{5.7}$$

we obtain:

$$\sum_{\substack{j=1 \\ j \neq k}}^{N} G_{ij}(Z)\, J_{jk}(Z) = -G_{ik}(Z) - \sum_{j=N+1}^{M} G_{ij}(Z)\, K_{jk}(Z), \tag{5.8}$$

$$k = 1, 2, \ldots, N, \ i = 1, 2 \ldots, M, \ i \neq k.$$

The solution of the N pieces, system of linear equations each with $(N-1)$ unknown value, supplies the functions $J_{jk}(Z)$ which were regarded as unknown.

With the functions $J_{jk}(Z)$ determined in the described way, the transfer matrices $\mathbf{L}(Z)$ and $\mathbf{W}(Z)$ are diagonal. Taking this into consideration, in view of (5.1), the expression for the elements in the main diagonal of the transfer matrix of the closed system is

$$W_{kk}(Z) = \frac{D_{kk}(Z) \sum_{j=1}^{M} G_{kj}(Z)\, J_{jk}(Z)}{1 + D_{kk}(Z) \sum_{j=1}^{M} G_{kj}(Z)\, J_{jk}(Z)}. \tag{5.9}$$

This relationship serves as the basis for designing.

b) The condition of continuous noninteracting

The expression of the modified discrete transfer matrix serving for the determination of the continuous controlled variable is given according to (1.16) by

$$\mathbf{W}(Z, \lambda) = \mathbf{G}(Z, \lambda)\ \mathbf{D}(Z)\ \mathbf{L}^{-1}(Z)\ \mathbf{W}(Z)\,. \tag{5.10}$$

Since $\mathbf{L}$ and $\mathbf{W}$ are diagonal matrices according to the preceding considerations, thus one element of the matrix $\mathbf{W}(Z, \lambda)$ can be written as

$$W_{ik}(Z, \lambda) = \sum_{j=1}^{M} G_{ij}(Z, \lambda)\, D_{jk}(Z) \frac{1}{1 + \sum\limits_{j=1}^{M} G_{kj}(Z)\, D_{jk}(Z)} =$$

$$= \frac{W_{kk}}{\sum\limits_{j=1}^{M} G_{ij}(Z)\, J_{jk}(Z)} \sum_{j=1}^{M} G_{ij}(Z, \lambda)\, J_{jk}(Z), \quad i, k = 1, 2, \ldots, N. \tag{5.11}$$

Now in view of (5.6) it is ensured that $\Sigma\, G_{ij}(Z)\, J_{jk}(Z) = 0$, if $i \neq k$. Generally from this it does not follow, that $\Sigma\, G_{ij}(Z, \lambda)\, J_{jk}(Z)$ figuring in the expression for $W_{ik}(Z, \lambda)$ is also zero. Consequently the system is not continuously noninteracting, the reasons for this have already been discussed.

Let us, however, assume that for one or more indices i the relationship

$$G_{ij}(Z, \lambda) = k_{ij}\, G_{ii}(Z, \lambda)\,, \quad j = 1, 2, \ldots, M \tag{5.12}$$

is valid, where k_{ij} is a constant, though in principle it can be an arbitrary function of Z too. The elements of the i-th row of the transfer matrix $\mathbf{G}(Z, \lambda)$ are proportional with one another and the proportion factors do not depend on λ. In this case, naturally, the relationship $G_{ij}(Z) = k_{ij}\, G_{ii}(Z)$ is also valid, hence (5.6) is simplified to the following form:

$$G_{ii}(Z) \sum_{j=1}^{M} k_{ij}\, J_{jk}(Z) = 0, \quad i \neq k. \tag{5.13}$$

At the same time the expression for the sum figuring in (5.11) is

$$\sum_{j=1}^{M} G_{ij}(Z, \lambda)\, J_{jk}(Z) = G_{ij}(Z, \lambda) \sum_{j=1}^{M} k_{ij}\, J_{jk}(Z). \tag{5.14}$$

Since the value of this is zero according to (5.13) in the case of $i \neq k$, thus the corresponding elements of the modified discrete transfer matrix are

$$W_{ii}(Z, \lambda) = W_{ii}(Z) \frac{G_{ii}(Z, \lambda)}{G_{ii}(Z)}\,, \tag{5.15}$$

$$W_{ik}(Z, \lambda) = 0, \qquad\qquad i \neq k.$$

Thus under these conditions the controlled variable y_i is noninteracting continuously too. If the condition (5.12) is fulfilled for any value of i, then the sampled noninteracting system is continuously noninteracting too. The existence of the relation (5.12) can be already decided on the basis of the examination of the transfer matrix $\mathbf{G}_S(s)$ of the controlled system. The condition is evidently that we should have

$$G_{S\,ij}(s) = k_{ij}\, G_{S\,ii}(s), \quad j = 1, 2, \ldots, M. \tag{5.16}$$

Here k_{ij} may be a constant or a rational function of e^{-sT}, but it may not be directly dependent on the variable s. This means essentially that all the components of the i-th output signal of the controlled system change with identical time constants and natural frequencies, thus the value of the mixed index components could actually be zero.

c) *The course of designing*

Let us summarize the course for designing a sampled noninteracting system.

1. Certain functions $K_{jk}(Z)$ are taken down which have no pole at $Z = 0$. The more indefinite coefficients these functions contain, the more possibilities shall we have for additional corrections, but the more difficult the calculation will become. A very simple, but effective choice is

$$K_{jk}(Z) = K_{jk}, \quad j = N+1, N+2, \ldots, M, k = 1, 2, \ldots, N, \tag{5.17}$$

where K_{jk} is an indefinite constant.

2. The system of N equations, each having $(N-1)$ unknown quantities given under (5.8), is solved. The coupling functions $J_{jk}(Z)$ contain the parameters of the functions $K_{jk}(Z)$, this means that in the case of choosing $K_{jk}(Z) = K_{jk}$, the coupling elements $J_{jk}(Z)$ are linear functions of the factors K_{jk}.

3. At the sampling instants, the controlled variables depend only on their own reference input. The discrete transfer function is given by (5.9). This can be interpreted so, that the system of N variables can be substituted with respect to the sampling instants by N pieces of equivalent subsystems (*Fig. 3*). The transfer functions of this are

$$G_k(Z) = \sum_{j=1}^{M} G_{kj}(Z)\, J_{jk}(Z), \tag{5.18}$$

$$D_k(Z) = D_{kk}(Z), \tag{5.19}$$

$$W_k(Z) = \frac{D_k(Z)\, G_k(Z)}{1 + D_k(Z)\, G_k(Z)}. \tag{5.20}$$

For designing these subsystems with one variable, all the methods can be employed which are based on the analysis of the signal values assumed at the sampling instants. Such method is typically the one with the aid of which the system with finite settling time and minimum statistical error (interpreted for the sampling instants) can be designed. This method has been elaborated in detail [8], thus we do not treat it here. If the transfer function $W_k(Z)$ has already been determined, then the transfer function of the impulse compensator of the subsystem is

$$D_k(Z) = \frac{1}{G_k(Z)} \cdot \frac{W_k(Z)}{1 - W_k(Z)} . \tag{5.21}$$

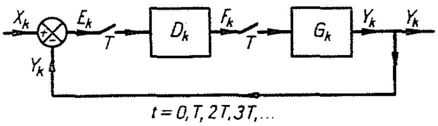

Fig. 3

4. Already during the examination of the equivalent subsystems, certain aspects can be asserted on determining the free parameters of the functions $K_{jk}(Z)$. It is advantageous if the numerator of the functions $G_k(Z)$ has as low a degree as possible, since thus a ripple-free system can be ensured with a shorter settling time. It is similarly advantageous if the degree of the denominator is reduced, since the consequence of this is that the transfer function of the impulse compensator is more simple and consequently (in most cases) the build-up of the compensator too.

During the analysis of the values assumed by the controlled variables at the sampling instants, further aspects can be obtained for the choice of the free parameters of the functions $K_{jk}(Z)$. In this respect these have a similar role to the eventually chosen free parameters of the function $W_k(Z)$. The essential difference is that every free parameter of the function $W_k(Z)$ (which is a polynomial) increases the settling time by one period, while the number of the free parameters of the function $K_{jk}(Z)$ is in no direct connection with the settling time. (As we have seen, the settling time can still be reduced.)

It should be mentioned here that the stability of the impulse compensator is advantageous, though not unconditionally necessary. The condition of this is that no function $D_{jk}(Z)$ should have a pole inside the unit circle. This can be ensured for the functions $D_{kk}(Z)$ by a suitable choice of $W_k(Z)$, (this precondition is automatically fulfilled when designing a ripple-free system), or by making sure that the functions $G_k(Z)$ have no zero inside the unit circle. The functions $J_{jk}(Z)$ may not have a pole inside the unit circle. This is a new aspect for choosing the parameters of the functions $K_{jk}(Z)$.

5. With the aid of the modified discrete transfer functions, the continuous controlled variables are determined:

$$W_{ik}(Z,\lambda) = \frac{W_k(Z)}{G_k(Z)} \sum_{j=1}^{M} G_{ij}(Z,\lambda)\, J_{jk}(Z). \tag{5.22}$$

First of all the components originating from the coupling are examined and we shall try to minimize their maximum value (or quadratic integral). To this end the free parameters of the functions $K_{jk}(Z)$ can similarly be utilized. The continuous overshoot of the own component can also be checked and (in the case of a not ripple-free system) the degree of the ripple after the settling time.

It should be mentioned here that if the conditions (5.12) and (5.16) are satisfied for certain indices i, then

$$W_{ii}(Z, \lambda) = W_{ii}(Z)\, \frac{G_{ii}(Z,\lambda)}{G_{ii}(Z)}\,, \tag{5.23}$$

$$W_{ik}(Z,\lambda) = 0, \quad i \neq k.$$

Accordingly the controlled variable y_i is noninteracting continuously too.

6. If we succeeded in choosing all the free parameters in such a way that the performance of the system is satisfactory in every aspect, then the transfer matrix of the impulse compensator can be determined on the basis of the relationships

$$D_{kk}(Z) = D_k(Z), \qquad D_{jk}(Z) = J_{jk}(Z)\, D_{kk}(Z)\,. \tag{5.24}$$

To sum up, the essence of the designing procedure is that the examination of a multivariable system is reduced to the examination of systems with a single variable. In comparing both a single-variable and symmetrical multivariable systems, what is new is that the functions $K_{jk}(Z)$ can be chosen. The number of such functions pertaining to an index k (subsystem) is $(M - N)$. If the recommended choice of $K_{jk}(Z) = K_{jk}$ is used, the number of free parameters is $N\,(M - N)$. With more complicated functions $K_{jk}(Z)$, as we have already mentioned, the calculation will be more difficult, but we have accordingly more parameters for considering the enumerated points of view.

The symmetrical system $(M = N)$ can be handled as the limit case of the excess system. The process is exactly the same, but in this case not a single function $K_{jk}(Z)$ may, and can, be chosen.

d) The effect of the disturbing variables

Let us now examine the effect of the disturbing variables in the case of symmetrical and excess multivariable systems. We shall assume that the disturbing variables are acting at the output of the system, or they are reduced there (Fig. 4).

When examining the effect of the disturbing variables at the sampling instants (assuming the reference inputs to be zero), the following relationships can be written on the basis of the block diagram in Fig. 4:

$$\mathbf{Y}(Z) - \mathbf{U}(Z) = \mathbf{G}(Z)\ \mathbf{F}(Z) = \mathbf{G}(Z)\ \mathbf{D}(Z)\,\mathbf{E}(Z) = = -\ \mathbf{G}(Z)\ \mathbf{D}(Z)\ \mathbf{Y}(Z)\,, \tag{5.25}$$

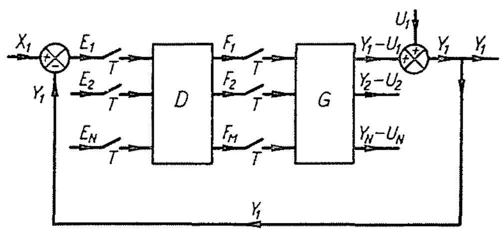

Fig. 4

where $\mathbf{U}(Z)$ denotes the N-row column matrix of the disturbing variables. Let us introduce the discrete transfer matrix $\mathbf{W}_U(Z)$ pertaining to the disturbing variable:

$$\mathbf{Y}(Z) = \mathbf{W}_U(Z)\,\mathbf{U}(Z)\,, \quad \text{if } \mathbf{X}(Z) = 0\,. \tag{5.26}$$

According to (5.25) and (1.15), we obtain

$$\mathbf{W}_U(Z) = [\mathbf{I} + \mathbf{L}(Z)]^{-1} = \mathbf{L}^{-1}(Z)\,\mathbf{W}(Z) = \mathbf{I} - \mathbf{W}(Z). \tag{5.27}$$

Since the matrix $\mathbf{W}(Z)$ is diagonal, also the matrix $\mathbf{W}_U(Z)$ is diagonal.

$$W_{Ukk}(Z) = 1 - W_k(Z); \quad W_{Uik}(Z) = 0\,, \quad i \neq k. \tag{5.28}$$

This means that the system is noninteracting at the sampling instants with respect not only to the reference input but also to the disturbing variables. A certain controlled variable depends at the sampling instants only on the disturbing variable acting on it but not on the other disturbing variables. Thus the equivalent subsystems with one variable are also valid for the disturbing variable at the sampling instants.

Accordingly, in the case of sampled noninteracting multivariable systems with finite settling time, the same can be said on the effect of the disturbing variables at the sampling instants, that being the case for finite settling time systems with one variable [9]. It is specially to be mentioned that the system has a finite settling time with respect to the disturbing variable, too. The order m_U of the compensated disturbing input is identical with the order m of the reference input.

A reduction of other effects of the disturbing variable, however, (overshoot, degree of coupling between the sampling instants, statistical error), is only possible at the expense of the characteristics of the reference input (e.g. settling time).

e) Compensation of the disturbing variable

In systems with one variable, the discrete transfer function pertaining to the reference input or to the disturbing variable can be prescribed separately

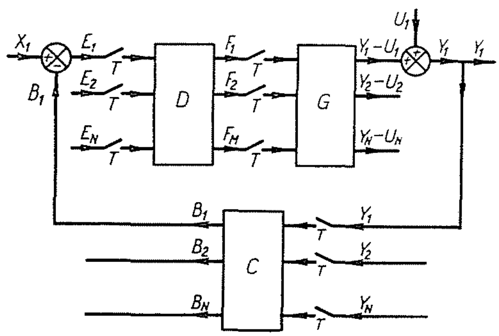

Fig. 5

with the aid of a compensator arranged in the feedback loop [5, 9]. In multivariable systems, this effect can evidently be achieved by an impulse compensator characterized by a transfer matrix. The transfer matrix of this compensator is designated by $\mathbf{C}(Z)$ in *Fig.* 5.

The column matrix of the modified discrete transform of the controlled variables can be written on the basis of the block diagram in Fig. 5, as

$$\mathbf{Y}(Z, \lambda) - \mathbf{U}(Z, \lambda) = \mathbf{G}(Z, \lambda)\,\mathbf{D}(Z)\,[\mathbf{X}(Z) - \mathbf{C}(Z)\,\mathbf{Y}(Z)]\,. \qquad (5.29)$$

Upon changing over to the (simple) discrete transforms and expressing the controlled variable, we obtain the discrete transfer matrices for the reference input and the disturbing variable, respectively:

$$\mathbf{W}(Z) = [\mathbf{I} + \mathbf{L}(Z)\,\mathbf{C}(Z)]^{-1}\,\mathbf{L}(Z) = \mathbf{L}(Z)\,[\mathbf{I} + \mathbf{C}(Z)\,\mathbf{L}(Z)]^{-1}, \quad (5.30)$$

$$\mathbf{W}_U(Z) = [\mathbf{I} + \mathbf{L}(Z)\,\mathbf{C}(Z)]^{-1} = \mathbf{W}(Z)\,\mathbf{L}^{-1}(Z) = \mathbf{L}^{-1}(Z)\,\mathbf{W}(Z)\,. \quad (5.31)$$

Naturally the requirement still remains that the system should be sampled noninteracting with respect to the reference inputs. It is advantageous both from the aspect of the functioning of the system and from the clear arrangement of the design, that the system be sampled noninteracting with respect to the disturbing variables too. These two conditions are satisfied if both $\mathbf{W}(Z)$ and $\mathbf{W}_U(Z)$ are diagonal matrices. This is, in turn, fulfilled if both $\mathbf{L}(Z) = \mathbf{G}(Z)\,\mathbf{D}(Z)$ and $\mathbf{L}(Z) = \mathbf{C}(Z)$ are diagonal. In this case, $\mathbf{C}(Z)$ necessarily must only be diagonal.

Accordingly the conditions of noninteracting are

$$L_{ik}(Z) = 0, \quad i \neq k\,, \quad C_{ik}(Z) = 0\,, \quad i \neq k\,. \quad (5.32)$$

The way of satisfying the first system of conditions was described in the preceding discussion. On the other hand, the diagonality of $\mathbf{C}(Z)$ means

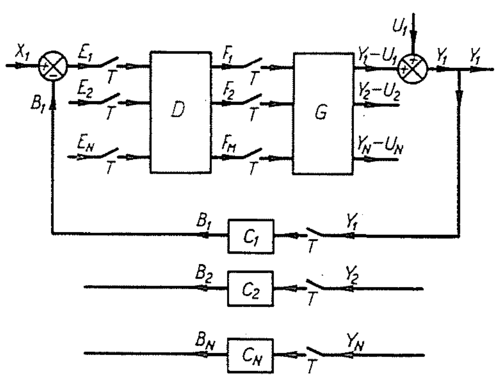

Fig. 6

that the impulse compensator with the transfer matrix $\mathbf{C}(Z)$ can simply be realized by N pieces of common (one variable) impulse compensators (Fig. 6), that is otherwise evident according to Fig. 5. This fact brings about a simplification in realization, and calculation work is reduced thereby. A drawback is, in turn, that we can decide only above a smaller number of free parameters, namely N pieces of transfer functions $C_i(Z) = C_{ii}(Z)$, than in the case of a compensator with several variables, characterized by an $N \times N$ element matrix.

Since both the transfer matrices $\mathbf{W}(Z)$ and $\mathbf{W}_U(Z)$ are diagonal, the multivariable system can be substituted with respect to the sampling instants by single variable subsystems (Fig. 7). The transfer functions of these, in view of (5.30) and (5.31), and the preceding sections, are

$$G_k(Z) = \sum_{j=1}^{M} G_{kj}(Z)\, J_{jk}(Z) \tag{5.33}$$

$$D_k(Z) = D_{kk}(Z)\,; \quad G_k(Z) = C_{kk}(Z)\,; \tag{5.34}$$

$$W_k(Z) = \frac{D_k(Z)\, G_k(Z)}{1 + C_k(Z)\, D_k(Z)\, G_k(Z)}\,, \tag{5.35}$$

$$W_{UK}(Z) = \frac{1}{1 + C_k(Z)\, D_k(Z)\, G_k(Z)}\,. \tag{5.36}$$

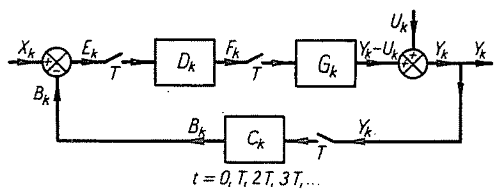

Fig. 7

The impulse compensators of the substituting subsystems can be designed by any method elaborated for one variable system, which are based on the analysis of values occurring at the sampling instants. In consequence of the existence of the second impulse compensator, the quality characteristics for the reference signal and the disturbing variable can be prescribed as essentially independent. That method should be mentioned specially with the aid of which a system of finite settling time and minimum statistical error can be designed with respect to both the reference input and the disturbing variable [9]. In this case, the settling time, the compensated limit order, the minimum of the statistical error can be independently chosen for the reference input and the disturbing variable, respectively. With the knowledge of the functions $D_k(Z)$ and $C_k(Z)$ the elements of the transfer matrices $\mathbf{D}(Z)$ and $\mathbf{C}(Z)$, respectively, can be determined:

$$D_{jk}(Z) = J_{jk}(Z)\, D_{kk}(Z)\,, \quad C_{kk}(Z) = C_k(Z)\,. \tag{5.37}$$

As a final check, it may be advisable to examine the continuous controlled variable pertaining to the reference input. If $\mathbf{U} = \mathbf{0}$, then according to (5.29) and (5.30), after matrix algebraical transformations, it is quite evident, that the modified discrete transfer matrix is

$$\mathbf{W}(Z, \lambda) = \mathbf{G}(Z, \lambda)\, \mathbf{D}(Z)\, [\mathbf{I} + \mathbf{C}(Z)\, \mathbf{L}(Z)]^{-1}, \tag{5.38}$$

that corresponds to the second form of (5.30).

One element of the modified transfer matrix is

$$W_{ik}(Z, \lambda) = \frac{D_{kk}(Z) \sum_{j=1}^{M} G_{ij}(Z, \lambda)\, J_{jk}(Z)}{1 + C_{kk}(Z)\, D_{kk}(Z) \sum_{j=1}^{M} G_{kj}(Z)\, J_{jk}(Z)} =$$

$$= W_{kk}(Z) \frac{\sum_{j=1}^{M} G_{ij}(Z, \lambda)\, J_{jk}(Z)}{\sum_{j=1}^{M} G_{ij}(Z)\, J_{jk}(Z)}. \tag{5.39}$$

These are completely analogous to the two forms of (5.11). Thus our statements concerning the continuous noninteracting in Section 5b are valid word by word, thus it is not worthwhile to repeat them here.

6. *Sampled noninteracting deficiency systems*

a) The ensurance of noninteracting

We have seen in Section 5a that the sampled noninteracting cannot be ensured with the aid of an impulse compensator arranged in the forward loop in the case of deficiency systems ($M < N$, the number of inputs of the controlled element being lower, than that of the controlled variables). The reason is to be found in the fact that the quadratic transfer matrix $\mathbf{W}(Z)$ is characterized by N^2 pieces of functions $W_{ik}(Z)$, while the number of elements $D_{ik}(Z)$ of the transfer matrix $\mathbf{D}(Z)$ is only MN. Thus in the case of $M < N$ we do not have the necessary number of functions which can be chosen.

Noninteracting can only be ensured, i.e. the transfer matrix $\mathbf{W}(Z)$ can be made diagonal only if a new organ is (or organs are) inserted into the system. In principle it would be sufficient, if the new organ were characterized by $(N - M)\, N$ pieces of transfer functions, this can however hardly be built logically into the system. It seems to be simpler if the new organ has N inputs and N outputs, consequently the characterizing N^2 pieces of transfer functions are sufficient themselves to ensure the desired form of the matrix $\mathbf{W}(Z)$. The role of the signal modifier inserted in the forward loop is basically reduced in this case to the production of M pieces of inputs of the controlled element from N pieces of actuating signals (by way of delaying and linear superposition). The task of the new impulse compensator is the suitable modification of the signals. (In the case of symmetrical and excess systems the impulse compensator having the transfer matrix $\mathbf{D}(Z)$ was able to perform these two tasks simultaneously.)

An organ of this kind can be arranged into the system in several ways. Naturally any solution is senseless which means no independent organ from the aspect of operation. E.g. a compensator connected in series or in parallel with the compensator in the forward loop does not represent a new organ, since these two compensators can be substituted from the aspect of operation by their resultant. The resultant organ has naturally N inputs and M outputs, thus it is characterized by NM transfer functions, consequently we do not obtain new transfer functions which can be chosen.

b) The difficulties of realization

The realization of the system built up in accordance with the above train of thoughts has a serious difficulty. Let us designate the transfer matrix of the impulse compensator serving to convert the number of variables by $\mathbf{D}(Z)$, while the transfer matrix of the actual signal modifier by $\mathbf{C}(Z)$. Let the latter be arranged e.g. in the feedback loop (cf. Fig. 5). The matrix $\mathbf{D}(Z)$ can be regarded for the time being as given, consequently the matrix $\mathbf{L}(Z) = \mathbf{G}(Z)\,\mathbf{D}(Z)$ is also regarded as known, the task being to determine the matrix $\mathbf{C}(Z)$.

The discrete transfer matrix of the closed system is, in accordance with (5.30),

$$\mathbf{W}(Z) = \mathbf{L}(Z)\,[\mathbf{I} + \mathbf{C}(Z)\,\mathbf{L}(Z)]^{-1} = \Big[[\mathbf{I} + \mathbf{C}(Z)\,\mathbf{L}(Z)]\,\mathbf{L}^{-1}(Z)\Big]^{-1} = [\mathbf{L}^{-1}(Z) + \mathbf{C}(Z)]^{-1}. \tag{6.1}$$

It is evident already from this, that

$$\mathbf{C}(Z) = \mathbf{W}^{-1}(Z) - \mathbf{L}^{-1}(Z). \tag{6.2}$$

This result is, however, only virtual, since the matrix $\mathbf{L}(Z) = \mathbf{G}(Z)\,\mathbf{D}(Z)$ cannot be inverted, as its determinant is zero, i.e. the ordinal is lower than the dimension. In our case, namely, $M < N$, consequently it is evident that the order of both the matrices $\mathbf{G}(Z)$ and $\mathbf{D}(Z)$ can be M at the maximum, i.e. $r_G \leqq M$, $r_D \leqq M$ (if the ordinal is designated by r). According to the estimation of the order of the product of matrices

$$r_L = r_{GD} \leqq \min(r_G, r_D) \leqq M < N. \tag{6.3}$$

The condition of the invertability of the matrix $\mathbf{L}(Z)$ in turn is $r_L = N$, which is certainly not satisfied in the case of a deficiency system, consequently the matrix $\mathbf{L}(Z)$ cannot be inverted, as we have already asserted.

Strictly speaking, we have proved by this only that the prescribed matrix $\mathbf{W}(Z)$ of the closed system cannot be realized by the system shown in Fig. 8. It is evident, however, that the requirement of the invertability of the matrix $\mathbf{L}(Z) = \mathbf{G}(Z)\,\mathbf{D}(Z)$ arises also in the case of other systems of similar structure. Accordingly the sampled noninteracting of deficiency multivariable systems can be ensured only by systems of basically different structures. The noninteracting with finite settling time, in turn, can be ensured by the method described in Chapter 4, in the case of both excess and deficiency systems.

7. *Illustrative example*

The manner and some aspects of designing excess systems is illustrated by a very simple example. Let the number of controlled variables be $N = 2$, while the number of inputs of the controlled element $M = 3$. Let the transfer matrix of the controlled system be

$$G_S(s) = \begin{bmatrix} \dfrac{1}{s+1} & \dfrac{0.5}{s+1} & 0 \\ \dfrac{0.2}{s+0.25} & \dfrac{1}{s+0.5} & \dfrac{0.1}{s+0.25} \end{bmatrix}. \tag{7.1}$$

The matrix has a very simple structure, since

$$G_{12} = \frac{1}{2} G_{11}, \quad G_{13} = 0, \quad G_{23} = \frac{1}{2} G_{21}. \tag{7.2}$$

Calculation is facilitated by these properties, and some considerations become more easy to survey. However, attention is called to the fact that the situation is generally more complicated.

By employing a zero order hold circuit, the discrete transfer matrix, by using the designation

$$p = e^{-1/4} = 0.77880 \tag{7.3}$$

will be

$$\mathbf{G}(Z) = \begin{bmatrix} \dfrac{(1-p^4)\,Z}{1-p^4 Z} & \dfrac{0.5(1-p^4)\,Z}{1-p^4 Z} & 0 \\ \dfrac{0.8(1-p)\,Z}{1-pZ} & \dfrac{2(1-p^2)\,Z}{1-p^2 Z} & \dfrac{0.4(1-p)\,Z}{1-pZ} \end{bmatrix}. \tag{7.4}$$

The independent elements of the modified discrete transfer matrix are

$$G_{11}(Z,\lambda) = Z\,\frac{1-p^{4\lambda}+(p^{4\lambda}-p^4)\,Z}{1-p^4 Z},$$

$$G_{21}(Z,\lambda) = 0.8\,Z\,\frac{1-p^{\lambda}+(p^{\lambda}-p)\,Z}{1-pZ},$$

$$G_{22}(Z,\lambda) = 2Z\,\frac{1-p^{2\lambda}+(p^{2\lambda}-p^2)\,Z}{1-p^2 Z}. \tag{7.5}$$

The other elements can be calculated on the basis of (7.2).

Let the task be to design such a system that follows the step-form reference input without steady-state error, with the shortest possible settling time. The statistical error and the effect of the disturbing variables is not discussed here. The designing of the finite settling time noninteracting system is not performed here.

b) The conditions of sampled noninteracting

The condition equations of the sampled noninteracting are very simple, in the present case, according to (5.8). The functions $K_{jk}(Z)$ are assumed to be constant, thus

$$(k=1,\ i=2)\quad G_{22}(Z)\,J_{21}(Z) = -\,G_{21}(Z) - G_{23}(Z)\,K_{31},$$
$$(k=2,\ i=1)\quad G_{11}(Z)\,J_{12}(Z) = -\,G_{12}(Z) - G_{13}(Z)\,K_{32}. \tag{7.6}$$

From this, the elements of the coupling matrix $\mathbf{J}(Z)$ are

$$J_{11}(Z) = 1, \qquad J_{12}(Z) = -\frac{1}{2},$$
$$J_{21}(Z) = -\frac{0.4}{1+p}\,\frac{2+K_{31}}{2}\,\frac{1-p^2Z}{1-pZ}, \qquad J_{22}(Z) = 1,$$
$$J_{31}(Z) = K_{31}, \qquad J_{32}(Z) = K_{32}. \tag{7.7}$$

The transfer functions of the equivalent subsystems are, according to (5.18),

$$G_1(Z) = G_{11}(Z)\,J_{11}(Z) + G_{12}(Z)\,J_{21}(Z) + G_{13}(Z)\,J_{31}(Z) =$$
$$= \frac{1-p^4}{1-p}\,Z\,\frac{(0.8+p-0.1\,K_{31}) - p\,(1+0.8p-0.1p\,K_{31})\,Z}{(1-p^4\,Z)\,(1-p\,Z)}, \tag{7.8}$$

$$G_2(Z) = G_{21}(Z)\;J_{12}(Z) + G_{22}(Z)\;J_{22}(Z) + G_{23}(Z)\;J_{32}(Z) =$$
$$= 2(1-p)\,Z\,\frac{(0.8+p+0.2\,K_{32}) - p\,(1+0.8p+0.2p\,K_{32})\,Z}{(1-p^2\,Z)\,(1-p\,Z)}. \tag{7.9}$$

c) The designing of a not ripple-free system

Let us restrict our considerations to the designing of a not ripple-free system and try to obtain a system of possibly rapid operation, i.e. having the minimum settling time. In this case the form of the transfer function of the equivalent subsystems is

$$W_1(Z) = Z, \quad W_2(Z) = Z. \tag{7.10}$$

The elements of the transfer matrix describing the continuous signal are very simple with respect to the controlled variable y_1, since for the elements of the first row of the matrix $\mathbf{G}$ the relationships (5.16) are valid. Thus by force of (5.23)

$$W_{11}(Z,\lambda) = W_{11}(Z)\,\frac{G_{11}(Z,\lambda)}{G_{11}(Z)} = \frac{1}{1-p^4}\,Z\,\left[(1-p^{4\lambda}) + (p^{4\lambda}-p^4)\,Z\right], \tag{7.11}$$
$$W_{12}(Z,\lambda) = 0.$$

The controlled variable y_1 is unambiguously determined, since it does not depend on the parameters K_{31} and K_{32}. If $x_1(t) = 1(t)$, then in the case of an arbitrary $x_2(t)$

$$Y_1(Z,\ \lambda) = 1.582(1-e^{-\lambda})\;Z + Z^2 + Z^3 + Z^4 + \ldots\;. \tag{7.12}$$

Somewhat more calculations are necessary for determining the modified discrete transfer functions pertaining to the controlled variable y_2. According to (5.22)

$$W_{21}(Z, \lambda) = \frac{W_1(Z)}{G_1(Z)} [G_{21}(Z, \lambda) J_{11}(Z) + G_{22}(Z, \lambda) J_{21}(Z) + G_{23}(Z, \lambda) J_{31}(Z)] =$$

$$= 0.4 \frac{2 + K_{31}}{1 - p^4} Z \frac{(1 - p^4 Z)(1 - Z) [p - (1 + p)p^{\lambda} + p^{2\lambda}]}{(0.8 + p - 0.1K_{31}) - p(1 + 0.8p - 0.1pK_{31}) Z} . \tag{7.13}$$

It is immediately evident, that the most advisable choice is

$$K_{31} = - 2, \tag{7.14}$$

since in this case

$$W_{21}(Z, \lambda) = 0, \tag{7.15}$$

which means that y_2 is noninteracting in continuous sense too, this cannot be achieved in the symmetrical case.

This result is, however, surprising. As in the chosen example $G_{13} = 0$, i.e. the controlled variable y_1 does not depend on the input f_3 of the controlled element, therefore, it would seem logical if f_3 in turn were not dependent on the actuating signal e_1. This would mean the choice of $D_{31} = 0$, that is $K_{31} = 0$. By examining the whole complexity of the problem, the choice of $K_{31} = - 2$ seemed to be the most advisable. The discrete transfer function of the equivalent controlled system is then simplified in accordance with (7.8):

$$G_1(Z) = \frac{(1 - p^4) Z}{1 - p^4 Z} . \tag{7.16}$$

The following steps are the calculation of $W_{22}(Z, \lambda)$ and the choice of K_{32}. In the present simple example, however, the order of operations can be reversed. The most obvious additive requirement, namely, is that y_2 should be ripple-free. This is the case if $W_2(Z)$ contains all the zeros of $G_2(Z)$. In our example this can be ensured if $G_2(Z)$ has no other zeros beyond $Z = 0$. This can be achieved e.g. by making the first degree term in the numerator of $G_2(Z)$ equal to zero, which means that on the basis of (7.9)

$$K_{32} = - \frac{1 + 0.8p}{0.2p} = - 10.4200. \tag{7.17}$$

Now for the subsystem

$$G_2(Z) = - \frac{2(1 - p^2)(1 - p)}{p} \frac{Z}{(1 - pZ)(1 - p^2 Z)} . \tag{7.18}$$

Forming the transfer matrix element $W_{22}(Z, \lambda)$

$$W_{22}(Z, \lambda) = \frac{W_2(Z)}{G_2(Z)} [G_{21}(Z, \lambda) J_{12}(Z) + G_{22}(Z, \lambda) J_{22}(Z) + G_{23}(Z, \lambda) J_{32}(Z)] =$$

$$= \frac{1}{(1 - p^2)(1 - p)} Z\big[\big(1 - (1 + p) p^{\lambda} + pp^{2\lambda}\big) - (1 + p)\big(p - (1 + p^2) p^{\lambda} -$$

$$pp^{2\lambda}\big) Z + p^2 \big(p - (1 + p) p^{\lambda} + p^2\big) Z^2\big]. \tag{7.19}$$

It is worth mentioning that no overshoot occurs at the sampling instants. It can be shown by some calculations, that in the case of $x_2(t) = 1(t)$

$$y_{2\max} = 1 + \frac{p^2}{4(1 + p)} = 1.085. \tag{7.20}$$

Ripple-free operation can, however, be ensured in other ways too. Let the value of K_{32} be chosen so that the numerator of $G_2(Z)$ is equal to the radical $(1 - pZ)$ of the denominator. (The other radical of the denominator cannot be eliminated in this way.) The condition of this is, by force of (7.9), that

$$0.8 + p + 0.2\, K_{32} = 1 + 0.8p + 0.2pK_{32}, \tag{7.21}$$

from which we obtain

$$K_{32} = 1. \tag{7.22}$$

Then the transfer function of the subsystem is given by

$$G_2(Z) = \frac{2(1 - p^2)\, Z}{1 - p^2 Z}\,. \tag{7.23}$$

The modified discrete transfer function is

$$W_{22}(Z, \lambda) = \frac{1}{1 - p^2} Z \left[(1 - p^{2\lambda}) + (p^{2\lambda} - p^2)\, Z\right]. \tag{7.24}$$

The modified discrete transform of the controlled variable corresponding to the reference input $x_2(t) = 1(t)$ is

$$Y_{22}(Z, \lambda) = 2.541(1 - e^{\lambda/2}) \quad Z + Z^2 + Z^3 + \ldots . \tag{7.25}$$

Accordingly the settling time is $T_S = T$ and the controlled variable attains its steady state without overshoot and keeps it without ripple. This very favourable solution was made possible by the special case $G_{23}(Z, \lambda) = 0.5\; G_{21}(Z, \lambda)$. If other aspects are not examined, evidently the second solution should be regarded as being the more favourable.

d) The impulse compensator

According to (5.21), the transfer function of the subsystems is, by using (7.16) and (7.23),

$$D_1(Z) = \frac{1}{G_1(Z)} \frac{W_1(Z)}{1 - W_1(Z)} = \frac{1}{1 - p^4} \frac{1 - p^4 Z}{1 - Z}\,,$$

$$D_2(Z) = \frac{1}{G_2(Z)} \frac{W_2(Z)}{1 - W_2(Z)} = \frac{1}{2(1 - p^2)} \frac{1 - p^2 Z}{1 - Z}\,. \tag{7.26}$$

Accordingly, on the basis of (5.24),

$$D_{11}(Z) = D_1(Z),$$

$$D_{21}(Z) = J_{21}(Z)\; D_{11}(Z) = 0,$$

$$D_{31}(Z) = K_{31}\, D_{11}(Z) = -\, 2D_{11}(Z),$$

$$D_{22}(Z) = D_2(Z),$$

$$D_{12}(Z) = J_{12}(Z)\, D_{22}(Z) = -\frac{1}{2} D_{22}(Z), \qquad D_{32}(Z) = K_{32}(Z)\, D_{22}(Z) = D_{22}(Z). \tag{7.27}$$

The realization of the impulse compensator is not the subject of the present paper. It is worth while, however, to call the attention of the reader to the point, that realization is considerably facilitated in the present case by the fact that the coupling coefficients J_{jk} are constants independent of Z. The detailed block diagram in Fig. 8. was composed in such a way that only one compensating organ pertains to every actuating signal, afterwards the signals should only be attenuated. The equations of the block diagram are:

$$D_{31}(Z) = -\frac{2}{1-p^4} \cdot \frac{1-p^4 Z}{1-Z},$$

$$D_{11}(Z) = -\frac{1}{2} D_{31}(Z), \qquad D_{21}(Z) = 0;$$

$$D_{32}(Z) = \frac{1}{2(1-p^2)} \cdot \frac{1-p^2 Z}{1-Z},$$

$$D_{12}(Z) = -\frac{1}{2} D_{32}(Z), \qquad D_{22}(Z) = D_{32}(Z). \qquad (7.28)$$

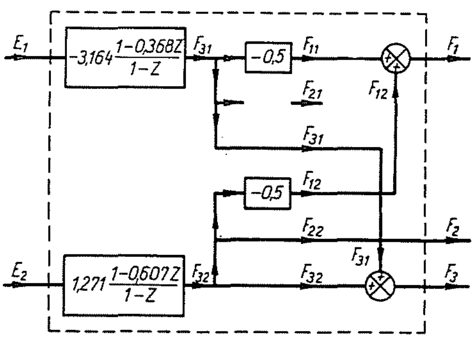

Fig. 8

In the case of a controlled system having a more general structure, or of more stringent requirements, the train of calculations will not be so simple and clearly arranged. First of all the choice of the coupling coefficients K_{jk} requires earnest considerations and detailed calculations. However, even this simple example was suitable for indicating some characteristics of nonsymmetrical systems.

Summary

Some theoretical problems and two designing methods of multivariable sampled-data control systems were discussed. The central problem is to ensure the noninteracting of the system, i.e. to attain, that all the controlled variables should depend only on their own reference input, and be independent of all the others, at least in a certain sense.

By employing one of the designing methods, only noninteracting finite settling time can be ensured: All the controlled variables are equal to their own reference input after a definite settling time, if the reference input is of the $(m-1)$th degree at the maximum. Since noninteracting can be ensured only for reference inputs of a certain type, the system is not noninteracting in the case of a reference input of the general type. Consequently the minimalization of the statistical error corresponding to the stochastic reference input is difficult in practice. The other drawback of this designing method is its formal character. It is especially difficult to predict the effect of the individual parameters on the build-up of the impulse

compensator. An advantage of the method is, in turn, that it can in principle be employed independently of the number of controlled variables and inputs of the controlled element.

By employing the other designing method, sampled noninteracting can be ensured: The controlled variables depend in the sampling instants only on their own reference input, whatever the change in time of the reference inputs may be. The multivariable system can be substituted with respect to the sampling instants by so many systems with one variable, as the number of the controlled variables. The designing of these equivalent systems can be performed by any method based on the analysis of the values occurring at the sampling instants.

The procedure can be generalized in such a way that the effect of the disturbing variables can be compensated independently of the reference inputs. To this end a second impulse compensator is required. The essence of the designing processes is, in this case too, the examination of the equivalent subsystems. This method is not suitable for designing deficiency systems ($M < N$).

The ideal continuous noninteracting cannot be ensured in a general case in sampled-data systems. If, however, individual rows in the transfer matrix of the controlled system have special characteristics, the controlled variables of the corresponding order number can be made continuously noninteracting.

The process is illustrated by a numerical example, in which some practical aspects of the designing methods are indicated.

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