SIMPLIFIED DERIVATION OF OPTIMUM TRANSFER FUNCTIONS FOR DIGITAL STOCHASTIC PROCESSES

By

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(Received March 11, 1965)

In some papers [1, 2, 3, 4, 5] a simplified derivation technique was presented for obtaining the optimum transfer functions in the WIENER—NEWTON sense for stationary ergodic stochastic continuous processes or signals. Here, the method is extended to strictly digital processes, that is, the input and output signals as well as the other signals are assumed to be in discrete form, or more precisely, we are dealing with sampled-data or pulsed-data signals (which may be treated as though they were number sequences) and only pulse-transfer functions are of interest. Hybrid systems are not considered here. In this paper only single variable cases are studied, multivariable systems will be investigated later.

On the other hand, not only the completely-free configuration but also the semi-free configuration and the semi-free configuration with constraints are considered.

As usual, here also the signals are assumed to be stationary and the ergodic hypothesis is adopted. As a basis of optimization the least mean square value of the error is taken.

As in the technical literature various definitions are given for the correlation functions of sampled-data signals [see e.g. 6, 7, 8, 9, 10] the definitions used here are summarized in an Appendix.

1. Completely-free configuration

The problem for the completely free configuration is demonstrated by *Fig. 1.* The reference signal $r(t)$ assumed as being a stationary ergodic stochastic process, contains an useful signal component $s(t)$ and a corrumping noise component $n(t)$. The reference signal $r(t)$ is sampled, thus the input signal is $r^*(t)$. The actual output signal $c^*(t)$ is also in sampled-data form, and is compared with the idealized or desired sampled-data output signal $i^*(t)$. Later this is obtained from the sampled-data signal component $s^*(t)$ through a discrete filter $y(t)$ which must not be physically realizable. The sampled-data error is

$$e^*(t) = i^*(t) - c^*(t) \tag{1.a}$$

or the corresponding number sequence is:

$$e(nT) = i(nT) - c(nT) \tag{1.b}$$

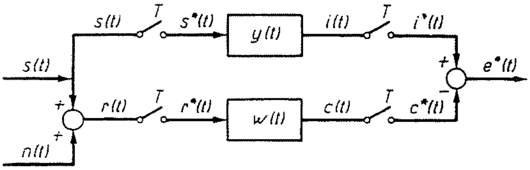

Fig. 1

Now the question arises, which is the weighting function $w(t)$ or the pulse-transfer function $W(z)$ minimizing the mean-square value of the number sequence of the error

$$\overline{e^2(nT)} = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} e^2(nT) \tag{2}$$

This mean-square value can be expressed as

$$\overline{e^2(nT)} = \varphi_{ee}(0T) = \frac{1}{2\pi j} \oint_{\Gamma_0} \Phi_{ee}(z)\, z^{-1}\, dz \tag{3}$$

where $\varphi_{ee}(kT)$ is the autocorrelation sequence (see Appendix 2) of the number sequence $e(nT)$:

$$\varphi_{ee}(kT) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} e(nT)\, e(nT + kT) \tag{4}$$

while $\Phi_{ee}(z)$ is the power spectrum (see Appendix 3):

$$\Phi_{ee}(z) = \sum_{k=-\infty}^{\infty} \varphi_{ee}(kT)\, z^{-k} \tag{5}$$

and Γ_0 is the unit circle in the z plane. Taking Eqs. (1) and (4) into consideration the autocorrelation sequence of the error can be expressed as follows:

$$\varphi_{ee}(kT) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \left[i(nT) - c(nT)\right] \cdot \left[i(nT + kT) - c(nT + kT)\right] \tag{6}$$

that is:

$$\varphi_{ee}(kT) = \varphi_{ii}(kT) - \varphi_{ic}(nT) - \varphi_{ci}(kT) + \varphi_{cc}(kT) \tag{7}$$

Similarly, the power spectrum is

$$\Phi_{ee}(z) = \Phi_{ii}(z) - \Phi_{ic}(z) - \Phi_{ci}(z) + \Phi_{cc}(z) \tag{8}$$

which, with the aid of the well-known index changing rule (see Appendix 3), can also be expressed as

$$\Phi_{ee}(z) = \Phi_{ii}(z) - \Phi_{ir}(z)\, W(z) - W(z^{-1})\, \Phi_{ri}(z) + W(z^{-1})\, \Phi_{rr}(z)\, W(z) \tag{9}$$

Here $W(z)$ is the z-transform of $w(t)$ or, more precisely, of $w^*(t)$. Let us define the following auxiliary pulse-transfer function

$$G(z) = \frac{\Phi_{ri}(z)}{\Phi_{rr}(z)} \tag{10}$$

Later this can be considered as a known function, as the power densities are our starting data. Taking $G(z)$ into consideration, the power spectrum of the error $\Phi_{ee}(z)$ can be expressed as follows:

$$\Phi_{ee}(z) = \Phi_{ii}(z) - G(z^{-1})\, \Phi_{rr}(z)\, G(z) +$$

$$+ [G(z^{-1}) - W(z^{-1})]\, \Phi_{rr}(z)\, [G(z) - W(z)] \tag{11}$$

It must be emphasized that only the last term of Eq. (11) contains the minimizing transfer function $W(z)$. The mean-square value of the number sequence of the error would be minimum, if and only if the last term was zero.

In this case the optimum pulse-transfer function must be

$$W_0(z) = G(z) \tag{12}$$

or according to Eq. (10):

$$W_0(z) = \frac{\Phi_{ri}(z)}{\Phi_{rr}(z)} \tag{13}$$

Unfortunately, $W_0(z)$ is, in general, physically unrealizable. The physically realizable optimum pulse-transfer function can be obtained in the following way. Rearranging Eq. (13)

$$\Phi_{rr}(z)\, W_0(z) - \Phi_{ri}(z) = 0 \tag{14}$$

Now, if $W_0(z)$ is substituted by the physically realizable optimum pulse-transfer function $W_m(z)$ then instead of Eq. (14) the following relation becomes valid

$$\Phi_{rr}(z)\, W_m(z) - \Phi_{ri}(z) = F_-(z) \tag{15}$$

where $F_-(z)$ is some function not yet known and having no poles or zeros inside the unit circle of the z plane. Performing the spectrum factorization the power spectrum $\Phi_{rr}(z)$ can be expressed as

$$\Phi_{rr}(z) = \Phi_{rr}^-(z)\, \Phi_{rr}^+(z) \tag{16}$$

where the factor $\Phi_{rr}^+(z)$ contains all the poles and zeros of $\Phi_{rr}(z)$ lying inside the unit circle, while the factor $\Phi_{rr}^-(z)$ contains all the poles and zeros of $\Phi_{rr}(z)$ outside the unit circle.

Thus, from Eq. (15)

$$\Phi_{rr}^+(z)\, W_m(z) = \frac{\Phi_{ri}(z)}{\Phi_{rr}^-(z)} + \frac{F_-(z)}{\Phi_{rr}^-(z)} \tag{17}$$

Separating each term into two components, the first having only poles inside the unit circle and thus belonging to positive-time functions, the second having only poles outside the unit circle and belonging to negative-time functions, the following two relations will be valid:

$$\begin{aligned} \Phi_{rr}^+(z)\, W_m(z) &= \left[\frac{\Phi_{ri}(z)}{\Phi_{rr}^-(z)}\right]_+ \\ 0 &= \left[\frac{\Phi_{rr}(z)}{\Phi_{rr}^-(z)}\right]_- + \frac{F_-(z)}{\Phi_{ri}^-(z)} \end{aligned} \tag{18}$$

Finally, from the first relation of Eqs. (18), the physically realizable optimum pulse-transfer function is

$$W_m(z) = \frac{\left[\dfrac{\Phi_{ri}(z)}{\Phi_{rr}^-(z)}\right]_+}{\Phi_{rr}^+(z)} \tag{19}$$

Additionally it must be mentioned that the separating procedure may be performed by the following calculus:

$$\left[\frac{\Phi_{ri}(z)}{\Phi_{rr}^-(z)}\right]_+ = \sum_{n=0}^{\infty} \left(\frac{1}{2\pi j} \oint_{\Gamma_0} \frac{\Phi_{ri}(\zeta)}{\Phi_{rr}^-(\zeta)}\, \zeta^{n-1}\, d\zeta\right) z^{-n} \tag{20}$$

that is, by inverse bilateral (two-sided) z transform, taking for the integration path the unit circle, and thereafter by the ordinary unilateral (one-sided) z transform, performing the summation only from $n = 0$ to ∞ (and not from $n = -\infty$ to ∞).

It must be once more emphasized that the term, figuring on the right-hand side of Eq. (20) is an unilateral z-transform belonging to a certain positive-time function. This is the meaning of the symbolism $[\ ]_+$. Thus it is a false imagination to assume, for example, in case of simple poles that this symbolism means the sum of the ordinary partial fractions, hence

$$\left[\frac{\Phi_{ri}(z)}{\Phi_{rr}^{-}(z)}\right]_+ \neq \sum_{\mu} \frac{A_\mu}{z - p_\mu^*}$$

where p_μ^* denotes the simple poles inside the unit circle Γ_0 of the z-plane. On the contrary, the term in question can be obtained as

$$\left[\frac{\Phi_{ri}(z)}{\Phi_{rr}^{-}(z)}\right]_+ = \sum_{\mu} \frac{A_\mu z}{z - p_\mu^*} = \sum_{\mu} \frac{A_\mu}{1 - p_\mu^* z^{-1}}$$

where

$$A_\mu = \lim_{z \to p_\mu^*} (z - p_\mu^*) \frac{\Phi_{ri}(z)}{z\Phi_{rr}^{-}(z)}$$

Hence, in case of simple poles p_μ^*

$$\left[\frac{\Phi_{ri}(z)}{\Phi_{rr}^{-}(z)}\right]_+ = \sum_{\mu} \frac{z}{z - p_\mu^*} \lim_{z \to p_\mu^*} (z - p_\mu^*) \frac{\Phi_{ri}(z)}{z\Phi_{rr}^{-}(z)} \tag{20'}$$

This remark is made because sometimes on can see false formulae, for example, instead of the true expression (19) the worse one

$$W_{m?}(z) = \frac{z\left[\dfrac{\Phi_{ri}(z)}{z\Phi_{rr}^{-}(z)}\right]_+}{\Phi_{rr}^{+}(z)} \tag{19?}$$

This mistake originates from the incorrect application of the symbolism $[\ ]_+$.

2. Semi-free configuration

Let us study the problem of the semi-free configuration but again for strictly digital processes only. This case is illustrated in *Fig. 2*. Everything is the same as in Fig. 1, the sole difference being that the link between the input and output now contains the cascade connection of the fixed elements with weighting function $w_f(t)$ and the compensating elements with weighting function $w_c(t)$. It is worthwhile to mention that between the two parts a sampler is figuring, thus, the corresponding weighting functions can be replaced by the pulse-transfer functions $W_f(z)$ and $W_c(z)$, respectively.

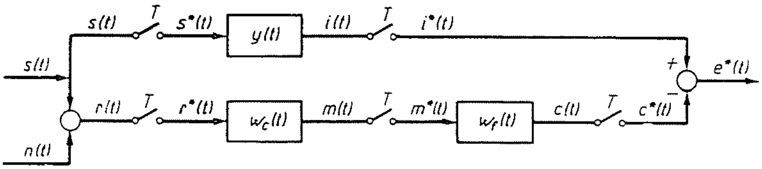

Fig. 2

Taking the index-changing rule into consideration, the power spectra can evidently be now expressed as follows:

$$\begin{aligned} &W_f(z^{-1})\,W_c(z^{-1})\,\Phi_{rr}(z)\,W_c(z)\,W_f(z) = \Phi_{cc}(z) \\ &W_f(z^{-1})\,W_c(z^{-1})\,\Phi_{ri}(z) = \Phi_{ci}(z) \\ &\Phi_{ir}(z)\,W_c(z)\,W_f(z) = \Phi_{ic}(z) \end{aligned} \tag{21}$$

Introducing the auxiliary power spectra $\Phi_{ff}(z)$, $\Phi_{if}(z)$, $\Phi_{fi}(z)$ by the following relations

$$\begin{aligned} W_f(z^{-1})\,\Phi_{rr}(z)\,W_f(z) &= \Phi_{ff}(z) \\ W_f(z^{-1})\,\Phi_{ri}(z) &= \Phi_{fi}(z) \\ \Phi_{ir}(z)\,W_f(z) &= \Phi_{if}(z) \end{aligned} \tag{22}$$

the power spectrum of the error can be expressed as

$$\begin{aligned} \Phi_{ee}(z) = \Phi_{ii}(z) - W_c(z^{-1})\,\Phi_{fi}(z) - \Phi_{if}(z)\,W_c(z) + \\ + W_c(z^{-1})\,\Phi_{ff}(z)\,W_c(z) \end{aligned} \tag{23}$$

Formally, this is the same expression as Eq. (9) but some other functions figure here. By the proper choice of an auxiliary pulse-transfer function

$$G_c(z) = \frac{\Phi_{fi}(z)}{\Phi_{ff}(z)} \tag{24}$$

the power spectrum of the error can be written as

$$\begin{aligned} \Phi_{ee}(z) = \Phi_{ii}(z) - G_c(z^{-1})\,\Phi_{ff}(z)\,G_c(z) + \\ + \left[G_c(z^{-1}) - W_c(z^{-1})\right]\Phi_{ff}(z)\left[G_c(z) - W_c(z)\right] \end{aligned} \tag{25}$$

According to the same line of reasoning as in the previous case of completely-free configuration, the physically unrealizable optimum pulse-transfer function of the cascade controller is:

$$W_{co}(z) = \frac{\Phi_{fi}(z)}{\Phi_{ff}(z)} \tag{26}$$

or taking into consideration Eqs. (22):

$$W_{co}(z) = \frac{W_f(z^{-1})\,\Phi_{ri}(z)}{W_f(z^{-1})\,\Phi_{rr}(z)\,W_f(z)} = \frac{\Phi_{ri}(z)}{\Phi_{rr}(z)\,W_f(z)} \tag{27}$$

Similarly, the physically realizable optimum pulse-transfer function of the cascade controller is

$$W_{cm}(z) = \frac{\left[\dfrac{\Phi_{fi}(z)}{\Phi_{ff}^{-}(z)}\right]_{+}}{\Phi_{ff}^{+}(z)} \tag{28}$$

or substituting Eqs. (22)

$$W_{cm}(z) = \frac{\left[\dfrac{W_f(z^{-1})\,\Phi_{ri}(z)}{[W_f(z^{-1})\,W_f(z)]^{-}\,\Phi_{rr}^{-}(z)}\right]_{+}}{[W_f(z^{-1})\,W_f(z)]^{+}\,\Phi_{rr}^{+}(z)} \tag{29}$$

If the pulse-transfer function of the fixed elements $W_f(z)$ has no zeros and poles outside the unit circle, then

$$[W_f(z^{-1})\,W_f(z)]^{-} = W_f(z^{-1}) \tag{30}$$

$$[W_f(z^{-1})\,W_f(z)]^{+} = W_f(z)$$

and the physically realizable optimum pulse-transfer function of the cascade controller can be more simply expressed

$$W_{cm}(z) = \frac{\left[\dfrac{\Phi_{ri}(z)}{\Phi_{rr}^{-}(z)}\right]_{+}}{W_f(z)\,\Phi_{rr}^{+}(z)} \tag{31}$$

or taking expression (19) also into consideration

$$W_{cm}(z) = \frac{W_m(z)}{W_f(z)} \tag{32}$$

3. Semi-free configuration with constraints

If some signals are limited, as they are in practically all control systems, then constraints arise in the optimization procedure. For the sake of simplicity, only one constraint is assumed, but it is not difficult to generalize this case for many constraints. The problem can be depicted by Fig. 3. Here the actuating signal $m^*(t)$ is indirectly constrained through a general weighting function $w_k(t)$. If, for example, $w_k(t) = \delta(t)$ then a direct constraint is prescribed on $m^*(t)$, on the other hand, if $w_k(t) = w_f(t)$ then the output $c^*(t)$ is directly constrained. Generally, let us assume the constraint as being expressed in the following form:

$$\overline{l^2(nT)} = \varphi_{ll}(0T) = \frac{1}{2\pi j} \oint_{\Gamma_0} \Phi_{ll}(z) z^{-1} dz \leqq \sigma_m^2 \tag{33}$$

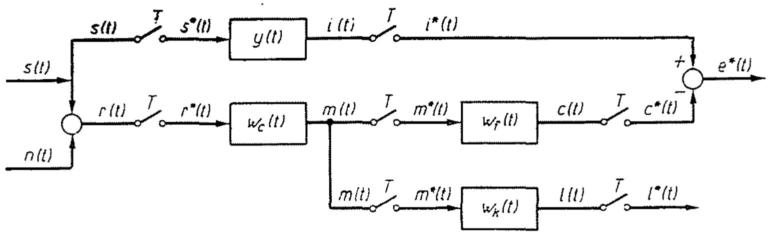

Fig. 3

This condition is called the unequality of constraint. The problem of the semi-free configuration with constraint can be solved by the Lagrangean conditional extremum technique. The function to be minimized now is

$$\overline{x^{*2}(t, \lambda)} = \overline{e^{*2}(t)} + \lambda \overline{l^{*2}(t)} \tag{34.a}$$

or in another form

$$\overline{x^2(nT, \lambda)} = \overline{e^2(nT)} + \lambda \overline{l^2(nT)} \tag{34.b}$$

This can be expressed through the first term of a proper correlation sequence or through the power spectra as follows:

$$\begin{aligned} \overline{x^2(nT, \lambda)} &= \varphi_{xx}(0T, \lambda) = \frac{1}{2\pi j} \oint_{\Gamma_0} \Phi_{xx}(z, \lambda) z^{-1} dz = \\ &= \frac{1}{2\pi j} \oint_{\Gamma_0} [\Phi_{ee}(z) + \lambda \Phi_{ll}(z)] z^{-1} dz \end{aligned} \tag{35}$$

The first power spectrum $\Phi_{ee}(z)$ figuring here is again given by Eq. (23) while the second can be expressed as follows:

$$\Phi_{ll}(z) = W_k(z^{-1}) W_c(z^{-1}) \Phi_{rr}(z) W_c(z) W_k(z) \tag{36}$$

Let us now define the following auxiliary power spectrum

$$\Phi_{aa}(z, \lambda) = [W_f(z^{-1}) W_f(z) + \lambda W_k(z^{-1}) W_k(z)] \Phi_{rr}(z) \tag{37}$$

and the following auxiliary pulse-transfer function:

$$G_a(z, \lambda) = \frac{\Phi_{fi}(z)}{\Phi_{aa}(z, \lambda)} \tag{38}$$

both being also functions of the undetermined parameter λ. Then the power spectrum $\Phi_{xx}(z, \lambda)$ can be expressed in the form:

$$\begin{aligned} \Phi_{xx}(z, \lambda) &= \Phi_{ee}(z) + \lambda \Phi_{ll}(z) = \\ &= \Phi_{ii}(z) - G_a(z^{-1})\, \Phi_{aa}(z, \lambda)\, G_a(z) + \\ &+ [G_a(z^{-1}) - W_c(z^{-1})]\, \Phi_{aa}(z, \lambda)\, [G_a(z) - W_c(z)] \end{aligned} \tag{39}$$

Following the same line of reasoning as previously, the physically unrealizable optimum pulse-transfer function now is

$$W_{co}(z, \lambda) = \frac{\Phi_{fi}(z)}{\Phi_{aa}(z, \lambda)} \tag{40}$$

or in an expanded form:

$$W_{co}(z, \lambda) = \frac{W_f(z^{-1})\, \Phi_{ri}(z)}{[W_f(^{-1})\, W_f(z) + \lambda W_k(z^{-1})\, W_k(z)]\, \Phi_{rr}(z)} \tag{41}$$

Again employing the spectrum factorization procedure, the physically realizable optimum pulse-transfer function of the cascade controller is

$$W_{cm}(z, \lambda) = \frac{\left[\dfrac{\Phi_{fi}(z)}{\Phi^{-}_{aa}(z, \lambda)} \right]_{+}}{\Phi^{+}_{aa}(z, \lambda)} \tag{42}$$

or in an expanded form:

$$W_{cm}(z, \lambda) = \frac{\left[\dfrac{W_f(z^{-1})\, \Phi_{ri}(z)}{[W_f(z^{-1})\, W_f(z) + \lambda\, W_k(z^{-1})\, W_k(z)]^{-}\, \Phi^{-}_{rr}(z)} \right]_{+}}{[W_f(z^{-1})\, W_f(z) + \lambda\, W_k(z^{-1})\, W_k(z)]^{+}\, \Phi^{+}_{rr}(z)} \tag{43}$$

In case of the semi-free configuration with constraint the undetermined Lagrangean multiplier λ does figure in the expression of the physically realizable optimum pulse-transfer function. For obtaining the final form, the parameter λ must be eliminated with the aid of the unequality of constraint: Eq. (33). First in the expression of $\Phi_{ll}(z)$ i.e. in Eq. (36) the pulse-transfer function $W_c(z)$ is replaced by the physically realizable optimum pulse-transfer function $W_{cm}(z, \lambda)$ and the mean-square value $\overline{l^2(nT)}$ of the sampled-data output signal $l^*(t)$ of the constraint is determined by the residue theorem, then the multiplier λ can be so adjusted that the equation of constraint Eq. (33) should be fulfilled. Substituting this value of λ into Eq. (43) we have the desired physically realizable optimum pulse-transfer function for the case of the semi-free configuration with constraint.

It is worthwhile to mention the following special cases: If $W_k(z) = 0$ then Eq. (43) is reduced to Eq. (29). Further, if $W_k(z) = 0$ and $W_f(z) = 1$ then Eq. (43) or Eq. (29) is reduced to Eq. (19). Thus, the semi-free configuration with constraint is the more general case.

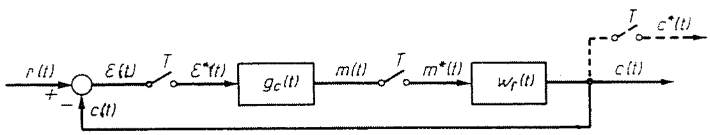

Fig. 4a

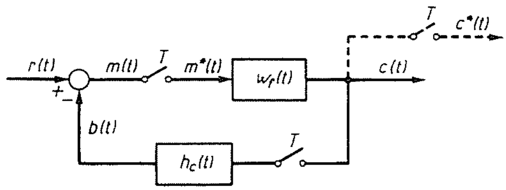

Fig. 4b

4. A complementary remark

Finally, if $W_{cm}(z)$ is known, then the optimum transfer function $G_{cm}(z)$ of the cascade controller in the closed loop *(Fig. 4a)* can be determined on the basis of

$$W_m(z) = W_{cm}(z)\, W_f(z) = \frac{G_{cm}(z)\, W_f(z)}{1 + G_{cm}(z)\, W_f(z)} \tag{44}$$

as

$$G_{cm}(z) = \frac{W_{cm}(z)}{1 - W_{cm}(z)\, W_f(z)} \tag{45}$$

On the other hand, the optimum transfer function $H_{cm}(z)$ of the feed-back controller in the closed loop *(Fig. 4b)* can be determined on the basis of

$$W_m(z) = W_{cm}(z)\, W_f(z) = \frac{W_f(z)}{1 + H_{cm}(z)\, W_f(z)} \tag{46}$$

as

$$H_{cm}(z) = \frac{1 - W_{cm}(z)}{W_{cm}(z)\, W_f(z)} = \frac{1}{W_{cm}(z)\, W_f(z)} - \frac{1}{W_f(z)} \tag{47}$$

Appendix

1. The bilateral (two-sided) z transform

First of all it is necessary to introduce the two-sided (bilateral) z transform. Let it be a sampled-data function $f^*(t)$ which is, in general, not zero for both positive and negative time:

$$f^*(t) = \sum_{n=-\infty}^{\infty} f(nT)\,\delta(t-nT) = \\ = \sum_{n=-\infty}^{-1} f(nT)\,\delta(t-nT) + \sum_{n=0}^{\infty} f(nT)\,\delta(t-nT) \tag{A1}$$

Thus, $f^*(t)$ may be decomposed into two components:

$$f^*(t) = f_-^*(t) + f_+^*(t) \tag{A2}$$

where $f_-(t) \equiv 0$ if $0 \leqq t$ and $f_+(t) \equiv 0$ if $t < 0$. Taking the z transforms:

$$F(z) = \sum_{n=-\infty}^{\infty} f(nT)\,z^{-n} = \sum_{n=-\infty}^{-1} f_-(nT)\,z^{-n} + \sum_{n=0}^{\infty} f_+(nT)\,z^{-n} \tag{A3}$$

Thus, $F(z)$ can be written as

$$F(z) = F_-(z) + F_+(z) \tag{A4}$$

This is an expression for the two-sided z transform. If $f_-(t) \equiv 0$ and $F_-(z) \equiv 0$ then the well-known unilateral z transform is obtained.

If the bilateral z transform is needed in a closed form, then first of all $f^*(t)$ must be expressed according to Eq. (A1) in the following form:

$$f^*(t) = f(t)\,i^*(t) = f(t)\,i_-^*(t) + f(t)\,i_+^*(t) = f_-^*(t) + f_+^*(t) \tag{A5}$$

where

$$i_-^*(t) = \sum_{n=\infty}^{-1} \delta(t-nT) \tag{A6}$$

and

$$i_+^*(t) = \sum_{n=0}^{\infty} \delta(t-nT) \tag{A7}$$

Employing the complex convolution theorem for the two-sided Laplace transform the following discrete Laplace transform is first obtained:

$$F_{+}^{*}(s) = \mathcal{T}[f_{+}^{*}(t)] = \mathcal{T}[f_{+}(t)\, i_{+}^{*}(t)] =$$

$$= \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F_{+}(p)\, I_{+}^{*}(s-p)\, dp =$$

$$= \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F_{+}(p) \sum_{n=0}^{\infty} e^{-(s-p)Tn}\, dp = \tag{A8}$$

$$= \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F_{+}(p) \frac{1}{1-e^{-(s-p)T}}\, dp$$

where

$$F_{+}(p) = \mathcal{T}[f_{+}(t)] = \mathcal{L}[f_{+}(t)] \tag{A9}$$

is the ordinary Laplace transform of the continuous positive-time function $f_{+}(t)$.

The choice of the value c must be performed in such a way that on the Re $p = c$ axis the unequality

$$\left| e^{-(s-p)T} \right| < 1 \tag{A 10}$$

must be fulfilled and there $F_{+}(p)$ must be regular. Let us assume that the poles of $F_{+}(p)$ lie on the left-hand of the imaginary axis at some distance because $f_{+}(t)$ is a positive-time function which decays *(Fig. A 1)*.

On the one hand, c must be so chosen that the integration path be to the right of the regularity limit of $F_{+}(p)$ and, on the other hand, to the left of the poles of $I_{+}^{*}(s-p)$. If Re $s = 0$ as in the case of $s = j\omega$, then $c < 0$.

Similarly

$$F_{-}^{*}(s) = \mathcal{T}[f_{-}^{*}(t)] = \mathcal{T}[f_{-}(t)\, i_{-}^{*}(t)] =$$

$$= \frac{1}{2\pi j} \int_{c'-j\infty}^{c'+j\infty} F_{-}(p)\, I_{-}^{*}(s-p)\, dp =$$

$$= \frac{1}{2\pi j} \int_{c'-j\infty}^{c'+j\infty} F_{-}(p) \sum_{n=-\infty}^{-1} e^{-(s-p)Tn}\, dp =$$

$$= \frac{1}{2\pi j} \int_{c'-j\infty}^{c'+j\infty} F_{-}(p) \sum_{n'=1}^{\infty} e^{(s-p)Tn'}\, dp =$$

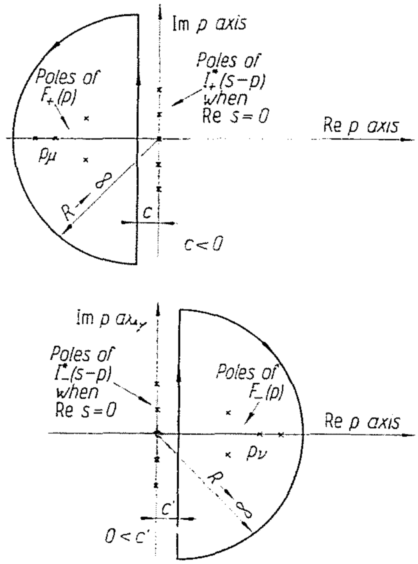

Fig. A1

$$= \frac{1}{2\pi j} \int_{c'-j\infty}^{c'+j\infty} F_-(p) \frac{e^{(s-p)T}}{1 - e^{(s-p)T}} dp =$$

$$= - \frac{1}{2\pi j} \int_{c'-j\infty}^{c'-j\infty} F_-(p) \frac{1}{1 - e^{-(s-p)T}} dp \tag{A11}$$

where

$$F_-(p) = \mathscr{T}[f_-(t)] \tag{A 12}$$

The value c' must be chosen in such a manner that on the $\operatorname{Re} p = c'$ axis the unequality

$$|e^{(s-p)T}| < 1 \tag{A13}$$

be fulfilled and there $F_-(p)$ must be regular. The poles of $F_-(p)$ lie to the right of the imaginary axis at some distance (Fig. A 1). Thus, c' must be so chosen that the integration path be to the left of the regularity limit of $F_-(p)$, and at the same time, to the right of the poles of $I^*_-(s-p)$. If $\operatorname{Re} s = 0$ because of $s = j\omega$, then $0 < c'$.

In the first case a closed integration path is made to the left with the aid of a semi-circle whose radius tends to infinity. In the second case the semi-circle of infinite radius is chosen in the right-half plane. Thus, in both cases the residue theorem can be applied (in the first case with a positive sign, in the second case, on the contrary, with a negative sign) giving:

$$F^*(s) = F_+^*(s) + F_-^*(s) =$$
$$= \sum_\mu \operatorname*{Res}_{p=p_\mu} \frac{F_+(p)}{1 - e^{-(s-p)T}} + \sum_\nu \operatorname*{Res}_{p=p_\nu} \frac{F_-(p)}{1 - e^{-(s-p)T}} \tag{A14}$$

where p_μ denotes the left-half-plane poles of $F_+(p)$ while p_ν denotes the right-half-plane poles of $F_-(p)$. Finally, substituting $e^{sT} = z$, the two-sided z transform is obtained:

$$F(z) = \sum_\mu \operatorname*{Res}_{p=p_\mu} \frac{F_+(p)}{1 - z^{-1} e^{pT}} + \sum_\nu \operatorname*{Res}_{p=p_\nu} \frac{F_-(p)}{1 - z^{-1} e^{pT}} \tag{A15}$$

If there are no many-fold poles, then the following formula is valid

$$F(z) = \sum_i \lim_{p \to p_i} (p - p_i) \frac{F(p)}{1 - z^{-1} e^{pT}} \tag{A16}$$

where the summation must be extended both on the right-half-plane poles as well as on the left-half-plane poles of $F(p)$.

2. *Correlation sequence and pulse-spectral density*

The correlation sequences may be defined in a similar manner to the definition of the well-known correlation functions. The autocorrelation sequence of a sampled-data signal $u^*(t)$ or a sequence $u(nT)$ is defined as

$$\varphi_{uu}(kT) = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} u(nT)\, u(nT + kT) \tag{A17}$$

The cross-correlation sequence between pulsed-data signals $u^*(t)$ and $v^*(t)$ or the corresponding sequences $u(nT)$ and $v(nT)$ is defined as

$$\varphi_{uv}(kT) = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} u(nT)\, v(nT + kT) \tag{A18}$$

The power-density spectrum or the power-spectral density is defined as the Fourier transform of the autocorrelation function of a continuous-data signal and the cross-power-density spectrum or the cross-power-spectral density is defined as the Fourier transform of the cross-correlation function between two continuous-data signals. Similarly, the two-sided z transform of the autocorrelation sequence of a pulsed-data signal is defined as the pulse-power-spectral density or briefly as the power spectrum. In a similar fashion the bilateral z transform of the cross-correlation sequence between two sampled-data signals is defined as the pulse-cross-power-spectral density or briefly as the cross-power spectrum. Thus, the power spectrum and the autocorrelation sequence are related by

$$\Phi_{uu}(z) = \sum_{k=-\infty}^{\infty} \varphi_{uu}(kT)\, z^{-k} \tag{A19}$$

$$\varphi_{uu}(kT) = \frac{1}{2\pi j} \oint_{\Gamma_0} \Phi_{uu}(z)\, z^{k-1}\, dz \tag{A20}$$

while the cross-power spectrum and the cross-correlation sequence are related by

$$\Phi_{uv}(z) = \sum_{k=-\infty}^{\infty} \varphi_{uv}(kT)\, z^{-k} \tag{A21}$$

$$\varphi_{uv}(kT) = \frac{1}{2\pi j} \oint_{\Gamma_0} \Phi_{uv}(z)\, z^{k-1}\, dz \tag{A22}$$

where $z = e^{j\omega T}$ and the contour of integration Γ_0 is the unit circle in the z plane.

It must be mentioned that the two-sided z transform is used in defining pulse-power-spectral densities, because the correlation sequences exist over all values of k from $-\infty$ to ∞.

Further, it is to be seen that when $k = 0$, the value of the autocorrelation sequence is equal to the mean-square value of the sampled-data signal $u^*(t)$ or the sequence $u(nT)$:

$$\varphi_{uu}(0T) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} u^2(nT) = \overline{u^2(nT)} \tag{A23}$$

Furthermore, letting $k = 0$ in Eq. (A20) relating the autocorrelation sequence and the power spectrum:

$$\varphi_{uu}(0T) = \frac{1}{2\pi j} \oint_{\Gamma_0} \Phi_{uu}(z)\, z^{-1}\, dz \tag{A24}$$

or taking into consideration Eq. (A23):

$$\overline{u^2(nT)} = \frac{1}{2\pi j} \oint_{\Gamma_0} \Phi_{uu}(z)\, z^{-1}\, dz \tag{A25}$$

3. *Properties of correlation sequence and pulse-power-spectral density*

Here some properties are summarized but without proof.

1° The autocorrelation sequence is an even function, thus

$$\varphi_{uu}(kT) = \varphi_{uu}(-kT) \tag{A26}$$

2° The cross-correlation sequences are not even functions but the following relation is valid

$$\varphi_{uv}(kT) = \varphi_{vu}(-kT) \tag{A27}$$

3° The pulse-power-spectral density (or power spectrum) has the following property:

$$\Phi_{uu}(z) = \Phi_{uu}(z^{-1}) \tag{A28}$$

4° The cross-power spectra are characterized by

$$\Phi_{uv}(z) = \Phi_{vu}(z^{-1}) \tag{A29}$$

5° If the response of the sampled-data control system with weighting function $w(t)$ and pulse transfer function $W(z)$ to an input $r^*(t)$ is $c^*(t)$, then the response of this system to an input $\varphi_{rr}(kT)$ is $\varphi_{rc}(kT)$ and the response of this system to an input $\varphi_{cr}(kT)$ is $\varphi_{cc}(kT)$. Thus

$$\sum_{n=-\infty}^{\infty} w(nT)\,\varphi_{rr}(kT - nT) = \varphi_{rc}(kT) \tag{A30}$$

$$\sum_{n=-\infty}^{\infty} w(nT)\,\varphi_{cr}(kT - nT) = \varphi_{cc}(kT) \tag{A31}$$

and so on.

6° The corresponding relations for the power spectra are

$$\Phi_{rr}(z)\ W(z) = \Phi_{rc}(z) \tag{A32}$$

$$\Phi_{cr}(z)\ W(z) = \Phi_{cc}(z) \tag{A33}$$

In a similar manner

$$W(z^{-1})\ \Phi_{rr}(z) = \Phi_{cr}(z) \tag{A34}$$

$$W(z^{-1})\ \Phi_{rc}(z) = \Phi_{cc}(z) \tag{A35}$$

Finally:

$$W(z^{-1})\ \Phi_{rr}(z)\ W(z) = \Phi_{cc}(z) \tag{A36}$$

These are the so-called index-changing rules.

Illustrative examples

For the sake of illustration two simple examples are given both concerned with the completely-free configuration only.

Example 1. For the sake of simplicity let us consider a control system without noise. Furthermore, let the autocorrelation function of the continuous signal be

$$\varphi_{rr}(\tau) = \varphi_{ss}(\tau) = B^2 e^{-2\nu|\tau|}$$

Then applying the two-sided Laplace or Fourier transform, the power-density spectrum (or power-spectral density) of the continuous signal is

$$\Phi_{rr}(s) = \Phi_{ss}(s) = \frac{4\nu B^2}{4\nu^2 - s^2}$$

Let us assume that the idealized or desired output signal is the integral function of the input signal. Thus, employing the index-changing rule for continuous signals

$$\Phi_{ri}(s) = \Phi_{si}(s) = \frac{1}{s}\,\Phi_{ss}(s) = \frac{4\nu B^2}{s(4\nu^2 - s^2)}$$

With due respect of the formula for obtaining the two-sided z transform Eq. (A 16) or Eq. (A 15), the power spectrum of the input is:

$$\Phi_{rr}(z) = \frac{B^2}{1 - z^{-1} e^{-2\nu T}} - \frac{B^2}{1 - z^{-1} e^{2\nu T}}$$

or in another form

$$\Phi_{rr}(z) = \frac{B^2 (1 - e^{-4\nu T})}{(1 - z^{-1} e^{-2\nu T})(1 - z\, e^{-2\nu T})}$$

Following the same method, the cross-power spectrum of the input and the idealized output is

$$\Phi_{ri}(z) = \frac{B^2}{2\nu}\left[\frac{2}{1 - z^{-1}} - \frac{1}{1 - z^{-1} e^{-2\nu T}} - \frac{1}{1 - z^{-1} e^{2\nu T}}\right]$$

or after some algebraic manipulations

$$\Phi_{ri}(z) = \frac{B^2}{2\nu}\;\frac{(1 - e^{-2\nu T})^2 (1 + z^{-1})}{(1 - z^{-1})(1 - z^{-1} e^{-2\nu T})(1 - z\, e^{-2\nu T})}$$

Performing the spectrum factorization

$$\Phi_{rr}^{-}(z) = \frac{B(1 - e^{-4\nu T})^{1/2}}{1 - z\, e^{-2\nu T}}$$

and

$$\Phi_{rr}^{+}(z) = \frac{B(1 - e^{-4\nu T})^{1/2}}{1 - z^{-1}\, e^{-2\nu T}}$$

Therefore

$$\frac{\Phi_{ri}(z)}{\Phi_{rr}^{-}(z)} = \frac{B}{2\nu} \frac{(1 - e^{-2\nu T})^2}{(1 - e^{-4\nu T})^{1/2}} \frac{1 + z^{-1}}{(1 - z^{-1})(1 - z^{-1} e^{-2\nu T})}$$

Specially in this case

$$\left[\frac{\Phi_{ri}(z)}{\Phi_{rr}^{-}(z)}\right]_{+} = \frac{\Phi_{ri}(z)}{\Phi_{rr}^{-}(z)}$$

Thus, the physically realizable optimum pulse-transfer function is:

$$W_m(z) = \frac{1}{2\nu} \frac{1 - e^{-2\nu T}}{1 + e^{-2\nu T}} \frac{1 + z^{-1}}{1 - z^{-1}}$$

It is interesting to note that in the case of $\nu \to 0$

$$\lim_{\nu \to 0} W_m(z) = \frac{T}{2} \frac{1 + z^{-1}}{1 - z^{-1}}$$

which is nothing else but the pulse-transfer function of the digital integrator working after the trapezoidal rule.

Example 2. Now let us assume that the autocorrelation function of the continuous useful signal component of the input is:

$$\varphi_{ss}(\tau) = B^2 e^{-2\nu|\tau|}$$

while the autocorrelation function of the corrumping white noise component is

$$\varphi_{nn}(\tau) = N^2 \delta(\tau)$$

Furthermore, it is assumed that the signal and noise component are uncorrelated, thus

$$\varphi_{ns}(\tau) = \varphi_{sn}(\tau) = 0$$

The desired idealized output is the same as the signal component of the input but advanced by the time kT, where k is an integer number and T is the sampling period. Thus, the physically unrealizable idealized transfer function is

$$Y_i(s) = e^{kTs}$$

Employing the Fourier transform the following power-density spectra are obtained

$$\Phi_{ss}(s) = \frac{4\nu B^2}{4\nu^2 - s^2}$$

$$\Phi_{nn}(s) = N^2$$

$$\Phi_{sn}(s) = 0$$

where $s = j\omega$.

On the other hand, the cross-power-density spectrum of the input and idealized output can be given as

$$\Phi_{ri}(s) = \Phi_{ss}(s)\, Y_i(s) = \frac{4\nu B^2}{4\nu^2 - s^2} e^{kTs}$$

Applying the bilateral z transformation

$$\Phi_{ss}(z) = \frac{B^2(1 - e^{-4\nu T})}{(1 - z^{-1} e^{-2\nu T})(1 - z\, e^{-2\nu T})}$$

$$\Phi_{nn}(z) = N^2$$

Thus

$$\Phi_{rr}(z) = \frac{B^2(1 - e^{-4\nu T}) + N^2\left[1 - (z^{-1} + z)\, e^{-2\nu T} + e^{-4\nu T}\right]}{(1 - z^{-1} e^{-2\nu T})(1 - z e^{-2\nu T})}$$

After introducing the following notations by

$$\left.\begin{matrix} P \\ Q \end{matrix}\right\} = \frac{1}{2}\left[\sqrt{N^2 + B^2 + 2N^2 e^{-2\nu T} + (N^2 - B^2)\, e^{-4\nu T}} \pm \right.$$

$$\left. \pm \sqrt{N^2 + B^2 - 2N^2 e^{-2\nu T} + (N^2 - B^2)\, e^{-4\nu T}}\right]$$

$$(P > Q)$$

the power spectrum can be rewritten as:

$$\Phi_{rr}(z) = \frac{(P - z^{-1} Q)(P - zQ)}{(1 + z^{-1} e^{-2\nu T})(1 - z\, e^{-2\nu T})}$$

Now, the spectrum factorization can easily be performed

$$\Phi_{rr}^{-}(z) = \frac{P - z\,Q}{1 - z\,e^{-2\nu T}}$$

$$\Phi_{rr}(z) = \frac{P - z^{-1}\,Q}{1 - z^{-1}\,e^{-2\nu T}}$$

Again applying the bilateral z transformation

$$\Phi_{ri}(z) = \frac{B^2\,(1 - e^{-4\nu T})\,z^k}{(1 - z^{-1}\,e^{-2\nu T})(1 - z\,e^{-2\nu T})}.$$

Thus,

$$\frac{\Phi_{ri}(z)}{\Phi_{rr}^{-}(z)} = \frac{B^2\,(1 - e^{-4\nu T})\,z^k}{(1 - z^{-1}\,e^{-2\nu T})\,(P - z\,Q)}$$

First, let us examine the case when $k = 0$. Then, after determining the partial fractions

$$\frac{\Phi_{ri}(z)}{\Phi_{rr}^{-}(z)} = \frac{B^2\,(1 - e^{-4\nu T})}{P - Q\,e^{-2\nu T}}\left[\frac{1}{1 - z^{-1}\,e^{-2\nu T}} + \frac{z\,Q}{P - z\,Q}\right]$$

the following expression is obtained:

$$\left[\frac{\Phi_{ri}(z)}{\Phi_{rr}^{-}(z)}\right]_{+} = \frac{B^2\,(1 - e^{-4\nu T})}{P - Q\,e^{-2\nu T}}\;\frac{1}{1 - z^{-1}\,e^{-2\nu T}}$$

Finally, according to Eq. (19) the physically realizable optimum pulse-transfer function is

$$W_m(z) = \frac{B^2\,(1 - e^{-4\nu T})}{P - Q\,e^{-2\nu T}}\;\frac{1}{P - z^{-1}\,Q}$$

Now, let us consider the case when the integer number $k > 0$. Assuming $|z^{-1}\,e^{-2\nu T}| < |z^{-1}| \leq 1$ the following relations are valid

$$\frac{z^k}{1 - z^{-1}\,e^{-2\nu T}} = z^k \sum_{n=0}^{\infty} z^{-n}\,e^{-2\nu T n} =$$

$$= \sum_{n=0}^{\infty} z^{(k-n)}\,e^{-2\nu T n} = \sum_{l=-k}^{\infty} z^{-l}\,e^{-2\nu T(l+k)}$$

Therefore separating the terms belonging to positive-time function:

$$\left[\frac{\Phi_{ri}(z)}{\Phi_{rr}^{-}(z)}\right]_{+} = \frac{B^2(1-e^{-4\nu T})}{P-Q^{-2\nu T}}\, e^{-2\nu Tk} \sum_{l=0}^{\infty} z^{-l} e^{-2\nu Tl} =$$

$$= \frac{B^2(1-e^{-4\nu T})}{P-Q\,e^{-2\nu T}} \; \frac{e^{-2\nu Tk}}{1-z^{-1}\,e^{-2\nu T}}$$

Finally, according to Eq. (19) the physically realizable optimum pulse-transfer function is

$$W_m(z) = \frac{B^2(1-e^{-4\nu T})}{P-Q\,e^{-2\nu T}} \; \frac{e^{-2\nu Tk}}{P-z^{-1}\,Q}$$

Summary

In this paper it is demonstrated how the so-called simplified derivation technique can be extended and applied to strictly digital stationary ergodic stochastic processes. Using the frequency domain technique, explicit solution formulae can be obtained in a relatively simple way. The physically realizable optimum pulse-transfer functions are determined not only for the completely-free configuration but also for the semi-free configuration and for the semi-free configuration with constraints. Two simple examples are also given for the sake of illustration.

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