OPTIMUM PULSE-TRANSFER FUNCTIONS FOR MULTIVARIABLE DIGITAL STOCHASTIC PROCESSES

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(Received July 11, 1965.)

In a previous paper [1] the so called simplified derivation technique [2, 3] was extended for strictly digital processes in case of single variable systems. Now, multivariable pulsed-data control systems will be treated. This paper is the generalization of some studies [4, 5, 6] which deal with multivariable continuous-data control systems. As usual, the inputs, outputs and manipulated variables are assumed to be stationary stochastic processes and the ergodic hypothesis is also adopted. As optimization criterion the least mean-square-sum of the error components is taken. The definitions and notions used here are the direct generalizations of that of the mentioned paper [1].

1. Optimum pulse-transfer-function matrix for the completely-free configuration

The notations in this paper will be the following: If both indices are variable this will mean a matrix, if one index is fixed or is missing and only the other index is variable we have a vector, finally, if both indices are fixed then a scalar quantity is represented. As concerns the vectors a variable first index will signify a column vector while a variable second index signifies a row vector (the other indices being fixed or missing).

A simple block-diagram representation of a multivariable pulsed-data system is depicted in Fig. 1. The pulsed-data inputs are represented by a row vector $r^*_{\cdot k}(t)$, where $k = 1, \ldots K$. The pulsed-data outputs are represented by a row vector $c^*_{\cdot l}(t)$, where $l = 1, \ldots L$. The weighting-function matrix of the multivariable system is $w_{kl}(t)$ $(k = 1, \ldots K;\ l = 1, \ldots L)$.

The number sequence of the pulsed-data output can be expressed by the following convolution sum:

$$c_{\cdot l}(nT) = \sum_{\nu=-\infty}^{\infty} r_{\cdot k}(nT - \nu T)\, w_{kl}(\nu T) = \sum_{\mu=-\infty}^{\infty} r_{\cdot k}(\mu T) w_{kl}(nT - \mu T) \qquad (1)$$

where n, ν, μ, are integers and the summation is performed on row vectors originating from the matrix multiplication of a $1 \times K$ row matrix and a $K \times L$ matrix.

Taking once more *Fig. 1* into consideration the problem of optimum syntesis of the pulsed-data system can be stated as follows. The input row vector $r^*_{.k}(t)$ has two components: the row vector of the useful signal components $s^*_{.k}(t)$ and the row vector of noise components $n^*_{.k}(t)$, $(k = 1, \dots K)$.

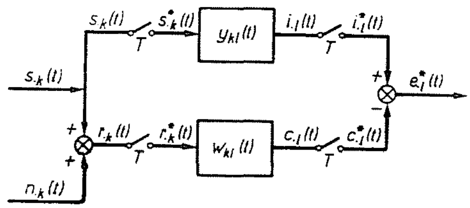

Fig. 1

The ideal or desired outputs are represented by $i^*_{.l}(t)$ $(l = 1, \dots L)$. In the way this row vectors or the number sequence $i_{.l}(nT)$ can be determined from the signal-component vector $s^*_{.k}(t)$, or more correctly, from the number sequence $s_{.k}(nT)$ by matrix multiplication with weighting-function matrix $y_{kl}(nT)$ and through one of the following convolution sums:

$$i_{.l}(nT) = \sum_{\nu=-\infty}^{\infty} s_{.k}(nT - \nu T)\, y_{kl}(\nu T) = \sum_{\mu=-\infty}^{\infty} s_{.k}(\mu T)\, y_{kl}(nT - \mu T). \qquad (2)$$

It is to be emphasized that the weighting-function matrix $y^*_{kl}(t)$, together with its elements, is generally not physically realizable the former being a fictitious one and merely serving for the origination of the idealized outputs.

The row vector of the pulsed-data error is nothing else but the difference of the row vectors of the ideal and real pulsed-data outputs:

$$e^*_{.l}(t) = i^*_{.l}(t) - c^*_{.l}(t). \qquad (3)$$

Taking the sum of the least-mean-square errors as a basis of optimization, the functional to be minimized can be expressed by the trace of a matrix or in other words by the sum of principal diagonal elements of a matrix, the latter being composed of the matrix multiplication of the column matrix

$e_{l'.}(nT)$ and the row matrix $e_{.l}(nT)$, and by an averaging process:

$$tr\,\overline{[e_{l'}.(nT)e._l(nT)]} = \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N} tr\,[e_{l'}.(nT)\,e._l(nT)] \tag{4}$$
$$= tr\,[\varphi_{e_{l'}e_l}(0T)] =$$
$$= \frac{1}{2\pi j}\oint_{\Gamma_0} tr\,[\Phi_{e_{l'}e_l}(z)]\,z^{-1}\,dz$$

where $e_{l'.}(nT)\,e_{.l}(nT)$ is an $L \times L$ matrix, while

$$\varphi_{e_{l'}e_l}(kT) = \overline{e_{l'}.(nT)e._l(nT+kT)} =$$
$$= \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N} e_{l'}.(nT)\,e._l(nT+kT) \tag{5}$$

is the $L \times L$ autocorrelation-sequence matrix of the errors. The power-spectrum matrix of the errors can be obtained from the correlation-sequence matrix by z transformation:

$$\Phi_{e_{l'}e_l}(z) = \mathfrak{Z}\,[\varphi_{e_{l'}e_l}(kT)] = \sum_{k=-\infty}^{\infty} \varphi_{e_{l'}e_l}(kT)\,z^{-k} \tag{6}$$

while the inverse relation is

$$\varphi_{e_{l'}e_l}(kT) = \mathfrak{Z}^{-1}\,[\Phi_{e_{l'}e_l}(z)] = \frac{1}{2\pi j}\oint_{\Gamma_0} \Phi_{e_{l'}e_l}(z)\,z^{k-1}dz. \tag{7}$$

Here and also in the following treatise $z = e^{sT} = e^{j\omega T}$ and Γ_0 is the unit circle of the z plane. Since evidently

$$\overline{e_{l'}.(nT)e._l(nT)} = \overline{[i_{l'}.(nT) - c_{l'}.(nT)]\,[i._l(nT+kT) - c._l(nT+kT)]}, \tag{8}$$

therefore, the autocorrelation-sequence matrix of the errors can be expressed as:

$$\varphi_{e_{l'}e_l}(kT) = \varphi_{i_{l'}i_l}(kT) - \varphi_{i_{l'}c_l}(kT) - \varphi_{c_{l'}i_l}(kT) + \varphi_{c_{l'}c_l}(kT) \tag{9}$$

and consequently the power-spectrum matrix of the error can be set up in the following form:

$$\Phi_{e_{l'}e_l}(z) = \Phi_{i_{l'}i_l}(z) - \Phi_{i_{l'}c_l}(z) - \Phi_{c_{l'}i_l}(z) + \Phi_{c_{l'}c_l}(z)\,. \tag{10}$$

Our task is to minimize the trace of the power-spectrum matrix $\Phi_{e_{l'}e_l}(z)$, as according to Eq. (4), this latter gives the sum of the mean-square errors through a contour integration around the unit circle. If the power-spectrum matrix $\Phi_{e_{l'} e_l}(z)$ itself can be minimized then evidently the sum of the mean-square errors is also minimized at the same time. Thus, the value of the integral in Eq. (4) becomes minimum if, and only if, the integrand is the least possible. As the index changing rule is also valid for pulsed-data systems [1] this can be generalized for multivariable systems, too. Thus, the power-spectrum matrix can be expressed as

$$\begin{aligned}\Phi_{e_{l'}e_l}(z) = \Phi_{i_{l'}i_l}(z) - \Phi_{i_{l'}r_k}(z)\, W_{kl}(z) - W_{l'k'}(z^{-1})\, \Phi_{r_{k'}i_l}(z) + \\ + W_{l'k'}(z^{-1})\, \Phi_{r_{k'}r_k}(z) W_{kl}(z) \\ (k, k' = 1, \ \dots \ K;\ l, l' = 1, \ \dots L)\end{aligned} \tag{11}$$

where $\Phi_{e_{l'}e_l}(z)$, $\Phi_{i_{l'}i_l}(z)$ are $L \times L$ matrices, $\Phi_{i_{l'}r_k}(z)$ is an $L \times K$ matrix, $\Phi_{r_{k'}i_l}(z)$ is a $K \times L$ matrix and $\Phi_{r_{k'}r_k}(z)$ is a $K \times K$ matrix. $W_{kl}(z)$ is a $K \times L$ matrix and $W_{l'k'}(z^{-1})$ is the adjoint, that is, the transposed complex conjugate matrix of the former ($e^{-j\omega T} = e^{-sT} = z^{-1}$).

Naturally, the pulsed-data-transfer-function matrix $W_{kl}(z)$ is nothing but the z transform of the weighting-function matrix $w_{kl}(t)$ (or more precisely of $w^*_{kl}(t)$).

Now, let us introduce a $K \times L$ auxiliary pulse-transfer-function matrix $G_{kl}(z)$ and its adjoint $L \times K$ matrix $G_{l'k'}(z^{-1})$ by the following relations

$$\begin{aligned}\Phi_{r_{k'}r_k}(z)\, G_{kl}(z) = \Phi_{r_{k'}i_l}(z) \\ G_{l'k'}(z^{-1})\, \Phi_{r_{k'}r_k}(z)\, \Phi_{i_{l'}r_{k'}}(z).\end{aligned} \tag{12}$$

The auxiliary matrix $G_{kl}(z)$ is, in general, physically unrealizable. As $\Phi_{r_{k'}r_k}(z)$ and $\Phi_{r_{k'}i_l}(z)$ can be considered as given matrices $G_{kl}(z)$ and $G_{l'k'}(z^{-1})$ are also given through Eq. (12). With the aid of the auxiliary matrices the power-spectrum matrix of errors takes the following form

$$\begin{aligned}\Phi_{e_{l'}e_l}(z) = \Phi_{i_{l'}i_l}(z) - G_{l'k'}(z^{-1})\, \Phi_{r_{k'}r_k}(z)\, W_{kl}(z) - \\ - W_{l'k'}(z^{-1})\, \Phi_{r_{k'}r_k}(z)\, G_{kl}(z) + \\ + W_{l'k'}(z^{-1})\, \Phi_{r_{k'}r_k}(z)\, W_{kl}(z)\end{aligned} \tag{13}$$

or after some algebraic manipulations:

$$\begin{aligned}\Phi_{e_{l'}e_l}(z) = \Phi_{i_{l'}i_l}(z) - G_{l'k'}(z^{-1})\, \Phi_{r_{k'}r_k}(z)\, G_{lk}(z) + \\ + [G_{l'k'}(z^{-1}) - W_{l'k'}(z^{-1})]\, \Phi_{r_{k'}r_k}(z)[G_{kl}(z) - W_{kl}(z)].\end{aligned} \tag{14}$$

It can be easily observed that the pulse-transfer-function matrix $W_{kl}(z)$ and its adjoint matrix $W_{l'k'}(z^{-1})$ are only contained in the last term of Eq. (14).

The power-spectrum matrix $\Phi_{e_l e_l}(z)$ is the least one, if the last term becomes a zero matrix. This is arrived at if, and only if,

$$G_{kl}(z) - W^{o}_{kl}(z) = 0$$

or

$$G_{l'k'}(z^{-1}) - W^{o}_{l'k'}(z^{-1}) = 0 \tag{15}$$

where $W^{o}_{kl}(z)$ is the optimum pulse-transfer-function matrix and $W^{o}_{l'k'}(z^{-1})$ is its adjoint.

With due respect to Eq. (15), Eqs. (12) give:

$$\begin{aligned} \Phi_{r_k, r_k}(z)\, W^{o}_{kl}(z) - \Phi_{r_k \cdot i_l}(z) = 0 \\ W^{o}_{l'k'}(z^{-1})\, \Phi_{r_k \cdot r_k}(z) - \Phi_{i_l, r_k}(z) = 0\,. \end{aligned} \tag{16}$$

Thus,

$$W^{o}_{kl}(z) = [\Phi_{r_k \cdot r_k}(z)]^{-1}\, \Phi_{\ k \cdot i_l}(z)\,. \tag{17}$$

Unfortunately, the optimum-pulse-transfer-function matrix is generally physically unrealizable, and therefore does not supply the solution of the optimization problem.

Let us separate the physically unrealizable transfer-function matrix into two parts:

$$W^{o}_{kl}(z) = W^{m}_{kl}(z) + W^{n}_{kl}(z) \tag{18}$$

where $W^{m}_{kl}(z)$ is the physically-realizable-component matrix with poles lying inside the unit circle of the z plane. $W^{n}_{kl}(z)$ is the remainder of the matrix. The latter is physically unrealizable, because the corresponding weighting-function matrix $w^{n}_{kl}(t)$ is a negative-time function, which becomes identically a zero matrix for the positive time. The poles of $W^{n}_{kl}(z)$ are lying outside the unit circle.

With due respect to the restriction of the physical realizability of the pulse-transfer-function matrix $W_{kl}(z)$, instead of the first relation of Eq. (16) at most the following equation is valid:

$$\Phi_{r_k, r_k}(z)\, W^{m}_{kl}(z) - \Phi_{r_k \cdot i_l}(z) = F_{k'l}(z) \tag{19}$$

where $F_{k'l}(z)$ is a still unknown matrix containig no poles inside the unit circle of the z plane. Evidently

$$\Phi_{r_k \cdot r_k}(z)\, W^{n}_{kl}(z) - \Phi_{r_k \cdot i_l}(z) = -F_{k'l}(z)\,. \tag{20}$$

It must be emphasized that after substituting the physically realizable matrix component in Eq. (14) the last term does not become a zero matrix, but with respect to the physical realizability, together with the power-spectrum matrix $\Phi_{e_l, e_l}(z)$, the last term becomes a minimum.

As is well known the continuous-data power-density-spectrum matrices, which are assumed to be real paraconjugate hermitian, can be factorized [8, 9]. This theorem may be generalized also for pulsed-data power-spectrum matrices. If $\Phi_{r_{k'}r_k}(z) = \Phi_{r_k r_{k'}}(z^{-1})$, then

$$\Phi^{-}_{r_{k'}r_{k''}}(z)\,\Phi^{+}_{r_{k''}r_k}(z) = \Phi_{r_{k'}r_k}(z) \tag{21}$$

where, on the one hand, the second matrix factor and its inverse matrix

$$\Phi^{+}_{r_{k''}r_k}(z)\,;\ [\Phi^{+}_{r_{k''}r_k}(z)]^{-1}$$

contain elements which have poles only inside the unit circle, while, on the other hand, the second matrix factor together with its inverse matrix:

$$\Phi^{-}_{r_{k'}r_{k''}}(z)\,;\ [\Phi^{-}_{r_{k'}r_{k''}}(z)]^{-1}$$

have only elements with poles outside the unit circle. By the way, the adjoint of the second matrix factor is just the same as the first matrix factor, and the adjoint of the first matrix factor is equal to the second matrix factor. This feature is also valid for the inverse matrices.

Taking Eq. (21) into consideration, we have from Eq. (19):

$$\Phi^{-}_{r_{k'}r_{k''}}(z)\,\Phi^{+}_{r_{k''}r_k}(z)\,W^{m}_{kl}(z) = \Phi_{r_{k'}i_l}(z) + F_{k'l}(z) \tag{22}$$

or in an other form:

$$\Phi^{+}_{r_{k''}r_k}(z)\,W^{m}_{kl}(z) = [\Phi^{-}_{r_{k'}r_{k''}}(z)]^{-1}\,\Phi_{r_{k'}i_l}(z) - [\Phi^{-}_{r_{k'}r_{k''}}(z)]^{-1}\,F_{k'l}(z). \tag{23}$$

Let us now decompose the matrices on both sides of Eq. (23) into physically realizable and unrealizable matrix components, belonging to positive-time and negative-time matrices, respectively:

$$\Phi^{+}_{r_{k''}r_k}(z)\,W^{m}_{kl}(z) = \{[\Phi^{-}_{r_{k'}r_{k''}}(z)]^{-1}\,\Phi_{r_{k'}i_l}(z)\}_{+} \tag{24}$$

$$0 = \{[\Phi^{-}_{r_{k'}r_{k''}}(z)]^{-1}\,\Phi_{r_{k'}i_l}(z)\}_{-} + [\Phi^{-}_{r_{k'}r_{k''}}(z)]^{-1}\,F_{k'l}(z) \tag{25}$$

where the lower index + (plus) denotes a matrix *component* with physically realizable elements, which have only poles inside the unit circle, while the subscript — (minus) denotes a matrix component with physically unrealizable elements having only poles outside the unit circle. The decomposition shown in Eqs. (24), (25) is evidently valid, because the second term on the right-hand side of Eq. (23) have poles only outside the unit circle and thus it can not supply any physically realizable matrix component. On the contrary, the term on the left-hand side of Eq. (23) have poles only inside the unit circle, and so it consists exclusively of a physically realizable matrix component.

From Eq. (24) the physically realizable optimum pulse-transfer-function matrix is obtained as a final solution of the problem concerning the completely-free configuration:

$$W_{kl}^{m}(z) = [\Phi_{r_{k'} r_k}^{-}(z)]^{-1} \{[\Phi_{r_{k'} r_{k'}}^{-}(z)]^{-1} \Phi_{r_{k'} i_l}\}_{+} \tag{26}$$

After substituting the optimum transfer-function matrix into Eq. (11), and thereafter the so obtained minimum power-spectrum matrix into Eq. (4), as a result, the least sum of the mean-square errors can be computed.

2. Optimum pulse-transfer-function matrix for the semi-free configuration

The completely-free configuration seems to be a rather special case. The semi-free configuration is much nearer to reality. This problem is depicted in Fig. 2. The pulsed-data system now consists of two parts. The fixed part, for example, the plant is represented by weighting-function matrix $w_{jl}^{f}(t)$ ($j = 1, \ldots J; l = 1, \ldots L$), while the multipole cascade controller is represented by weighting-function matrix $w_{kj}^{c}(t)$ ($k = 1, \ldots K; j = 1, \ldots J$).

These two matrices must be physically realizable in all cases. Our task now is to find the optimum pulsed-data cascade controller when the plant is a priori given. Between the two parts of the system the pulsed-data manipulated variables are acting which are represented by a row vector $m_j^*(t)$ ($j = 1, \ldots J$). The other parts of the system, demonstrated in Fig. 2, are just the same as in Fig. 1. Everywhere synchronuous samplers are assumed.

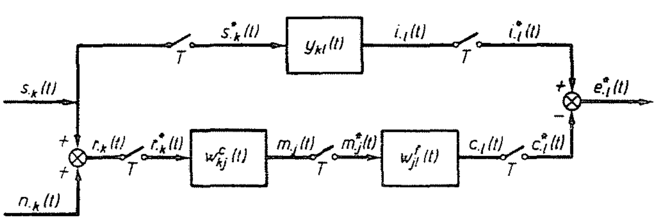

Fig. 2

Taking the generalization of the index changing rule [1] into consideration and observing that the input first penetrates the controller and thereafter the plant, the power-spectrum matrix in Eq. (10) can now be expressed as:

$$\begin{aligned}\Phi_{e_{l'} e_l}(z) = \Phi_{i_{l'} i_l}(z) - \Phi_{i_{l'} r_k}(z) W_{kj}^{c}(z) W_{jl}^{f}(z) - \\ - W_{l'j'}^{f}(z^{-1}) W_{j'k'}^{c}(z^{-1}) \Phi_{r_{k'} i_l}(z) + \\ + W_{l'j'}^{f}(z^{-1}) W_{j'k'}^{c}(z^{-1}) \Phi_{r_{k'} r_k}(z) W_{kj}^{c}(z) W_{jl}^{f}(z)\end{aligned} \tag{27}$$

$(k, k' = 1, \ldots K; \quad j, j' = 1, \ldots J; \quad l, l' = 1, \ldots L)$

where $W^{c}_{kj}(z)$ and $W^{f}_{jl}(z)$ are the pulse-transfer-function matrices of the controller and the plant, respectively, obtained from the corresponding weighting-function matrices performing z transformations.

Similarly to Eqs. (12) let us introduce another auxiliary $K \times J$ matrix $G^{c}_{kj}(z)$ and its adjoint matrix $G^{c}_{j'k'}(z^{-1})$ with the following relations:

$$\begin{aligned} \Phi_{r_{k'} r_k}(z)\, G^{c}_{kj}(z)\, W^{f}_{jl}(z) &= \Phi_{r_{k'} i_l}(z) \\ W^{f}_{l'j'}(z^{-1})\, G^{c}_{j'k'}(z^{-1})\, \Phi_{r_{k'} r_k}(z) &= \Phi_{i_{l'} r_k}(z). \end{aligned} \tag{28}$$

Since the power-spectrum matrices and the transfer-function matrix of the plant can be considered as given, the auxiliary matrices are also known.

On the basis of Eqs. (28) the power-spectrum matrix of the errors in Eq. (27) can also be written as

$$\begin{aligned} \Phi_{e_{l'} e_l}(z) = {} & \Phi_{i_{l'} i_l}(z) - W^{f}_{l'j'}(z^{-1})\, G^{c}_{j'k'}(z^{-1})\, \Phi_{r_{k'} r_k}(z)\, G^{c}_{kj}(z)\, W^{f}_{jl}(z) + \\ & + [W^{f}_{l'j'}(z^{-1})\, G^{c}_{j'k'}(z^{-1}) - W^{f}_{l'j'}(z^{-1})\, W^{c}_{j'k'}(z^{-1})] \times \\ & \times \Phi_{r_{k'} r_k}(z)\, [G^{c}_{kj}(z)\, W^{f}_{jl}(z) - W^{c}_{kj}(z)\, W^{f}_{jl}(z)]. \end{aligned} \tag{29}$$

The transfer-function matrix of the controller $W^{c}_{kj}(z)$ and its adjoint matrix $W^{c}_{j'k'}(z^{-1})$ are contained in the last term only. The sum of the mean-square errors will be minimum if the last term becomes a zero matrix. From this sufficient and necessary condition and from Eq. (28) the physically unrealizable optimum pulsed-data transfer-function matrix of the controller can be obtained:

$$W^{co}_{kj}(z) = [\Phi_{r_{k'} r_k}(z)]^{-1}\, \Phi_{r_{k'} i_l}(z)\, [W^{f}_{jl}(z)]^{-1}. \tag{30}$$

Since here the inversion of the plant matrix does figure, it is necessary to have $J = L$. Naturally the physically unrealizable transfer-function matrix is not acceptable. If, instead of $W^{co}_{kj}(z)$ the physically realizable optimum pulse-transfer-function matrix $W^{cm}_{kj}(z)$ is employed, then instead of

$$\Phi_{r_{k'} r_k}(z)\, W^{co}_{kj}(z) W^{f}_{jl}(z) = \Phi_{r_{k'} i_l}(z) \tag{31}$$

the following relation will be valid:

$$\Phi_{r_{k'} r_k}(z)\, W^{cm}_{kj}(z)\, W^{f}_{jl}(z)\, W^{f}_{l'j'}(z^{-1}) = \Phi_{r_{k'} i_l}(z)\, W^{f}_{l'j'}(z^{-1}) + F^{c}_{k'j'}(z) \tag{32}$$

where $F^{c}_{k'j'}(z)$ is a still unknown matrix with transfer-function elements having only poles outside the unit circle. Eq. (32) is the direct generalization of Eq. (19). It must be emphasized that the matrix factor $W^{f}_{l'j'}(z^{-1})$ is inevitable in Eq. (32) as $W^{f}_{jl}(z)\, W^{f}_{l'j'}(z^{-1})$ must be treated as a power-spectrum matrix, and must be factorized:

$$W^{f}_{jl}(z)\, W^{f}_{l'j'}(z^{-1}) = \left(W^{f}_{jl}(z)\, W^{f}_{l'j'}(z^{-1})\right)^{+} \left(W^{f}_{jl}(z)\, W^{f}_{l'j'}(z^{-1})\right)^{-}. \tag{33}$$

Taking Eq. (21) and Eq. (33) into consideration, substituting the factorized matrices into Eq. (32), furthermore separating the physically realizable and unrealizable matrix components on both sides of the equation, finally, after some algebraic manipulations the physically realizable optimum pulse-transfer-function matrix of the pulsed-data cascade controller can be obtained:

$$
\begin{aligned}
W_{kj}^{\mathrm{cm}}(z) = {} & [\Phi^{+}_{k' r_k}(z)]^{-1} \times \\
& \times \left\{[\Phi^{-}_{k' k}(z)]^{-1} \Phi_{r_{k'} i_l}(z)\, W^{\mathrm{f}}_{l' j'}(z^{-1}) \left[\left(W^{\mathrm{f}}_{jl}(z)\, W^{\mathrm{f}}_{l' j'}(z^{-1})\right)^{-}\right]^{-1}\right\}_{+} \times \\
& \times \left[\left(W^{\mathrm{f}}_{jl}(z)\, W^{\mathrm{f}}_{l' j'}(z^{-1})\right)^{+}\right]^{-1}.
\end{aligned}
\tag{34}
$$

This is the final explicit solution formula for the pulsed-data control system with semi-free configuration. Substituting $W_{kj}^{\mathrm{cm}}(z)$ backwards in Eq. (27) in place of $W_{kj}^{\mathrm{c}}(z)$ and employing Eq. (4), the minimum value of the sum of the mean-square errors can be computed.

3. Optimum pulse-transfer-function matrix for the semi-free configuration with constraints

Now, it will be shown how the optimum multipole pulsed-data cascade controller can be determined for multivariable systems with constraints. The convention of notations is the same as previously and the problem is demonstrated in Fig. 3. It is assumed that even the manipulated variables,

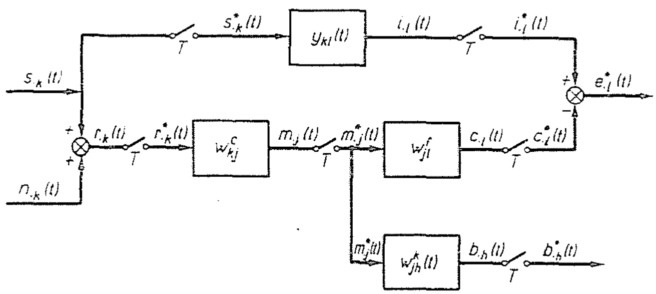

Fig. 3

acting between the controller and the plant, are submitted to constraints. Generally, an indirect manner can be taken as a basis, and for this purpose some constraint weighting-function matrix $w^{k}_{jh}(t)$ is constructed ($j = 1, \ldots J$; $h = 1, \ldots H$; $H \leq L$). The pulsed-data output row vector $b^{*}_{.h}(t)$ ($h = 1, \ldots H$) of this transfer link represents the indirect variables, or in other words, the modified manipulated variables to be constrained.

Let us assume that the sum of the mean-square values of the indirect variables is limited. Then the unequality of constraint can be expressed in the following form

$$tr\left[\overline{b_{h'.}(nT)\,b_{.h}(nT)}\right] = \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N} tr\left[\overline{b_{h'.}(nT)b_{.h}(nT)}\right] =$$

$$= tr\left[\varphi_{b_{h'}b_h}(0T)\right] = \frac{1}{2\pi j}\oint_{\Gamma_0} tr\left[\Phi_{b_{h'}b_h}(z)\right] z^{-1}\,dz \leqq \sigma^2 \qquad (35)$$

$$(h, h' = 1, \dots H)$$

where $b_{h'.}(nT)\,b_{.h}(nT)$ is a symmetrical $H \times H$ matrix composed of the matrix multiplication of column vector $b_{h'.}(nT)$ and row vector $b_{.h}(nT)$, which are $H \times 1$ and $1 \times H$ matrices, respectively, while "tr" denotes the trace that is the sum of the principal diagonal elements of the matrix. Further, $\varphi_{b_{h'}b_h}(0T)$ is the autocorrelation-sequence matrix with zero shifting time, while $\Phi_{b_{h'}b_h}(z)$ is the corresponding power-spectrum matrix.

As a minimization criterion, again the sum of the mean-square-error components is adopted. This sum can be expressed once more with Eq. (4). Taking Eq. (4) and Eq. (35) into consideration just as if one would apply the Lagrangean conditional extremum technique, the optimum design of the pulsed-data cascade controller is reduced to the minimization of the following expression

$$tr\left[\varphi_{x_{l'}x_l}(0T,\lambda)\right] = tr\left[\overline{x_{l'.}(nT,\lambda)x_{.l}(nT,\lambda)}\right] = \frac{1}{2\pi j}\oint_{\Gamma_0} tr[\Phi_{x_{l'}x_l}(z,\lambda)]z^{-1}\,dz \qquad (36)$$

where the matrices figuring here are the following

$$x_{l'.}(nT,\lambda)\,x_{.l}(nT,\lambda) = e_{l'.}(nT)\,e_{.l}(nT) + \lambda\, b_{h'.}(nT)\,b_{.h}(nT)$$

and

$$\varphi_{x_{l'}x_l}(nT,\lambda) = \varphi_{e_{l'}e_l}(nT) + \lambda\,\varphi_{b_{h'}b_h}(nT)$$

and

$$\Phi_{x_{l'}x_l}(z,\lambda) = \Phi_{e_{l'}e_l}(z) + \lambda\,\Phi_{b_{h'}b_h}(z).$$

Our task is again to find an expression for the power-spectrum matrix $\Phi_{x_{l'}x_l}(z,\lambda)$ and then to minimize the latter. Applying the generalized index-changing rule and considering Fig. 3, the power-spectrum matrix of the errors can be expressed as in Eq. (27), while the power-spectrum matrix of the modified manipulated variables is

$$\Phi_{b_{h'}b_h}(z) = W^{k}_{h'j'}(z^{-1})\,W^{c}_{j'k'}(z^{-1})\,\Phi_{r_{k'}r_k}\,W^{c}_{kj}(z)\,W^{k}_{jh}(z) \qquad (37)$$

$$(k, k' = 1, \dots K\,;\; j, j' = 1, \dots J\,;\; h, h' = 1, \dots H)$$

where $W^{k}_{jh}(z)$ is the $J \times H$ pulse-transfer-function matrix of the constraint determined from the weighting-function matrix $w^{k*}_{jh}(t)$ performing z transformation. $W^{k}_{h'j'}(z^{-1})$ is the adjoint $H \times J$ matrix of the former. In most cases $K = J = L = H$ can be assumed without loss of generality.

Let us now introduce an auxiliary power-spectrum matrix $\Phi_{a_{k'}a_k}(z, \lambda)$ implicitly defined by the following relation:

$$\begin{aligned} & W^{f}_{l'j'}(z^{-1})\, W^{c}_{j'k'}(z^{-1})\, \Phi_{r_{k'}r_k}(z)\, W^{c}_{kj}(z)\, W^{f}_{jl}(z) + \\ & + \lambda\, W^{k}_{h'j'}(z^{-1})\, W^{c}_{j'k'}(z^{-1})\, \Phi_{r_{k'}r_k}(z)\, W^{c}_{kj}(z)\, W^{f}_{jl}(z) = \\ & = W^{f}_{l'j'}(z^{-1})\, W^{c}_{j'k'}(z^{-1})\, \Phi_{a_{k'}a_k}(z, \lambda)\, W^{c}_{kj}(z)\, W^{f}_{jl}(z)\,. \end{aligned} \tag{38}$$

It can be shown that the auxiliary power-spectrum matrix $\Phi_{a_{k'}a_k}(z, \lambda)$ is uniquely determined by Eq. (38) if it does not depend on the choice of transfer-function matrix $W^{c}_{kj}(z)$ of the cascade controller itself. (See later in "4. Some supplementary remarks".) Now, taking Eqs. (27), (37) and (38) into consideration the power-spectrum matrix figuring in Eq. (36) can be expressed as follows:

$$\begin{aligned} \Phi_{x_{l'}x_l}(z, \lambda) = \Phi_{i_{l'}i_l} & - \Phi_{i_{l'}r_k}(z)\, W^{c}_{kj}(z) W^{f}_{jl}(z) - \\ & - W^{f}_{l'j'}(z^{-1})\, W^{c}_{j'k'}(z^{-1})\, \Phi_{r_{k'}i_l}(z) + \\ & + W^{f}_{l'j'}(z^{-1})\, W^{c}_{j'k'}(z^{-1})\, \Phi_{a_{k'}a_k}(z, \lambda)\, W^{c}_{kj}(z)\, W^{f}_{jl}(z)\,. \end{aligned} \tag{39}$$

The form of this expression is quite similar to Eq. (27). Thus, the same technique can be employed as in the case of the semi-free configuration without constraints.

Therefore let us introduce an auxiliary $K \times J$ matrix $G^{ck}_{kj}(z, \lambda)$ and its adjoint $G^{ck}_{j'k'}(z^{-1}, \lambda)$ with the following implicit definitions

$$\begin{aligned} & \Phi_{a_{k'}a_k}(z, \lambda)\, G^{ck}_{kj}(z, \lambda)\, W^{f}_{jl}(z) = \Phi_{r_{k'}i_l}(z) \\ & W^{f}_{l'j'}(z^{-1}) G^{ck}_{j'k'}(z^{-1}, \lambda)\, \Phi_{a_{k'}a_k}(z, \lambda) = \Phi_{i_{l'}r_k}(z)\,. \end{aligned} \tag{40}$$

As $\Phi_{a_{k'}a_k}(z, \lambda)$ does not only depend on z but also on λ, thus, the auxiliary matrix is also a two-variable function of z and λ. The physical realizability of $G^{ck}_{kj}(z, \lambda)$ is, of course, not garanteed, on the contrary, it is physically unrealizable. Substitution of Eqs. (40) into Eq. (39) gives

$$\begin{aligned} \Phi_{x_{l'}x_l}(z, \lambda) = \Phi_{i_{l'}i_l}(z) & - W^{f}_{l'j'}(z^{-1})\, G^{ck}_{j'k'}(z^{-1}, \lambda)\, \Phi_{a_{k'}a_k}(z, \lambda)\, G^{ck}_{kj}(z, \lambda)\, W^{f}_{jl}(z) + \\ & + [W^{f}_{l'j'}(z^{-1})\, G^{ck}_{j'k'}(z^{-1}, \lambda) - W^{f}_{l'j'}(z^{-1})\, W^{c}_{j'k'}(z^{-1})] \times \\ & \times \Phi_{a_{k'}a_k}(z, \lambda)\, [G^{ck}_{kj}(z, \lambda)\, W^{f}_{jl}(z) - W^{c}_{kj}(z)\, W^{f}_{jl}(z)]\,. \end{aligned} \tag{41}$$

Since the pulse-transfer-function matrix $W^{ck}_{kj}(z)$ and its adjoint are contained only in the last term, the power-spectrum matrix $\Phi_{x_{l'}x_l}(z, \lambda)$ together with the trace of it, will be minimum if the last term becomes a zero matrix. From this condition and on the basis of Eq. (40) the physically unrealizable optimum pulse-transfer-function matrix and its adjoint can be obtained from the following relations:

$$\begin{aligned} &\Phi_{a_{k'}a_k}(z,\lambda)\, W^{\mathrm{cko}}_{kj}(z,\lambda)\, W^{\mathrm{f}}_{jl}(z) = \Phi_{r_{k'}i_l}(z) \\ &W^{\mathrm{f}}_{l'j'}(z^{-1})\, W^{\mathrm{cko}}_{j'k'}(z^{-1},\lambda)\, \Phi_{a_{k'}a_k}(z,\lambda) = \Phi_{i_{l'}r_k}(z)\,. \end{aligned} \tag{42}$$

From the first relation the optimum matrix $W^{\mathrm{cko}}_{kj}(z,\lambda)$ itself can easily be determined, but this step is not necessary at all, because unrealizable transfer-function matrices give no solutions. Let us substitute a physically realizable optimum pulse-transfer-function matrix $W^{\mathrm{ckm}}_{kj}(z,\lambda)$ instead of $W^{\mathrm{cko}}_{kj}(z,\lambda)$ into Eq. (42) then the following relation can be derived:

$$\Phi_{a_{k'}a_k}(z,\lambda)\, W^{\mathrm{ckm}}_{kj}(z,\lambda)\, W^{\mathrm{f}}_{jl}(z)\, W^{\mathrm{f}}_{l'j'}(z^{-1}) = \Phi_{r_{k'}i_l}(z)\, W^{\mathrm{f}}_{l'j'}(z^{-1}) + F^{\mathrm{ck}}_{k'j'}(z,\lambda) \tag{43}$$

where $F^{\mathrm{ck}}_{k'j'}(z,\lambda)$ is still an unknown matrix with transfer-function elements having only poles outside the unit circle. This expression is quite similar to Eq. (32) and therefore, the same technique can be applied as was previously used.

Taking the spectrum-factorization relation

$$\Phi^{-}_{a_{k'}a_{k''}}(z,\lambda)\, \Phi^{+}_{a_{k''}a_k}(z,\lambda) = \Phi_{a_{k'}a_k}(z,\lambda) \tag{44}$$

and Eq. (32) into consideration, Eq. (43) may assume the following form:

$$\begin{aligned} &\Phi^{+}_{a_{k''}a_k}(z,\lambda)\, W^{\mathrm{ckm}}_{kj}(z,\lambda)\big(W^{\mathrm{f}}_{jl}(z)\, W^{\mathrm{f}}_{l'j'}(z^{-1})\big)^{+} = \\ &= [\Phi^{-}_{a_{k'}a_{k''}}(z,\lambda)]^{-1}\, \Phi_{r_{k'}i_l}(z)\, W^{\mathrm{f}}_{l'j'}(z^{-1})\big[\big(W^{\mathrm{f}}_{jl}(z)\, W^{\mathrm{f}}_{l'j'}(z^{-1})\big)^{-}\big]^{-1} + \\ &+ [\Phi^{-}_{a_{k'}a_{k''}}(z,\lambda)]^{-1}\, F^{\mathrm{ck}}_{k'j'}(z,\lambda)\big[\big(W^{\mathrm{f}}_{jl}(z)\, W^{\mathrm{f}}_{l'j'}(z^{-1})\big)^{-}\big]^{-1} \end{aligned} \tag{45}$$

Separating again the physically realizable and unrealizable matrix components on both sides of Eq. (45) the next two relations can be obtained:

$$\begin{aligned} &\Phi^{+}_{a_{k''}a_k}(z,\lambda)\, W^{\mathrm{ckm}}_{kj}(z,\lambda)\big(W^{\mathrm{f}}_{jl}(z)\, W^{\mathrm{f}}_{l'j'}(z^{-1})\big)^{+} = \\ &= \big\{[\Phi^{-}_{a_{k'}a_{k''}}(z,\lambda)]^{-1}\, \Phi_{r_{k'}i_l}(z)\, W^{\mathrm{f}}_{l'j'}(z^{-1})\big[\big(W^{\mathrm{f}}_{jl}(z)\, W^{\mathrm{f}}_{l'j'}(z^{-1})\big)^{-}\big]^{-1}\big\}_{+} \end{aligned} \tag{46}$$

and

$$\begin{aligned} 0 = &\big\{[\Phi^{-}_{a_{k'}a_{k''}}(z,\lambda)]^{-1}\, \Phi_{r_{k'}i_l}(z)\, W^{\mathrm{f}}_{l'j'}(z^{-1})\big[\big(W^{\mathrm{f}}_{jl}(z)\, W^{\mathrm{f}}_{l'j'}(z^{-1})\big)^{-}\big]^{-1}\big\}_{-} + \\ &+ [\Phi^{-}_{a_{k'}a_{k''}}(z,\lambda)]^{-1}\, F^{\mathrm{ck}}_{k'j'}(z,\lambda)\big[\big(W^{\mathrm{f}}_{jl}(z)\, W^{\mathrm{f}}_{l'j'}(z^{-1})\big)^{-}\big]^{-1} \end{aligned} \tag{47}$$

where the symbolism used is the same as in connection with Eqs. (24) and (25). Finally, from Eq. (46) the physically realizable optimum pulse-transfer-function matrix of the pulsed-data cascade controller in case of constraints can be expressed as follows:

$$
\begin{aligned}
W_{kj}^{\mathrm{ckm}}(z, \lambda) = {} & [\Phi_{a_k a_k}^{+}(z, \lambda)]^{-1} \times \\
& \times \left\{ [\Phi_{a_{k'} a_k}^{-}(z, \lambda)]^{-1} \Phi_{r_{k'} i_l}(z) W_{l' j'}^{\mathrm{f}}(z^{-1}) \left[\left(W_{jl}^{\mathrm{f}}(z) W_{l' j'}^{\mathrm{f}}(z^{-1}) \right)^{-} \right]^{-1} \right\}_{+} \times \\
& \times \left[\left(W_{jl}^{\mathrm{f}}(z) W_{l' j'}^{\mathrm{f}}(z^{-1}) \right)^{+} \right]^{-1} .
\end{aligned}
\tag{48}
$$

The expression of $W_{kj}^{\mathrm{c'm}}(z, \lambda)$ may now be substituted into the condition of constraint. This can be performed by first substituting $W_{kj}^{\mathrm{c'm}}(z, \lambda)$ and its adjoint matrix instead of $W_{kj}^{\mathrm{c}}(z)$ and its adjoint, respectively, into Eq. (37). Hence the power-spectrum matrix $\Phi_{b_{h'} b_h}(z, \lambda)$ is obtained. Substituting the latter matrix into Eq. (35) the parameter λ can be adjusted so that the condition of constraint, that is, even the unequality (35) will be satisfied.

After having determined the proper value of parameter λ, the latter can be substituted backwards into Eq. (48). Thus, as a result of this technique the physically realizable optimum pulse-transfer-function matrix of the pulsed-data cascade controller $W_{kj}^{\mathrm{c\,m}}(z)$ is obtained. Naturally, following the procedure outlined previously the parameter λ is already missing from the final solution formula. $W_{kj}^{\mathrm{c\,m}}(z)$ is the explicit solution of the problem in case of the semi-free configuration with constraints.

Substituting the so obtained matrix expression of $W_{jk}^{\mathrm{c'm}}(z)$ and its adjoint into Eq. (27) instead of $W_{kj}^{\mathrm{c}}(z)$ and $W_{j'k'}^{\mathrm{c}}(z^{-1})$, respectively, the power-spectrum matrix of the error can be computed. Henceforth, applying Eq. (4) the least sum of the mean-square-error components can be determined.

4. Some supplementary remarks

Evidently, the semi-free configuration with constraints is the most general case, and from the final solution formula of the latter, that of the semi-free configuration without constraints can easily be obtained. It is only necessary to substitute $\lambda = 0$ into Eq. (48), which is then reduced to Eq. (34), as in the same time from Eq. (38)

$$\Phi_{a_{k'} a_k}(z, \lambda) = \Phi_{r_{k'} r_k}(z) . \tag{49}$$

The final solution formula of the completely-free configuration, Eq. (26) can also be obtained from Eq. (48) after substituting $\lambda = 0$ and taking an unity matrix instead of the pulse-transfer-function matrix $W_{jl}^{\mathrm{f}}(z)$ of the fixed part.

All cases of the single variable systems can readily be obtained from the corresponding cases of the multivariable systems, too.

Let us now concentrate our attention to the most complex problem, that is, to the semi-free configuration with constraints. It must be emphasized that in the latter case the success of the proposed method depends on the condition that the auxiliary power-spectrum matrix $\Phi_{a_{k'}a_k}(z, \lambda)$ must be independently determined from the choice of the pulse-transfer-function matrices $W^{c}_{kj}(z)$ and $W^{c}_{j'k'}(z^{-1})$. Unfortunately, this condition is not guaranteed a priori.

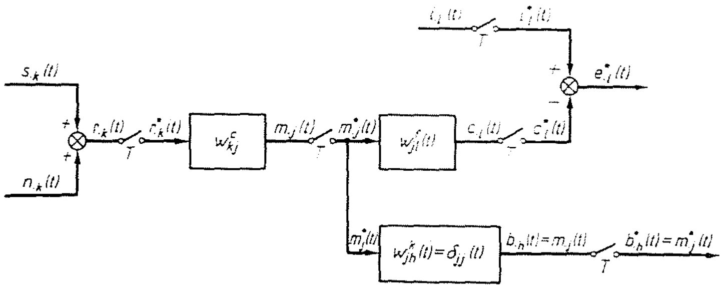

Fig. 4

From Eq. (38) the auxiliary power-spectrum matrix can be explicitly expressed as

$$\Phi_{a_{k'}a_k}(z, \lambda) = \Phi_{r_{k'}r_k}(z) + \lambda\,[W^{c}_{j'k'}(z^{-1})]^{-1}\,[W^{f}_{l'j'}(z^{-1})]^{-1} \times$$
$$\times\, W^{k}_{h'j'}(z^{-1})\, W^{c}_{j'k'}(z^{-1})\, \Phi_{r_{k'}r_k}(z)\, W^{c}_{kj}(z)\, W^{k}_{jh}(z)\, [W^{f}_{jl}(z)]^{-1}\, [W^{c}_{kj}(z)]^{-1}.$$

It can be seen that for arbitrary power-spectrum matrix $\Phi_{r_{k'}r_k}(z)$ the independence of the power-spectrum matrix $\Phi_{a_{k'}a_{k'}}(z, \lambda)$ is ensured only when

$$W^{k}_{jh}(z)\,[W^{f}_{jl}(z)]^{-1} = G(z)\, I_{jj} \tag{50}$$
$$(j = 1, \ldots J,\; h = l = 1, \ldots H = L)$$

where $G(z)$ is some scalar pulse-transfer function (or exceptionally a polynomial), while I_{jj} is an unity matrix of the $J \times J$ type. This condition yieldes the following auxiliary matrix

$$\Phi_{a_{k'}a_k}(z, \lambda) = [1 + \lambda\, G(z^{-1})\, G(z)]\, \Phi_{r_{k'}r_k}(z) \tag{51}$$

which is indeed independent of $W^{c}_{kj}(z)$ and $W^{c}_{j'k'}(z^{-1})$. This circumstance emphasizes the fact that for a given plant matrix the constraint matrix can not be optionally chosen but there most be an interdependency between them to ensure the independency of the auxiliary matrix. This is in contrast to the single variable systems where Eq. (50) is inherently guarantied, because for $j = 1$ being $I_{jj} = 1$, that is a scalar quantity. Thus, in single variable systems the necessary independence is a priori ensured.

In the following we will always assume that Eq. (50) is valid. Now, let us concentrate our attention to the constraint matrix $W^{k}_{jh}(z)$. If the manipulated variables are directly constrained then $W^{k}_{jh}(z)$ must be an unity matrix I_{jh} $(J = H)$, and $w^{k}_{jh}(t)$ must be a diagonal delta-function matrix $\delta_{jj}(t)$ (Fig. 4). If, on the other hand, the manipulated variables are indirectly constrained the pulse-transfer-function matrix $W^{k}_{jh}(z)$ may assume quite a gener-

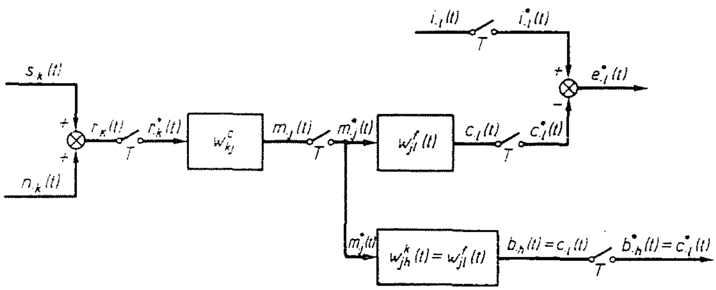

Fig. 5

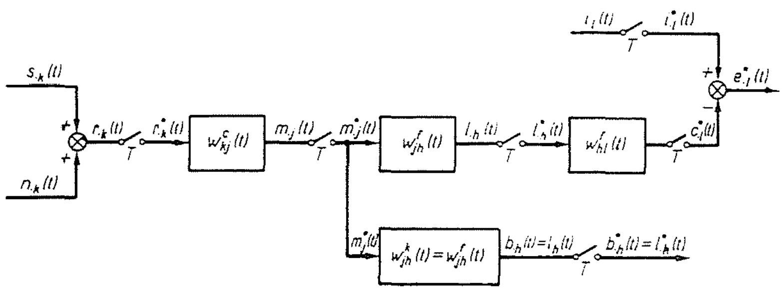

Fig. 6

al form. For example, if even the sum of the mean-square values of the controlled variables are limited, then the constraint matrix $W^{k}_{jh}(z)$ becomes the very same as the plant matrix $W^{f}_{jl}(z)$ (Fig. 5).

If the plant matrix $W^{f}_{jl}(z)$ can be expressed as the matrix multiplication of two corresponding pulse-transfer-function matrices

$$W^{f}_{jl}(z) = W^{f}_{jh}(z)\, W^{f}_{hl}(z) \tag{52}$$

and even the sum of the mean-square-value of the pulsed-data variables acting between the two control links mentioned above has to be limited, then $W^{k}_{jh}(z)$ must be taken as identical to $W^{f}_{jh}(z)$ (Fig. 6).

Now the results obtained may easily be generalized also for multiple constraints. For example, if the manipulated variables are simultaneously submitted to two or more constraints (Fig. 7) then instead of unequality (35) a system of unequalities is valid:

$$\frac{1}{2\pi j}\oint_{\Gamma_0} tr\,[\Phi_{b_{h'}b_{h(i)}}(z)]\, z^{-1}\, dz \leqq \sigma_i^2. \tag{35*}$$

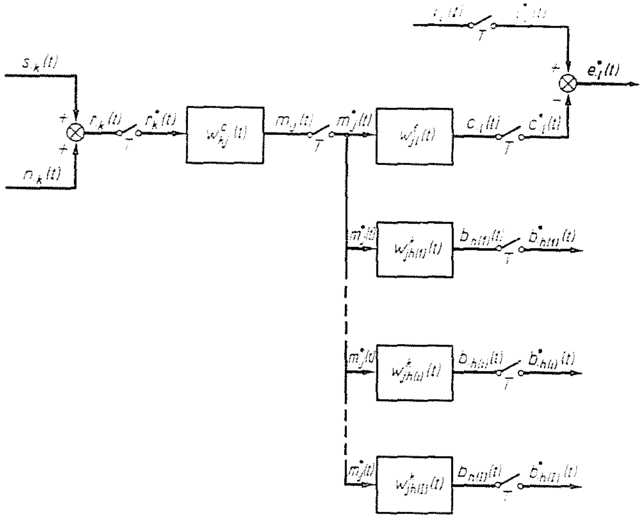

Fig. 7

Furthermore, instead of Eq. (36) we have the following relation:

$$tr\,[\varphi_{x_{l'}x_l}(0T, \lambda_1 \ldots \lambda_I)] = tr\,[\overline{x_{l'}(nT, \lambda_1 \ldots \lambda_I)\, x_l(nT, \lambda_1 \ldots \lambda_I)}] =$$

$$= \frac{1}{2\pi j}\oint_{\Gamma_0} tr\left[\Phi_{e_{l'}e_l}(z) + \sum_{i=1}^{I} \lambda_i\, \Phi_{b_{h'}b_{h(i)}}(z)\right] z^{-1}\; dz. \tag{36*}$$

Here the corresponding power-spectrum matrices are

$$\Phi_{b_{h'}b_{h(i)}}(z) = W^{\mathrm{k}}_{h'j'(i)}(z^{-1})\, W^{\mathrm{c}}_{j'k'}(z^{-1})\, \Phi_{r_{k'}r_k}(z)\, W^{\mathrm{c}}_{kj}(z)\, W^{\mathrm{k}}_{jh(i)}$$

where $W^{\mathrm{k}}_{jh\,(i)}(z)$ $(i = 1, \ldots I)$ are the corresponding constraint matrices. In case of the semi-free configuration with many constraints the conditions of independence

$$W^{\mathrm{k}}_{jh(i)}(z)[W^{\mathrm{f}}_{jl}(z)]^{-1} = G_{(i)}(z)\, I_{ll} \tag{50*}$$

$$(i = 1, \ldots I)$$

must be fulfilled, where $G_{(i)}(z)$ $(i = 1, \ldots I)$ are some scalar pulse-transfer functions (or polynomials) and I_{jj} is an unity matrix of the $J \times J$ type.

Naturally, in this case both the auxilliary power-spectrum matrix $\Phi_{a_k, a_k}(z, \lambda_1, \ldots \lambda_I)$ and the physically realizable optimum pulse-transfer-function matrix $W_{kj}^{c\,m}(z, \lambda_1, \ldots \lambda_I)$ becomes a multivariable function of the parameters λ_i. Accordingly, instead of Eq. (51) we obtain

$$\Phi_{a_k a_k}(z, \lambda_1, \ldots \lambda_I) = \left| 1 + \sum_{i=1}^{I} \lambda_i G_{(i)}(z^{-1}) G_{(i)}(z) \right| \Phi_{k' k}(z) \tag{51*}$$

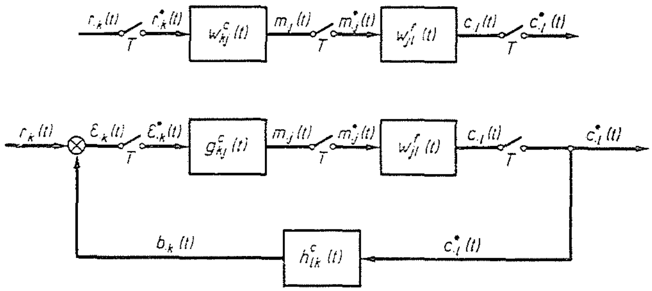

Fig. 8

Finally, the explicit solution formula for the semi-free configuration with multiple constraints becomes:

$$\begin{aligned} W_{kj}^{ckm}(z, \lambda_1, \ldots \lambda_I) &= [\Phi_{a_k a_k}^{+}(z, \lambda_1, \ldots \lambda_I)]^{-1} \times \\ &\times \{[\Phi_{a_k, a_k}^{-}(z, \lambda_1, \ldots \lambda_I)]^{-1} \Phi_{k' i_l}(z) W_{l' j'}^{f}(z^{-1}) \times \\ &\times [(W_{jl}^{f}(z) W_{l' j'}^{f}(z^{-1}))^{-}]^{-1}\} + [(W_{jl}^{f}(z) W_{l' j'}^{f}(z^{-1}))^{+}]^{-1}. \end{aligned} \tag{48*}$$

The adjustment procedure of the parameters λ_i must now be performed in such a way that the most rigorous of the unequalities (35*) can be fulfilled.

After having determined on the basis of Fig. 8. the optimum pulse-transfer-function matrix of the pulsed-data-cascade controller, the pulse-transfer-function matrix of the series controller in the closed loop can be expressed as

$$G_{kj}^{c}(z) = [I - W_{kj}^{c}(z) W_{jl}^{f}(z)]^{-1} W_{kj}^{c}(z)$$

if there is no feed-back controller. If, on the other hand, the series controller is missing, the pulse-transfer-function matrix of the feed-back controller is

$$H_{lk}^{c}(z) = [W_{kj}^{c}(z) W_{jl}^{f}(z)]^{-1} - [W_{jl}^{f}(z)]^{-1}$$
$$(k = l = j = 1, \ldots K = L = J).$$

5. Illustrative examples

To illustrate the proposed methods only a very simple two-variable pulsed-data system serves as an example. Each of the two inputs contains a useful signal component and the second one is also influenced by a white noise component. The useful signal components and the noise component are not correlated, while the signal components themselves are correlated. The ideal or desired outputs are assumed as identical with the signal components of the inputs. Only the completely-free and the semi-free configurations are treated here.

Example 1. Let the continuous-data power-density-spectrum matrices be

$$\Phi_{s_k s_k}(s) = \begin{bmatrix} \dfrac{4}{1 - s^2} & \dfrac{2A}{(1-s)(2+s)} \\ \dfrac{2A}{(2-s)(1+s)} & \dfrac{1}{4-s^2} \end{bmatrix} ; \quad \Phi_{n_k n_k}(s) = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix}$$

where $0 \leqq A \leqq 1$, and

$$\Phi_{s_k n_k}(s) = \Phi_{n_k s_k}(s) = \mathbf{0}$$

Correspondingly, the pulsed-data power-spectrum matrices are

$$\Phi_{s_k, s_k}(z) = \begin{bmatrix} \dfrac{2(1-e^{-2T})}{(1-z e^{-T})(1-z^{-1} e^{-T})} & \dfrac{2A}{3} \cdot \dfrac{(1-e^{-3T})}{(1-ze^{-T})(1-z^{-1} e^{-2T})} \\ \dfrac{2A}{3} \cdot \dfrac{(1-e^{-3T})}{(1-ze^{-2T})(1-z^{-1} e^{-T})} & \dfrac{1}{4} \cdot \dfrac{(1-e^{-4T})}{(1-ze^{-2T})(1-z^{-1} e^{-2T})} \end{bmatrix}$$

and

$$\Phi_{n_k, n_k}(z) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus, after addition the power-spectrum matrix of the inputs is

$$\Phi_{r_k r_k}(z) = \begin{bmatrix} \dfrac{2(1-e^{-2T})}{(1-ze^{-T})(1-z^{-1}e^{-T})} & \dfrac{2A}{3} \cdot \dfrac{(1-e^{-T})}{(1-ze^{-T})(1-z^{-1} e^{-2T})} \\ \dfrac{2A}{3} \cdot \dfrac{(1-e^{-3T})}{(1-ze^{-2T})(1-z^{-1} e^{-T})} & \dfrac{(P-zQ)(P-z^{-1}Q)}{(1-ze^{-2T})(1-z^{-1} e^{-2T})} \end{bmatrix}$$

where

$$\left.\begin{matrix} P \\ Q \end{matrix}\right\} = \frac{1}{2}\left[\sqrt{\frac{5}{4} + 2e^{-2T} + \frac{3}{4}e^{-4T}} \pm \sqrt{\frac{5}{4} - 2e^{-2T} + \frac{3}{4}e^{-4T}}\right]; \; (P > Q).$$

On the other hand

$$\Phi_{r_{k'} i_l}(z) = \Phi_{s_{k'} i_l}(z) + \Phi_{n_{k'} i_l}(z) = [\Phi_{s_{k'} s_k}(z) + \Phi_{n_{k'} s_k}(z)]\, Y_{kl}(z) = \Phi_{s_k s_k}(z)$$

because $Y_{kl}(z) = I$.

On the basis of Eq. (21) in the present case

$$\Phi^{-}_{r_{k'} r_{k'}}(z) = \begin{bmatrix} \dfrac{\sqrt{2\,(1-e^{-2T})}}{1-ze^{-2T}} & 0 \\[2ex] \dfrac{\dfrac{2A}{3}\cdot\dfrac{1-e^{-3T}}{\sqrt{2\,(1-e^{-2T})}}}{1-ze^{-2T}} & \dfrac{U-zV}{1-ze^{-2T}} \end{bmatrix}$$

and

$$\Phi^{+}_{r_k r_k}(z) = \begin{bmatrix} \dfrac{\sqrt{2\,(1-e^{-2T})}}{1-z^{-1}\,e^{-2T}} & \dfrac{\dfrac{2A}{3}\cdot\dfrac{1-e^{-3T}}{\sqrt{2\,(1-e^{-2T})}}}{1-z^{-1}\,e^{-2T}} \\[2ex] 0 & \dfrac{U-z^{-1}V}{1-z^{-1}\,e^{-2T}} \end{bmatrix}$$

where

$$\left.\begin{matrix} U \\ V \end{matrix}\right\} = \frac{1}{2}\left[\sqrt{P^2+2\,PQ+Q^2-K^2} \pm \sqrt{P^2-2\,PQ+Q^2-K^2}\right];\ (U > V).$$

Here

$$K = \frac{2A}{3}\cdot\frac{(1-e^{-3T})}{\sqrt{2\,(1-e^{-2T})}}$$

is substituted.

Correspondingly,

$$[\Phi^{-}_{r_k r_{k'}}(z)]^{-1} = \begin{bmatrix} \dfrac{1-z\,e^{-T}}{\sqrt{2\,(1-e^{-2T})}} & 0 \\[2ex] -\dfrac{A}{3}\cdot\dfrac{(1-e^{-3T})\,(1-ze^{-T})}{(1-e^{-2T})\,(U-z\,V)} & \dfrac{1-ze^{-2T}}{U-z\,V} \end{bmatrix}$$

and

$$[\Phi^{-}_{r_k r_k}(z)]^{-1} = \begin{bmatrix} \dfrac{1-z^{-1}\,e^{-T}}{\sqrt{2(1-e^{-2T})}} & -\dfrac{A}{3}\cdot\dfrac{(1-e^{-3T})(1-z^{-1}\,e^{-T})}{(1-e^{-2T})(U-z^{-1}\,V)} \\[2ex] 0 & \dfrac{1-z^{-1}\,e^{-2T}}{U-z^{-1}\,V} \end{bmatrix}$$

Hence

$$[\Phi^{-}_{r_{k'}r_{k'}}(z)]^{-1}\,\Phi_{\;k' i_l}(z) =$$

$$= \begin{bmatrix} \dfrac{\sqrt{2(1-e^{-2T})}}{1-z^{-1}e^{-T}} & \dfrac{2\,A}{3}\cdot\dfrac{(1-e^{-3T})}{\sqrt{2\,(1-e^{-2T})}}\cdot\dfrac{1}{(1-z^{-1}e^{-2T})} \\ 0 & \begin{matrix}\left[\dfrac{1}{4}(1-e^{-4T})-\dfrac{2\,A^2}{9}\,\dfrac{(1-e^{-3T})^2}{(1-e^{-2T})}\right]\times \\ \times\,\dfrac{1}{(U-z\,V)(1-z^{-1}e^{-2T})}\end{matrix} \end{bmatrix}$$

Separating the physically realizable matrix component:

$$\{[\Phi^{-}_{r_{k'}r_{k'}}(z)]^{-1}\,\Phi_{r_{k'}\,i_l}(z)\}_{+} =$$

$$= \begin{bmatrix} \dfrac{\sqrt{2\,(1-e^{-2T})}}{1-z^{-1}e^{-T}} & \dfrac{2\,A}{3}\cdot\dfrac{(1-e^{-3T})}{\sqrt{2\,(1-e^{-2T})}}\cdot\dfrac{1}{(1-z^{-1}e^{-2T})} \\ 0 & \begin{matrix}\left[\dfrac{1}{4}(1-e^{-4T})-\dfrac{2\,A^2}{9}\cdot\dfrac{(1-e^{-3T})^2}{(1-e^{-2T})}\right]\times \\ \times\,\dfrac{1}{U-e^{-2T}\,V}\cdot\dfrac{1}{1-z^{-1}e^{-2T}}\end{matrix} \end{bmatrix}.$$

Finally, for the case of the completely-free configuration the physically realizable optimum pulse-transfer-function matrix can be obtained according to Eq. (26) as

$$\mathbf{W}^{m}_{kl}(z) = \begin{bmatrix} 1 & \begin{matrix}\dfrac{A}{3}\cdot\dfrac{(1-e^{-3T})}{(1-e^{-2T})}\left[1-\left(\dfrac{1}{4}(1-e^{-4T})-\dfrac{2\,A^2}{9}\,\dfrac{(1-e^{-3T})^2}{(1-e^{-2T})}\right)\times\right. \\ \left.\times\,\dfrac{1}{U-e^{-2T}\,V}\cdot\dfrac{1}{U-z^{-1}\,V}\right]\dfrac{1-z^{-1}e^{-T}}{1-z^{-1}e^{-2T}}\end{matrix} \\ 0 & \left[\dfrac{1}{4}(1-e^{-4T})-\dfrac{2\,A^2}{9}\,\dfrac{(1-e^{-3T})^2}{(1-e^{-2T})}\right]\dfrac{1}{(U-e^{-2T}\,V)}\cdot\dfrac{1}{U-z^{-1}\,V} \end{bmatrix}.$$

Example 2. Taking the same starting data as in example 1, let us look at the following plant matrix of nonminimum phase

$$\mathbf{F}^{i}_{jl}(z) = \begin{bmatrix} Bz^{-\beta} & 0 \\ 0 & Cz^{-\gamma} \end{bmatrix}$$

where β and γ are positive integer numbers. Now the more complicated formula (34) must be applied. The steps of the calculation are:

$$\Phi_{r_{k'} i_l}(z)\, W^{\dagger}_{l'j'}(z^{-1}) =$$

$$= \begin{bmatrix} \dfrac{2(1-e^{-2T})\,Bz^{\beta}}{(1-ze^{-T})(1-z^{-1}e^{-T})} & \dfrac{2A}{3}\cdot\dfrac{(1-e^{-3T})\,Cz^{\gamma}}{(1-ze^{-T})(1-z^{-1}e^{-2T})} \\ \dfrac{2A}{3}\cdot\dfrac{(1-e^{-3T})\,Bz^{\beta}}{(1-ze^{-2T})(1-z^{-1}e^{-T})} & \dfrac{1}{4}\cdot\dfrac{(1-e^{-4T})\,Cz^{\gamma}}{(1-ze^{-2T})(1-z^{-1}e^{-2T})} \end{bmatrix}.$$

It can easily be shown that

$$\left[\left(W^{\dagger}_{jl}(z)\, W^{\dagger}_{l'j'}(z^{-1})\right)^{-}\right]^{-1} = \begin{vmatrix} B^{-1} & 0 \\ 0 & C^{-1} \end{vmatrix} = \left[\left(W^{\dagger}_{jl}(z)\, W^{\dagger}_{l'j'}(z^{-1})\right)^{+}\right]^{-1}$$

thus

$$\Phi_{r_{k'} i_l}(z)\, W^{\dagger}_{l'j'}(z^{-1}) \left[\left(W^{\dagger}_{jl}(z)\, W^{\dagger}_{l'j'}(z^{-1})\right)^{-}\right]^{-1} =$$

$$= \begin{bmatrix} \dfrac{2(1-e^{-2T})\,z^{\beta}}{(1-ze^{-T})(1-z^{-1}e^{-T})} & \dfrac{2A}{3}\cdot\dfrac{(1-e^{-3T})\,z^{\gamma}}{(1-ze^{-T})(1-z^{-1}e^{-2T})} \\ \dfrac{2A}{3}\cdot\dfrac{(1-e^{-3T})\,z^{\beta}}{(1-ze^{-2T})(1-z^{-1}e^{-T})} & \dfrac{1}{4}\cdot\dfrac{(1-e^{-4T})\,z^{\gamma}}{(1-ze^{-2T})(1-z^{-1}e^{-2T})} \end{bmatrix}$$

and the physically realizable component is

$$\left\{\left[\Phi^{-}_{r_{k'} r_{k''}}(z)\right]^{-1} \Phi_{r_{k'} i_l}(z)\, W^{\dagger}_{l'j'}(z^{-1}) \left[\left(W^{\dagger}_{jl}(z)\, W^{\dagger}_{l'j'}(z^{-1})\right)^{-}\right]^{-1}\right\}_{+} =$$

$$= \begin{bmatrix} \dfrac{\sqrt{2(1-e^{-2T})}\,e^{-\beta T}}{1-z^{-1}e^{-T}} & \dfrac{2A}{3}\cdot\dfrac{(1-e^{-3T})}{\sqrt{2(1-e^{-2T})}}\cdot\dfrac{e^{-2\gamma T}}{(1-z^{-1}e^{-2T})} \\ 0 & \begin{array}{c}\left[\dfrac{1}{4}(1-e^{-4T}) - \dfrac{2A^2}{9}\cdot\dfrac{(1-e^{-3T})^2}{(1-e^{-2T})}\right]\times \\ \times\dfrac{e^{-2\gamma T}}{U-e^{-2T}V}\cdot\dfrac{1}{1-z^{-1}e^{-2T}}\end{array} \end{bmatrix}.$$

Finally, following Eq. (34), the pulse-transfer-function matrix of the physically realizable optimum pulsed-data cascade controller can be determined in the form:

$$W_{kj}^{\mathrm{cm}}(z) =$$

$$= \begin{bmatrix} \frac{1}{B} e^{-\beta T} & \frac{A}{3C} \cdot \frac{(1-e^{-3T})}{(1-e^{-2T})} e^{-2\gamma T} \left[1 - \left(\frac{1}{4}(1-e^{-4T}) - \right.\right. \\ & \left.\left. - \frac{2A^2}{9} \cdot \frac{(1-e^{-3T})^2}{(1-e^{-2T})} \right) \frac{1}{(U - e^{-2T} V)} \cdot \frac{1}{U - z^{-1} V} \right] \frac{1 - z^{-1} e^{-T}}{1 - z^{-1} e^{-2T}} \\ 0 & \frac{1}{C} \left[\frac{1}{4}(1-e^{-4T}) - \frac{2A^2}{9} \cdot \frac{(1-e^{-3T})^2}{(1-e^{-2T})} \right] \frac{e^{-2\gamma T}}{(U - e^{-2T} V)} \cdot \frac{1}{U - z^{-1} V} \end{bmatrix}.$$

Hence, the overall pulse-transfer-function matrix of the whole system is

$$W_{kl}^{\mathrm{m}}(z) = W_{kj}^{\mathrm{cm}}(z)\, W_{jl}^{\mathrm{i}}(z) =$$

$$= \begin{bmatrix} e^{-\beta T} z^{-\beta} & \frac{A}{3} \cdot \frac{(1-e^{-3T})}{(1-e^{-2T})} e^{-2\gamma T} \left[1 - \left(\frac{1}{4}(1-e^{-4T}) - \right.\right. \\ & \left.\left. - \frac{2A^2}{9} \frac{(1-e^{-3T})^2}{(1-e^{-2T})} \right) \frac{1}{(U - e^{-2T} V)} \cdot \frac{1}{U - z^{-1} V} \right] \frac{1 - z^{-1} e^{-T}}{1 - z^{-1} e^{-2T}} z^{-\gamma} \\ 0 & \left[\frac{1}{4}(1-e^{-4T}) - \frac{2A^2}{9} \cdot \frac{(1-e^{-3T})^2}{(1-e^{-2T})} \right] \frac{e^{-2\gamma T}}{(U - e^{-2T} V)} \cdot \frac{1}{U - z^{-1} V} z^{-\gamma} \end{bmatrix}.$$

In case of $\beta = 0$, $\gamma = 0$ this expression is reduced to the form given for the completely-free configuration in example 1.

Summary

It is shown here how the results obtained for continuous-data multivariable control systems can be generalized for the case of strictly digital or pulsed-data multivariable systems. The inputs and outputs are assumed as pulsed-data stationary ergodic stochastic processes. The optimization criterion is the least sum of the mean-square errors between the sets of actual and ideal outputs. Applying the so-called simplified derivation technique explicit solution formulae can be obtained in a relatively simple way. Thus, this technique which was formerly proposed by the author is now extended for multivariable pulsed-data systems, too. Not only the case of the completely-free configuration but also the cases of the semi-free configuration as well as the case of the semi-free configuration with constraints is investigated. Two simple examples are also given for the sake of illustration.

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