SIMPLIFIED DERIVATION OF THE OPTIMUM MULTIPOLE CASCADE CONTROLLER FOR RANDOM PROCESSES

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1. Introduction

In a previous paper a synopsis was constructed concerned with the statistical design methods for obtaining the optimum transfer functions. This synopsis based on references [1 to 10] can be found in paper [12] as Table 1. The so-called simplified derivation method was also mentioned which is based exclusively on frequency domain notions avoiding convolution integrals, integral equations, as well as variational calculus. As was shown in previous papers [5, 6] using consequently the frequency domain technique explicit solution formulae can readily be obtained in a relatively simple way for the cases of completely free configuration, semi-free configuration and semi-free configuration with constraints. All these formulae are concerned with single-variable systems. On the other hand paper [12] demonstrated, how easily the results of references [7, 10] can be obtained by the generalization of the simplified method for the case of completely free multivariable systems. In this paper the case of semi-free configuration is treated. (The multivariable systems in case of semi-free configuration with constraints will be treated in a following paper, to be published later.) The author pretends to priority only in the consequent application of the simplified method, as the results are similar to the results of references [7] and [13] obtained by integral equations and variational calculus.

As in the references also mentioned in this paper the following assumptions are adopted: The stochastic processes are stationary, the ergodic hypothesis holds and the criterion of performance is the minimization of the sum of the least-mean-square errors between the set of actual outputs and the set of the desired ideal outputs.

In the same way as in paper [12] here also the matrix method is used. According to reference [8] it is assumed that the spectrum factorization of power-spectrum-density matrices can be performed in the practice. Finally, according to the semi-free configuration the adoption is that one part of the control system is fixed and imposes additional restrictions on the overall performance of the system.

2. The proposed method

The convention of notations is the same as in paper [12]. The problem is demonstrated in Fig. 1, where $s_{\cdot k}(t)$ $(k = 1, \ldots K)$ is the row vector composed from the useful signal components, while $n_{\cdot k}(t)$ $(k = 1, \ldots K)$ is the row vector of the corrumping noise signals. On the other hand, $r_{\cdot k}(t)$ being the sum of $s_{\cdot k}(t)$ and $n_{\cdot k}(t)$ is the input row vector. All these and also the other signals are assumed to be stationary ergodic stochastic processes. The fixed part

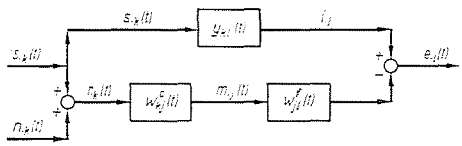

Fig. 1

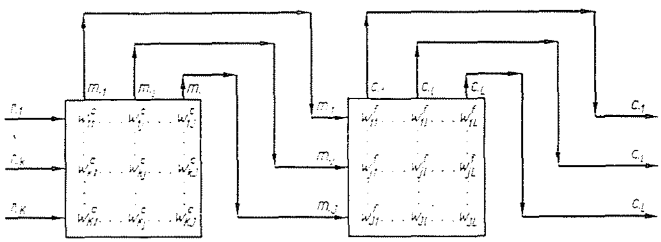

Fig. 2

of the system (the plant) is represented by weighting-function matrix $w^{\mathrm{f}}_{jl}(t)$ $(j = 1, \ldots J\,;\; l = 1, \ldots L)$ while the multipole cascade controller is represented by weighting-function matrix $w^{\mathrm{c}}_{kj}(t)$ $(k = 1, \ldots K\,;\; j = 1, \ldots J)$. This part of the system is detailed in Fig. 2, where $m_{\cdot j}(t)$ $(j = 1, \ldots J)$ represents the row vector of the manipulated variables this being the output of the cascade controller and the input of the plant. Finally, the error is given by the row vector $e_{\cdot l}(t)$ $(l = 1, \ldots L)$ which is the difference of the ideal-signal vector $i_{\cdot l}(t)$ and the real output vector $c_{\cdot l}(t)$. By the way, it should be mentioned that the ideal-signal vector can be obtained from the useful-signal vector $s_{\cdot k}(t)$ through the weighting function matrix $y_{kl}(t)$. This latter can also be physically unrealizable, but $w^{\mathrm{c}}_{kj}(t)$ and $w^{\mathrm{f}}_{jl}(t)$ must be physically realizable in every case. The latter may have non-minimum phase properties, too.

Adopting as minimization criterion the sum of the mean-square error components this latter can be written as the mean value of the trace (i.e. of the sum of the diagonal elements) of the matrix composed of the matrix multiplication of column vector $e_{l.}(t)$ and row vector $e_{.l}(t)$:

$$\mathrm{tr}\,[\overline{e_{l.}(t)\,e_{.l}(t)}] = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \mathrm{tr}\,[e_{l.}(t)\,e_{.l}(t)]\,dt \tag{1}$$

As is well known, this latter can also be expressed by the power-density spectrum as

$$\mathrm{tr}\,[\overline{e_{l.}(t)\,e_{.l}(t)}] = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \mathrm{tr}\,[\Phi_{e_{l'}e_l}(s)]\,ds \tag{2}$$

where $s = j\omega$. The problem is now reduced to the minimization of the integral in Eq. (2) i.e. it is necessary to find the minimalizing trace of the power-spectrum-density matrix. Since the latter is a function of s^2 or ω^2, the complex-variable integral can be reduced to a real-variable integral. As the following relations are valid

$$e_{.l}(t) = i_{.l}(t) - c_{.l}(t);\;\; e_{l.}(t) = i_{l.}(t) - c_{l.}(t) \tag{3}$$

thus,

$$\overline{e_{l.}(t)\,e_{.l}(t)} = \overline{i_{l.}(t)\,i_{.l}(t) - i_{l.}(t)\,c_{.l}(t) - c_{l.}(t)\,i_{.l}(t) + c_{l.}(t)\,c_{.l}(t)} = \tag{4}$$

$$= \varphi_{i_{l'}i_l}(0) - \varphi_{i_{l'}c_l}(0) - \varphi_{c_{l'}i_l}(0) + \varphi_{c_{l'}c_l}(0)$$

where $\varphi_{i_{l'}i_l}(0)$, $\varphi_{i_{l'}c_l}(0)$, $\varphi_{c_{l'}i_l}(0)$ and $\varphi_{c_l,c_l}(0)$ denote the corresponding correlation-function matrices after substituting for the shifting time τ the value $\tau = 0$.

The power-density-spectrum matrix in question can be expressed as:

$$\Phi_{e_{l'}e_l}(s) = \Phi_{i_{l'}i_l}(s) - \Phi_{i_{l'}c_l}(s) - \Phi_{c_{l'}i_l}(s) + \Phi_{c_{l'}c_l}(s) \tag{5}$$

Taking the generalization of the index-changing rule [11] according to Fig. 1 into consideration the power-density-spectrum matrix in Eq. (5) can also be expressed as:

$$\begin{aligned}\Phi_{e_{l'}e_l}(s) = {} & \Phi_{i_{l'}i_l}(s) - \Phi_{i_{l'}r_k}(s)\,W^{c}_{kj}(s)\,W^{\mathfrak{f}}_{jl}(s) - \\ & - W^{\mathfrak{f}}_{l'j'}(-s)\,W^{c}_{j'k'}(-s)\,\Phi_{r_{k'}i_l}(s) + \\ & + W^{\mathfrak{f}}_{l'j'}(-s)\,W^{c}_{j'k'}(-s)\,\Phi_{r_{k'}r_k}(s)\,W^{c}_{kj}(s)\,W^{\mathfrak{f}}_{jl}(s)\end{aligned}$$

$$(k, k' = 1, \ldots K;\;\; j, j' = 1, \ldots J;\;\; l, l' = 1, \ldots L) \tag{6}$$

where $W^{c}_{kj}(s)$ and $W^{\mathfrak{f}}_{jl}(s)$ are the transfer-function matrices of the controller and the plant, respectively, corresponding to the weighting function matrices of the controller and the plant, respectively, and obtained from the latter by Fourier or Laplace transformation. When constructing expression (6) and applying the index-changing rule, it must be taken into consideration that the signal $r_{.k}(t)$ first goes through the controller with weighting-function matrix $w_{kj}^{c}(t)$ and transfer-function matrix $W_{kj}^{c}(s)$ (which are $K \times J$ matrices) and only thereafter through the plant with weighting-function matrix $w_{jl}^{f}(t)$ and transfer-function matrix $W_{jl}^{f}(s)$ which are $J \times L$ matrices.

It must be emphasized that expression (6) is the direct generalization of Eq. (12) figuring in reference [12]. In expression (6) $\Phi_{e_l e_l}(s)$ and $\Phi_{i_l i_l}(s)$ are $L \times L$ matrices, while $\Phi_{i_{l'} r_k}(s)$ is a $L \times K$ matrix and $\Phi_{r_k i_l}(s)$ is $K \times L$ matrix, finally $\Phi_{r_{k'} r_k}(s)$ is a $K \times K$ matrix.

Let us now introduce an auxiliary $K \times J$ matrix $G_{kj}^{c}(s)$ and its transposed conjugate complex (that is its adjoint) matrix $G_{j'k'}^{c}(-s)$ by the following two relations:

$$\Phi_{k' r_k}(s)\, G_{kj}^{c}(s)\, W_{jl}^{f}(s) = \Phi_{r_k i_l}(s) \tag{7}$$

and

$$W_{l'j'}^{f}(-s)\, G_{j'k'}^{c}(-s)\, \Phi_{r_{k'} r_k}(s) = \Phi_{i_{l'} r_k}(s)$$

where the elements of the transfer-function matrix $G_{kj}^{c}(s)$ are not always physically realizable. As here the matrices figuring in Eqs. (7) are known therefore the auxiliary matrices can also be considered as known.

Taking into consideration Eqs. (7) expression (6) can also be written as:

$$\begin{aligned}\Phi_{e_l e_l}(s) = \Phi_{i_l i_l}(s) - W_{l'j'}^{f}(-s)\, G_{j'k'}^{c}(-s)\, \Phi_{r_{k'} r_k}(s)\, G_{kj}^{c}(s)\, W_{jl}^{f}(s) + \\ + [W_{l'j'}^{f}(-s)\, G_{j'k'}^{c}(-s) - W_{l'j'}^{f}(-s)\, W_{j'k'}^{c}(-s)] \times \\ \times \Phi_{r_{k'} r_k}(s)\, [G_{kj}(s)\, W_{jl}^{f}(s) - W_{kj}^{c}(s)\, W_{jl}^{f}(s)]\end{aligned} \tag{8}$$

The transfer-function matrix $W_{kj}^{c}(s)$ and its adjoint matrix (which is nothing else but the complex-conjugate of its transposed matrix) $W_{j'k'}^{c}(-s)$ are contained only in the last term of Eq. (8). The trace of the power-density-spectrum matrix will be minimum if this last term became zero. The sufficent and necessary conditions are evidently:

$$\begin{aligned}W_{kj}^{co}(s) = G_{kj}^{c}(s) \\ W_{j'k'}^{co}(-s) = G_{j'k'}^{c}(-s)\end{aligned} \tag{9}$$

Substituting Eqs. (9) in Eqs. (7) we obtain:

$$\begin{aligned}\Phi_{r_{k'} r_k}(s)\, W_{kj}^{co}(s)\, W_{jl}^{f}(s) = \Phi_{r_k i_l}(s) \\ W_{l'j'}^{f}(-s)\, W_{j'k'}^{co}(-s)\, \Phi_{r_{k'} r_k}(s) = \Phi_{i_{l'} r_k}(s)\end{aligned} \tag{10}$$

Thus, the physically unrealizable optimum transfer-function matrix is:

$$W_{kj}^{co}(s) = [\Phi_{r_{k'} r_k}(s)]^{-1}\, \Phi_{r_{k'} i_l}(s)\, [W_{jl}^{f}(s)]^{-1} \tag{11}$$

or its adjoint matrix is:

$$W_{j'k'}^{co}(-s) = [W_{l'j'}^{f}(-s)]^{-1}\, \Phi_{i_{l'} r_k}(s)\, [\Phi_{r_{k'} r_k}(s)]^{-1}$$

By the way, it is necessary to have $J = L$ for the inversion of the transfer-function matrices $W^{f}_{jl}(s)$ and $W^{f}_{l'j'}(-s)$ figuring here.

Now the question arises how to obtain the physically realizable transfer-function matrix. If, instead of the physically unrealizable transfer-function matrix $W^{co}_{kj}(s)$ the physically realizable transfer-function matrix $W^{cm}_{kj}(s)$ is applied then instead of the first equation of Eqs. (10) the following will be valid

$$\Phi_{r_k r_k}(s) W^{cm}_{kj}(s) W^{f}_{jl}(s) W^{f}_{l'j'}(-s) = \\ = \Phi_{r_{k'} i_l}(s) W^{f}_{l'j'}(-s) + F^{c}_{k'j'}(s) \tag{12}$$

where $F^{c}_{k'j'}(s)$ is a still unknown matrix with transfer-function elements having only right-half-plane poles. This equation is the direct generalization of Eq. (20) in reference [12]. It must be noted that the matrix factor $W^{f}_{l'j'}(-s)$ is inevitable in Eq. (12) as $W^{f}_{jl}(s)\, W^{f}_{l'j'}(-s)$ must be treated as a power-density-spectrum matrix.

Now let us introduce the following spectrum-factorization relations:

$$\Phi^{-}_{r_k r_{k''}}(s)\, \Phi^{+}_{r_{k''} r_k}(s) = \Phi_{r_{k'} r_k}(s) \tag{13}$$

and

$$\left(W^{f}_{jl}(s) W^{f}_{l'j'}(-s)\right) = \left(W^{f}_{jl}(s) W^{f}_{l'j'}(-s)\right)^{+} \left(W^{f}_{jl}(s) W^{f}_{l'j'}(-s)\right)^{-} \tag{14}$$

where the upper index, that is, the superscript — (minus), denotes a matrix factor whose elements and the elements of its inverse matrix have only right-half-plane poles and zeros, while the upper index + (plus) denotes a matrix factor whose elements together with the elements of its inverse matrix have only left-half-plane poles and zeros. Thus Eq. (12) can be written in the following form:

$$\Phi^{+}_{r_{k'} r_k}(s) W^{cm}_{kj}(s) \left(W^{f}_{jl}(s) W^{f}_{l'j}(-s)\right)^{+} = \\ = \left[\Phi^{-}_{r_{k'} r_{k''}}(s)\right]^{-1} \Phi_{r_k i_l}(s) W^{f}_{l'j'}(-s) \left[\left(W^{f}_{jl}(s) W^{f}_{l'j'}(-s)\right)^{-}\right]^{-1} + \\ + \left[\Phi^{-}_{r_{k'} r_{k''}}(s)\right]^{-1} F^{c}_{k'j'}(s) \left[\left(W^{f}_{jl}(s) W^{f}_{l'j'}(-s)\right)^{-}\right]^{-1} \tag{15}$$

Separating the physically realizable and unrealizable matrix components on both sides of Eq. (15) the following two relations can be obtained:

$$\Phi^{+}_{r_{k'} r_k}(s) W^{cm}_{kj}(s) \left(W^{f}_{jl}(s) W^{f}_{l'j'}(-s)\right)^{+} = \\ = \left\{\left[\Phi^{-}_{r_{k'} r_{k''}}(s)\right]^{-1} \Phi_{r_k i_l}(s) W^{f}_{l'j'}(-s) \times \\ \times \left[\left(W^{f}_{jl}(s) W^{f}_{l'j'}(-s)\right)^{-}\right]^{-1}\right\}_{+} \tag{16}$$

and

$$0 = \left\{\left[\Phi^{-}_{r_{k'} r_{k''}}(s)\right]^{-1} \Phi_{r_{k'} i_l}(s) W^{f}_{l'j'}(-s) \times \\ \times \left[\left(W^{f}_{jl}(s) W^{f}_{l'j'}(-s)\right)^{-}\right]^{-1}\right\}_{-} + \\ + \left[\Phi^{-}_{k' r_{k''}}(s)\right]^{-1} F^{c}_{k'j'}(s) \left[\left(W^{f}_{jl}(s) W^{f}_{l'j'}(-s)\right)^{-}\right]^{-1} \tag{17}$$

where the lower index that is, the subscript + (plus) denotes a matrix *component* with physically realizable elements, while the lower index — (minus) denotes a matrix component with physically unrealizable elements.

Finally, from Eq. (16) the physically realizable optimum transfer-function matrix of the cascade controller can be expressed as

$$W_{kj}^{\mathrm{cm}}(s) = [\Phi_{r_{k''}r_k}^{+}(s)]^{-1} \times$$
$$\times \left\{ [\Phi_{r_{k'}r_{k''}}^{-}(s)]^{-1} \Phi_{r_{k'}i_l}(s) W_{l'j'}^{\mathrm{f}}(-s) \left[(W_{jl}^{\mathrm{f}}(s) W_{l'j'}^{\mathrm{f}}(-s))^{-} \right]^{-1} \right\}_{+} \times$$
$$\times \left[(W_{jl}^{\mathrm{f}}(s) W_{l'j'}^{\mathrm{f}}(-s))^{+} \right]^{-1} \tag{18}$$

This is the final explicit solution formula for the case of semi-free configuration.

Substituting the obtained expression of $W_{kj}^{\mathrm{cm}}(s)$ in Eq. (6) or in Eq. (8) the power-density-spectrum matrix can be obtained and by formula (2) the mean-square error can be computed.

3. Some special cases

From the general explicit formula (18) also some special formulae can be obtained.

If the elements of the plant matrix are of minimum phase, then

$$(W_{jl}^{\mathrm{f}}(s) W_{l'j'}^{\mathrm{f}}(-s))^{-} = W_{l'j'}^{\mathrm{f}}(-s)$$
$$(W_{jl}^{\mathrm{f}}(s) W_{l'j'}^{\mathrm{f}}(-s))^{+} = W_{lj}^{\mathrm{f}}(s) \tag{19}$$

and in this case formula (18) can be simplified. The optimum transfer-function matrix of the multivariable cascade controller is:

$$W_{kj}^{\mathrm{cm}}(s) = [\Phi_{r_{k''}r_{k'}}^{+}(s)]^{-1} \times$$
$$\times \left\{ [\Phi_{r_{k'}r_{k''}}^{-}(s)]^{-1} \Phi_{r_k i_l}(s) \right\}_{+} [W_{jl}^{\mathrm{f}}(s)]^{-1} \tag{20}$$

In other words,

$$W_{kj}^{\mathrm{cm}}(s) = W_{kl}^{\mathrm{m}}(s) [W_{jl}^{\mathrm{f}}(s)]^{-1} \tag{21}$$

where $W_{kl}^{\mathrm{m}}(s)$ is given by formula (23). Eq. (21) signifies nothing else, than for minimum-phase systems the optimum transfer-function matrix of the cascade controller can simply be obtained by multiplying the optimum overall transfer-function matrix with the inverse matrix of the plant.

If on the other hand the system is of completely free configuration, then the following substitution can be made:

$$W_{lj}^{\mathrm{f}}(s) = I$$

and

$$W_{kj}^{\mathrm{cm}}(s) = W_{kl}^{\mathrm{m}}(s) \qquad (j = l) \tag{22}$$

Thus, from Eq. (20):

$$W^{\mathrm{m}}_{kl}(s) = \left[\Phi^{+}_{r_{k''}r_{k'}}(s)\right]^{-1} \left\{\left[\Phi^{-}_{r_{k'}r_{k''}}(s)\right]^{-1} \Phi_{r_{k'}i_l}(s)\right\}_{+} \tag{23}$$

which formula completely corresponds to formula (34) in reference [12]. The latter expression is the explicit solution for the optimum transfer-function matrix of the whole system.

If we had a single-variable system, then from expression (18) the following formula can be obtained:

$$W^{\mathrm{cm}}(s) = \frac{\left[\dfrac{\Phi_{ri}(s)\, W^{\mathrm{f}}(-s)}{\Phi^{-}_{rr}(s)\left[W^{\mathrm{f}}(s)\, W^{\mathrm{f}}(-s)\right]^{-}}\right]_{+}}{\Phi^{+}_{rr}(s)\left[W^{\mathrm{f}}(s)\, W^{\mathrm{f}}(-s)\right]^{+}} \tag{24}$$

This expression of the cascade controller is quite similar to formula (25) in reference [6].

If we had a single-variable system with a fixed element of minimum phase then formula (24) could be reduced to

$$W^{\mathrm{cm}}(s) = \frac{\left[\dfrac{\Phi_{ri}(s)}{\Phi^{-}_{rr}(s)}\right]_{+}}{\Phi^{+}_{rr}(s)\, W^{\mathrm{f}}(s)} \tag{25}$$

as relations (19) are also valid for 1×1 matrices, that is, for scalars.

Finally, if the single-variable system is of completely free configuration, then formula (23) can be reduced as follows:

$$W^{\mathrm{m}}(s) = \frac{\left[\dfrac{\Phi_{ri}(s)}{\Phi^{-}_{rr}(s)}\right]_{+}}{\Phi^{+}_{rr}(s)} \, . \tag{26}$$

This is the same expression as in reference [6] formula (15).

4. Examples

For the sake of simplicity the same initial data are assumed as in the illustrative example of reference [12], that is:

$$\Phi_{s_{k'}s_k}(s) = \begin{bmatrix} \dfrac{4}{1-s^2} & \dfrac{2A}{(1-s)(2+s)} \\ \dfrac{2A}{(2-s)(1+s)} & \dfrac{1}{4-s^2} \end{bmatrix}$$

$$\Phi_{n_{k'}n_k}(s) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \Phi_{n_{k'}s_k}(s) = \Phi_{s_{k'}n_k}(s) = 0$$

thus,

$$\Phi_{r_k \cdot r_k}(s) = \begin{bmatrix} \dfrac{4}{1-s^2} & \dfrac{2A}{(1-s)(2+s)} \\ \dfrac{2A}{(2-s)(1+s)} & \dfrac{5-s^2}{4-s^2} \end{bmatrix}.$$

On the other hand:

$$\Phi_{r_k \cdot i_l}(s) = [\Phi_{s_k \cdot s_k}(s) + \Phi_{n_k \cdot k}(s)]\, Y_{kl}(s) = \Phi_{s_k \cdot s_k}(s)$$

as

$$Y_{kl}(s) = I$$

First, let us consider a plant of minimum-phase property. Let it be the transfer-function matrix of the fixed part of the system:

$$W^{\mathrm{f}}_{jl}(s) = \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{1+s}{2+s} \end{bmatrix}.$$

Its inverse matrix is:

$$[W^{\mathrm{f}}_{jl}(s)]^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{2+s}{1+s} \end{bmatrix}.$$

Now, the transfer-function matrix of the physically realizable optimum cascade controller can be determined by formula (21):

$$W^{\mathrm{cm}}_{kj}(s) = W^{\mathrm{m}}_{kl}(s)\, [W^{\mathrm{f}}_{jl}(s)]^{-1} =$$

$$= \begin{bmatrix} 1 & \dfrac{A}{2}\left[1 - \dfrac{1-A^2}{(\sqrt{5-A^2}+2)(\sqrt{5-A^2}+s)}\right] \\ 0 & \dfrac{2+s}{1+s}\,\dfrac{1-A^2}{(\sqrt{5-A^2}+2)(\sqrt{5-A^2}+s)} \end{bmatrix}$$

since

$$W^{\mathrm{m}}_{kl}(s) = \begin{bmatrix} 1 & \dfrac{A}{2}\,\dfrac{1+s}{2+s}\left[1 - \dfrac{1-A^2}{(\sqrt{5-A^2}+2)(\sqrt{5-A^2}+s)}\right] \\ 0 & \dfrac{1-A^2}{(\sqrt{5-A^2}+2)(\sqrt{5-A^2}+s)} \end{bmatrix}$$

was already determined in reference [12].

Because the overall optimum transfer-function matrix is the same as in the illustrative example of reference [12] the mean-square errors must also be the same, that is, their sum is:

$$\left[1 + \frac{1-A^2}{(\sqrt{5-A^2}+2)^2}\right] \frac{1-A^2}{2\sqrt{5-A^2}}$$

As a second simple illustrative example, let us see the following plant matrix of non-minimum phase:

$$W^{f}_{jl}(s) = \begin{bmatrix} Be^{-bs} & 0 \\ 0 & Ce^{-cs} \end{bmatrix}.$$

Now the complicated formula (18) must be applied. The following details are calculated; first

$$\Phi_{r_k\, i_l}(s)\, W^{f}_{lj}(-s) = \begin{bmatrix} \dfrac{4}{1-s^2} Be^{bs} & \dfrac{2A}{(1-s)(2+s)} Ce^{cs} \\ \dfrac{2A}{(2-s)(1+s)} Be^{bs} & \dfrac{1}{4-s^2} Ce^{cs} \end{bmatrix}$$

then

$$\left(W^{f}_{jl}(s)\, W^{f}_{l'j'}(-s)\right) = \begin{bmatrix} B^2 & 0 \\ 0 & C^2 \end{bmatrix}$$

thus

$$\left(W^{f}_{jl}(s)\, W^{f}_{l'j'}(-s)\right)^{-} = \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix} = \left(W^{f}_{jl}(s)\, W^{f}_{l'j'}(-s)\right)^{+}$$

and

$$\left[\left(W^{f}_{jl}(s) W^{f}_{l'j'}(-s)\right)^{-}\right]^{-1} = \begin{bmatrix} \dfrac{1}{B} & 0 \\ 0 & \dfrac{1}{C} \end{bmatrix} = \left[\left(W^{f}_{jl}(s) W^{f}_{l'j'}(-s)\right)^{+}\right]^{-1}$$

As

$$\left[\Phi^{-}_{r_k\, r_{k'}}(s)\right]^{-1} = \begin{bmatrix} \dfrac{1-s}{2} & 0 \\ -\dfrac{A}{2}\, \dfrac{1-s}{\sqrt{5-A^2}-s} & \dfrac{2-s}{\sqrt{5-A^2}-s} \end{bmatrix}$$

has been already computed in reference [12], the expression in brackets can now be determined without any difficulty:

$$\left\{\left[\Phi^{-}_{r_{k'}\, k'}(s)\right]^{-1} \Phi_{r_{k'} i_l}(s)\, W^{f}_{l'j'}(-s) \left[\left(W^{f}_{jl}(s)\, W^{f}_{l'j'}(-s)\right)^{-}\right]^{-1}\right\} =$$

$$= \begin{bmatrix} \dfrac{2}{1+s} e^{bs} & \dfrac{A}{2+s} e^{bs} \\ 0 & \dfrac{1-A^2}{(\sqrt{5-A^2}-s)(2+s)} e^{cs} \end{bmatrix}.$$

In case of $b \geqq 0$, $c \geqq 0$, the physically realizable component of the latter is:

$$\{\ \}_+ = \begin{bmatrix} \dfrac{2}{1+s}\, e^{-b} & \dfrac{A}{2+s}\, e^{-b} \\ 0 & \dfrac{1-A^2}{(\sqrt{5-A^2}+2)(2+s)}\, e^{-2c} \end{bmatrix}.$$

Finally, taking into consideration the expression:

$$[\Phi^{+}_{r_k \cdot r_{k'}}(s)]^{-1} = \begin{bmatrix} \dfrac{1+s}{2} & -\dfrac{A}{2}\,\dfrac{1+s}{\sqrt{5-A^2}+s} \\ 0 & \dfrac{2+s}{\sqrt{5-A^2}+s} \end{bmatrix}$$

after all the transfer-function matrix of the physically realizable optimum cascade controller can be determined as:

$$W^{\mathrm{cm}}_{kj}(s) = \begin{bmatrix} \dfrac{1}{B}\, e^{-b} & \dfrac{A}{2C}\,\dfrac{1+s}{2+s}\left[e^{-2b} - \dfrac{1-A^2}{\sqrt{5-A^2}+2}\, e^{-2c}\,\dfrac{1}{\sqrt{5-A^2}+s}\right] \\ 0 & \dfrac{1-A^2}{C(\sqrt{5-A^2}+2)}\, e^{-2c}\,\dfrac{1}{\sqrt{5-A^2}+s} \end{bmatrix}.$$

Thus, the overall transfer-function matrix of the whole system is the following:

$$W^{\mathrm{m}}_{kl}(s) = W^{\mathrm{cm}}_{kj}(s)\, W^{\mathrm{f}}_{jl}(s) =$$

$$= \begin{bmatrix} e^{-b}\, e^{-bs} & \dfrac{A}{2}\,\dfrac{1+s}{2+s}\left[e^{-2b} - \dfrac{1-A^2}{\sqrt{5-A^2}+2}\, e^{-2c}\,\dfrac{1}{\sqrt{5-A^2}+s}\right] e^{-cs} \\ 0 & \dfrac{1-A^2}{\sqrt{5-A^2}+2}\, e^{-2c}\,\dfrac{1}{\sqrt{5-A^2}+s}\; e^{-cs} \end{bmatrix}.$$

In case of $b = 0$, $c = 0$ this expression can easily be reduced to the form given for the completely free configuration in the illustrative example of reference [12].

Summary

For stationary ergodic stochastic processes and taking as performance criterion the sum of the least-mean-square errors between the sets of actual and ideal outputs an explicit formula is derived for the multivariable optimum cascade controller. Only frequency domain technique: the so-called simplified method is used in connection with matrix calculus. Two simple illustrative examples are also given. Finally it is shown, how the general formula can be reduced for some special cases.

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