SIMPLIFIED DERIVATION OF THE OPTIMUM MULTIPOLE CASCADE CONTROLLER FOR MULTIVARIABLE SYSTEMS WITH CONSTRAINTS

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1. Introduction

This report is the direct continuation of two papers published previously in this periodical [12, 14] which are concerned with the optimum design of continuous linear multivariable control systems for stationary ergodic stochastic signals. This paper also applies the so-called simplified derivation technique presented first in references [5, 6] in connection with single variable systems. The purpose of this study is to demonstrate how the problem of the semi-free configuration with constraints defined first in book [2] and simplified derived in references [5, 6] can be generalized from single variable systems to multivariable control systems.

The advantages of the proposed method are ever better thrown into relief as the configuration of the systems becomes more complicated. While the papers mentioned before [5, 6, 12, 14] and based merely on frequency domain notions have only been arrived in a much simpler way at the well-known results as given, for example, in [1, 2, 3, 4] and [7, 9, 10, 13], the present paper examines the somewhat more complicated case of semi-free configuration with constraints which, according to the authors knowledge, is up till now not treated as a whole generality for multivariable systems. Thus, the present results can be considered as conform to the priority. It must be mentioned, however, that in a report on the second IFAC Congress [15] a dual-input system with one saturation constraint was analyzed. But this problem is only a special case of the multivariable systems with many constraints. On the occasion of the discussion of report [15] the author of the present paper has demonstrated briefly the main results of the general case. These results and the details are to be found here now.

As in papers [12, 14] here also the matrix method of computation is used. According to reference [8] it is assumed that the spectrum factorization of power-spectrum-density matrices can be performed. It must be noted, however, that in our case the spectrum factorization will be more complicated, because one or more parameters also figure in the corresponding power-spectrum-density matrix to be factorized. According to the case of the semi-free configuration with constraints, it is so adopted that one part of the control system is fixed, this is for example the *plant*, while the other part of the system i.e. the *controller* must be designed according to the least-mean-square error criterion, the error being taken between the actual and the ideal (or desired) outputs. In case of multivariable systems, this criterion means the least mean value of the sum of squared error components. For the sake of simplicity, first it is also assumed that even the manipulated variables acting between the controller and the plant are either directly or indirectly submitted to constraints.

2. The proposed method

The convention of notations is the same as in papers [12, 14]. The problem is depicted in *Fig. 1.* All the signals (the variables) in the control system are assumed to be stationary ergodic stochastic processes. Here $s_{\cdot k}(t)$ is the row vector composed of the useful signal components, while $n_{\cdot k}(t)$ is the row vector of the corrumping noise components ($k = 1, \ldots K$). Their sum forms the input row vector $r_{\cdot k}(t)$, being $r_{\cdot k}(t) = s_{\cdot k}(t) + n_{\cdot k}(t)$. The complete input signal first penetrates the cascade controller, the latter being represented by weighting-function matrix $w^{c}_{kj}(t)$ ($k = 1, \ldots K$; j $= 1, \ldots$ J). The outputs of the controller are the manipulated variables. Taking the latter ones as components the row vector $m_{\cdot j}(t)$ of the manipulated variables can be formed ($j = 1, \ldots$ J). It is assumed that even the manipulated variables are submitted to constraints. In general an indirect manner can be taken as a basis. For this purpose some constraint weighting-function matrix $w^{k}_{jh}(t)$ is constructed ($j = 1, \ldots$ J; $h = 1, \ldots H$; $H \leqq L$) . The output row vector $b_{\cdot h}(t)$ ($h = 1, \ldots H$) of this transfer link represents the indirect variables: the so-called modified manipulated variables to be constrained.

Let us assume that the sum of the mean-square values of the indirect variables is limited. This condition the so-called *unequality of constraint* can be expressed as follows

$$\operatorname{tr}\left[\overline{b_{h\cdot}(t)\, b_{\cdot h}(t)}\right] = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \operatorname{tr}\left[\overline{b_{h\cdot}(t)\, b_{\cdot h}(t)}\right] dt = \operatorname{tr}\left[\varphi_{b_{h\cdot}\, b_h}(0)\right] \leqq \sigma^2 \qquad (1)$$

Here $b_{h\cdot}(t)\; b_{\cdot h}(t)$ is a symmetrical matrix composed of the matrix multiplication of column vector $b_{h\cdot}(t)$ and row vector $b_{\cdot h}(t)$, while "tr" denotes the trace that is the sum of the diagonal elements of the matrix. The latter can also be expressed by a correlation matrix with zero shifting time.

As is well known, the unequality of constraint can also be expressed by the power-density-spectrum matrix as

$$\operatorname{tr}\left[\overline{b_{h.}(t)\, b_{.h}(t)}\right] = \operatorname{tr}\left[\varphi_{b_{h'} b_h}(0)\right] = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \operatorname{tr}\left[\Phi_{b_{h'} b_h}(s)\right] ds \leqq \sigma^2 \qquad (2)$$

where $s = j\omega$ and $h, h' = 1, \ldots H$.

Returning again to Fig. 1 it can be observed that the manipulated variables enter into the fixed part of the system that is into the plant, while the outputs of the latter are the controlled variables. The plant is represented by weighting-function matrix $w_{jl}^{f}(t)$ $(j = 1, \ldots J;\ l = 1, \ldots L)$.

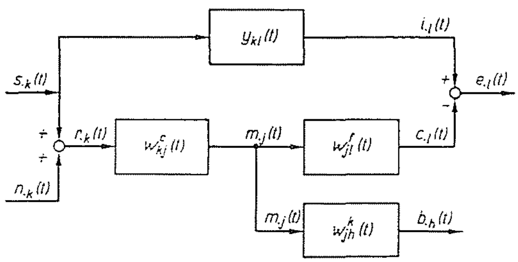

Fig. 1

From the controlled variables as components the row vector $c_{.l}(t)$ is constructed $(l = 1, \ldots L)$.

The row vector of the error $e_{.l}(t)$ is nothing else but the difference of the ideal or desired signal vector $i_{.l}(t)$ and the actual output vector $c_{.l}(t)$ $(l = 1, \ldots L)$. If needed the ideal output vector $i_{.l}(t)$ can be obtained from the useful signal vector $s_{.k}(t)$ by the weighting-function matrix $y_{kl}(t)$ which can exceptionally be physically unrealizable.

Now let us adopt as minimization criterion the *sum* of the mean-square-error components. This latter can be expressed as the mean value of the trace of the matrix composed of the matrix multiplication of column vector $e_{.l}(t)$ and row vector $e_{.l}(t)$ and obviously can also be expressed by the corresponding correlation matrix or power-density-spectrum matrix:

$$\operatorname{tr}\left[\overline{e_{l.}(t)\, e_{.l}(t)}\right] = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \operatorname{tr}\left[\overline{e_{l.}(t)\, e_{.l}(t)}\right] dt =$$
$$= \operatorname{tr}\left[\varphi_{e_{l'} e_l}(0)\right] = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \operatorname{tr}\left[\Phi_{e_{l'} e_l}(s)\right] ds. \qquad (3)$$

Applying the Lagrangean conditional extremum technique our problem is reduced to the minimization of the following expression:

$$\begin{aligned} \mathrm{tr}\,[\varphi_{x_{l},x_{l}}(0)] &= \mathrm{tr}\left[\overline{x_{l.}(t)\ x_{.l}(t)}\right] = \mathrm{tr}\left[\overline{e_{l.}(t)\ e_{.l}(t)}\right] + \lambda\ \mathrm{tr}\left[\overline{b_{h.}(t)\ b_{.h}(t)}\right] = \\ &= \mathrm{tr}\left[\overline{e_{l.}(t)\ e_{.l}(t)} + \lambda\ \overline{b_{h.}(t)\ b_{.h}(t)}\right] = \mathrm{tr}\ [\varphi_{e_{l},e_{l}}(0) + \lambda\varphi_{b_{h},b_{h}}(0)] \end{aligned} \qquad (4)$$

The latter can also be expressed as

$$\mathrm{tr}\ [\varphi_{x_{l},x_{l}}(0)] = \mathrm{tr}\left[\overline{x_{l.}(t)\ x_{.l}(t)}\right] = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \mathrm{tr}\ [\Phi_{e_{l},e_{l}}(s) + \lambda\Phi_{b_{h},b_{h}}(s)]\,ds \qquad (5)$$

Therefore, the task in question is the minimization of the integral on the right side of Eq. (5) or in other words it is necessary to find the minimalizing trace of the resultant power-density spectrum matrix

$$\Phi_{x_{l},x_{l}}(s,\lambda) = \Phi_{e_{l},e_{l}}(s) + \lambda\Phi_{b_{h},b_{h}}(s) \qquad (6)$$

which is a function of the variable s and parameter λ. As naturally, both the power-density-spectrum matrices in Eq. (6) are only functions of s^2 or ω^2, thus the complex variable integral in Eq. (5) can readily be reduced to a real-variable integral.

Evidently the following relation is valid [12, 14]:

$$\Phi_{e_{l},e_{l}}(s) = \Phi_{i_{l},i_{l}}(s) - \Phi_{i_{l},c_{l}}(s) - \Phi_{c_{l},i_{l}}(s) + \Phi_{c_{l},c_{l}}(s) \qquad (7)$$

Applying the generalization of the index-change rule [11] and taking Fig. 1. into consideration the latter matrix can also be expressed as

$$\begin{aligned} \Phi_{e_{l'},e_{l}}(s) = \Phi_{i_{l'},i_{l}}(s) &- \Phi_{i_{l'},r_{k}}(s)\ W^{c}_{kj}(s)\ W^{f}_{jl}(s) - \\ &- W^{f}_{l'j'}(-s)\ W^{c}_{j'k'}(-s)\ \Phi_{r_{k'},i_{l}}(s) + \\ &+ W^{f}_{l'j'}(-s)\ W^{c}_{j'k'}(s)\ \Phi_{r_{k'},r_{k}}(s)\ W^{c}_{kj}(s)\ W^{f}_{jl}(s) \end{aligned} \qquad (8)$$

and similarly

$$\Phi_{b_{h'},b_{h}}(s) = W^{k}_{h'j'}(-s)\ W^{c}_{j'k'}(-s)\ \Phi_{r_{k'},r_{k}}(s)\ W^{c}_{kj}(s)\ W^{k}_{jh}(s) \qquad (9)$$

($k, k' = 1, \ldots K$; $j, j' = 1, \ldots J$; l, $l' = 1, \ldots L$; $h, h' = 1, \ldots H$) where $W^{c}_{kj}(s)$, $W^{f}_{jl}(s)$ and $W^{k}_{jh}(s)$ are the transfer-function matrices of the controller, the plant and the constraint, respectively, determined from the corresponding weighting-function matrices by Fourier or Laplace transformation. Transfer function matrices $W^{c}_{j'k'}(-s)$, $W^{f}_{l'j'}(-s)$ and $W^{k}_{h'j'}(-s)$ are the adjoint, that is, the conjugate complex transposed matrices of transfer-function matrices $W^{c}_{kj}(s)$, $W^{f}_{jl}(s)$ and $W^{k}_{jh}(s)$ these latter being $K \times J$, $J \times L$ and $J \times H$ matrices, respectively. In most cases $K = J = L \geqq H$ can be assumed without loss of generality.

Let us now introduce an auxiliary power-density-spectrum matrix $\Phi_{a_{k'}a_k}(s, \lambda)$ implicitly defined in the following relation:

$$W^{f}_{l'j'}(-s)\, W^{c}_{j'k'}(-s)\, \Phi_{r_{k'}r_k}(s)\, W^{c}_{kj}(s)\, W^{f}_{jl}(s) + \\ + \lambda\, W^{k}_{h'j'}(-s)\, W^{c}_{j'k'}(-s)\, \Phi_{r_{k'}r_k}(s)\, W^{c}_{kj}(s)\, W^{k}_{jh}(s) = \\ = W^{f}_{l'j'}(-s)\, W^{c}_{j'k'}(-s)\, \Phi_{a_{k'}a_k}(s, \lambda)\, W^{c}_{kj}(s)\, W^{f}_{jl}(s) \tag{10}$$

With the aid of inverse matrices the auxiliary power-density spectrum matrix can, of course, also be expressed explicitly

$$\Phi_{a_{k'}a_k}(s, \lambda) = \Phi_{r_{k'}r_k}(s) + \lambda\, [W^{c}_{j'k'}(-s)]^{-1}\, [W^{f}_{l'j'}(-s)]^{-1} \times \\ \times W^{k}_{h'j'}(-s)\, W^{c}_{j'k'}(-s)\, \Phi_{r_{k'}r_k}(s)\, W^{c}_{kj}(s)\, W^{k}_{jh}(s)\, [W^{f}_{jl}(s)]^{-1}\, [W^{c}_{kj}(s)]^{-1} \tag{11}$$

It can be shown that the auxiliary power-density spectrum matrix $\Phi_{a_{k'}a_k}(s, \lambda)$ is uniquely determined by Eq. (10) or (11) and this matrix does not depend on the choice of transferfunction matrix $W^{c}_{kj}(s)$ of the controller. Now, taking Eqs. (8), (9) and (10) into consideration the power-density-spectrum matrix figuring in Eq. (6) can be expressed as

$$\Phi_{x_{l'}x_l}(s, \lambda) = \Phi_{i_{l'}i_l}(s) - \Phi_{i_{l'}r_k}(s)\, W^{c}_{kj}(s)\, W^{f}_{jl}(s) - \\ - W^{f}_{l'j'}(-s)\, W^{c}_{j'k'}(-s)\, \Phi_{r_{k'}i_l}(s) + \\ + W^{f}_{l'j'}(-s)\, W^{c}_{j'k'}(-s)\, \Phi_{a_{k'}a_k}(s, \lambda)\, W^{c}_{kj}(s)\, W^{f}_{jl}(s) \tag{12}$$

This form of Eq. (12) is quite similar to Eq. (6) figuring in reference [14]. But this means nothing else than that our present problem is reduced to the problem of the semi-free configuration without constraint and the same technique can be used as in the previous paper [14].

Therefore, let us introduce an auxiliary $K \times J$ transfer-function matrix $G^{ck}_{kj}(s, \lambda)$ and its adjoint matrix $G^{ck}_{j'k'}(-s, \lambda)$ by the following implicit relations

$$\Phi_{a_{k'}a_k}(s, \lambda)\, G^{ck}_{kj}(s, \lambda)\, W^{f}_{jl}(s) = \Phi_{r_{k'}i_l}(s)$$

and

$$W^{f}_{l'j'}(-s)\, G^{ck}_{j'k'}(-s, \lambda)\, \Phi_{a_{k'}a_k}(s, \lambda) = \Phi_{i_{l'}r_k}(s) \tag{13}$$

If necessary the auxiliary matrices can also be expressed explicitly:

and

$$
\begin{aligned}
G^{\mathrm{ck}}_{kj}(s,\lambda) &= [\Phi_{a_{k'}a_k}(s,\lambda)]^{-1}\,\Phi_{r_{k'}i_l}(s)\,[W^{\mathrm{f}}_{jl}(s)]^{-1}\\
G^{\mathrm{ck}}_{j'k'}(-s,\lambda) &= [W^{\mathrm{f}}_{l'j'}(-s)]^{-1}\,\Phi_{i_{l'}r_k}(s)\,[\Phi_{a_{k'}a_k}(s,\lambda)]^{-1}
\end{aligned}
\tag{14}
$$

These relations clearly show that in the auxiliary transfer-function matrices the parameter λ must also figure. Naturally, the physical realizability of the first auxiliary transfer-function matrix is a priori not guaranteed, on the contrary, in general the matrix $G^{\mathrm{ck}}_{kj}(s,\lambda)$ is physically unrealizable.

Substituting expression (13) into Eq. (12) the power-density-spectrum matrix in question takes the following form

$$
\begin{aligned}
\Phi_{x_{l'}x_l}(s,\lambda) = {} & \Phi_{i_{l'}i_l}(s) - \\
& - W^{\mathrm{f}}_{l'j'}(-s)\,G^{\mathrm{ck}}_{j'k'}(-s,\lambda)\,\Phi_{a_{k'}a_k}(s,\lambda)\,G^{\mathrm{ck}}_{kj}(s,\lambda)\,W^{\mathrm{f}}_{jl}(s) + \\
& + [W^{\mathrm{f}}_{l'j'}(-s)\,G^{\mathrm{ck}}_{j'k'}(-s,\lambda) - W^{\mathrm{f}}_{l'j'}(-s)\,W^{\mathrm{c}}_{j'k'}(-s)] \times \\
& \times \Phi_{a_{k'}a_k}(s,\lambda)[G^{\mathrm{ck}}_{kj}(s,\lambda)\,W^{\mathrm{f}}_{jl}(s) - W^{\mathrm{c}}_{kj}(s)\,W^{\mathrm{f}}_{jl}(s)]\,.
\end{aligned}
\tag{15}
$$

The transfer-function matrix $W^{\mathrm{ck}}_{kj}(s)$ and its adjoint $W^{\mathrm{ck}}_{j'k'}(-s)$ are contained only in the last term of Eq. (15). The trace of the power-density spectrum matrix $\Phi_{x_{l'}x_l}(s,\lambda)$ will obviously be minimum if this last term becomes zero. The sufficient and necessary conditions are

$$
\begin{aligned}
W^{\mathrm{cko}}_{kj}(s,\lambda) &= G^{\mathrm{ck}}_{kj}(s,\lambda)\\
W^{\mathrm{cko}}_{j'k'}(-s,\lambda) &= G^{\mathrm{ck}}_{j'k'}(-s,\lambda)
\end{aligned}
\tag{16}
$$

where the upper index o signifies the optimum. The optimum transfer-function matrix of the cascade controller in case of constraints figuring in Eq. (16) is now a two-variable function of $s = j\,\omega$ and the parameter λ.

Substituting Eq. (16) in Eq. (14) we obtain the physically unrealizable transfer-function matrix of the cascade controller and its adjoint matrix:

$$
\begin{aligned}
W^{\mathrm{cko}}_{kj}(s,\lambda) &= [\Phi_{a_{k'}a_k}(s,\lambda)]^{-1}\,\Phi_{r_{k'}i_l}(s)\,[W^{\mathrm{f}}_{jl}(s)]^{-1}\\
W^{\mathrm{cko}}_{j'k'}(-s,\lambda) &= [W^{\mathrm{f}}_{l'j'}(-s)]^{-1}\,\Phi_{i_{l'}r_k}(s)\,[\Phi_{r_{k'}r_k}(s)]^{-1}
\end{aligned}
\tag{17}
$$

In order to perform matrix inversion $J = L$ must be valid. By the way, instead of the explicit relations (17) the following implicit relations can also be written

$$
\begin{aligned}
\Phi_{a_{k'}a_k}(s,\lambda)\,W^{\mathrm{cko}}_{kj}(s,\lambda)\,W^{\mathrm{f}}_{jl}(s) &= \Phi_{r_{k'}i_l}(s)\\
W^{\mathrm{f}}_{l'j'}(-s)\,W^{\mathrm{cko}}_{j'k'}(-s)\,\Phi_{a_{k'}a_k}(s,\lambda) &= \Phi_{i\ \ k}(s)
\end{aligned}
\tag{18}
$$

as obtained from Eq. (17) or after substituting Eq. (16) in Eq. (13).

Naturally, physically unrealizable transfer-function matrix does not solve our problem and we must search for a physically realizable one. Let us assume that $W_{kj}^{\mathrm{ckm}}(s, \lambda)$ is the physically realizable optimum transfer-function matrix of the cascade controller in case of constraints. Substituting this matrix instead of $W_{kj}^{cko}(s, \lambda)$ then from the first relation of Eq. (18) the following expression can be derived:

$$\begin{aligned} \Phi_{a_{k'}a_k}(s, \lambda)\, W_{kj}^{\mathrm{ckm}}(s, \lambda)\, W_{jl}^{\mathrm{f}}(s)\, W_{l'j'}^{\mathrm{f}}(-s) = \\ = \Phi_{r_{k'}i_l}(s)\, W_{l'j'}^{\mathrm{f}}(-s) + F_{k'j'}^{\mathrm{ck}}(s, \lambda) \end{aligned} \tag{19}$$

where $F_{k'j'}^{\mathrm{ck}}(s, \lambda)$ is still an unknown matrix with transfer-function elements having only right-half-plane poles. In this equation the matrix factor $W_{l'j'}^{\mathrm{f}}(-s)$ is inevitable as $W_{lj}^{\mathrm{f}}(s)\, W_{l'j'}^{\mathrm{f}}(-s)$ must be treated as a power-density-spectrum matrix. Otherwise Eq. (19) here plays the same role as Eq. (12) in reference [14] or Eq. (20) in reference [12].

Now let us introduce the following spectrum-factorization relations:

$$\Phi_{a_{k'}a_{k''}}^{-}(s, \lambda)\, \Phi_{a_{k''}a_k}^{+}(s, \lambda) = \Phi_{a_{k'}a_k}(s, \lambda) \tag{20}$$

and

$$W_{jl}^{\mathrm{f}}(s)\, W_{l'j'}^{\mathrm{f}}(-s) = \left(W_{jl}^{\mathrm{f}}(s)\, W_{l'j'}^{\mathrm{f}}(-s)\right)^{+} \left(W_{jl}^{\mathrm{f}}(s)\, W_{l'j'}^{\mathrm{f}}(-s)\right)^{-} \tag{21}$$

where the upper index — (minus) denotes a matrix *factor* whose elements, and the elements of the inverse matrix, have only right-half-plane poles and zeros, while the upper index + (plus) denotes a matrix factor whose elements together with the elements of its inverse matrix have only left-half-plane poles and zeros.

Taking Eqs. (20) and (21) into consideration then Eq. (19) may assume the following form:

$$\begin{aligned} \Phi_{a_{k''}a_k}^{+}(s, \lambda)\, W_{kj}^{\mathrm{cm}}(s, \lambda) \left(W_{jl}^{\mathrm{f}}(s)\, W_{l'j'}^{\mathrm{f}}(-s)\right)^{+} = \\ = [\Phi_{a_{k'}a_{k''}}^{-}(s, \lambda)]^{-1}\, \Phi_{r_{k'}i_l}(s)\, W_{l'j'}^{\mathrm{f}}(-s) \left[\left(W_{jl}^{\mathrm{f}}(s)\, W_{l'j'}^{\mathrm{f}}(-s)\right)^{-}\right]^{-1} + \\ + [\Phi_{a_{k'}a_{k''}}^{-}(s, \lambda)]^{-1}\, F_{k'j'}^{\mathrm{ck}}(s, \lambda) \left[\left(W_{jl}^{\mathrm{f}}(s) W_{l'j'}^{\mathrm{f}}(-s)\right)^{-}\right]^{-1} \end{aligned} \tag{22}$$

Separating the physically realizable and unrealizable matrix components on both sides of Eq. (22) the following two relations can be obtained:

$$\begin{aligned} \Phi_{a_{k''}a_k}^{+}(s, \lambda)\, W_{kj}^{\mathrm{ckm}}(s, \lambda) \left(W_{jl}^{\mathrm{f}}(s)\, W_{l'j'}^{\mathrm{f}}(-s)\right)^{+} = \\ = \left\{[\Phi_{a_{k'}a_{k''}}^{-}(s, \lambda)]^{-1}\, \Phi_{r_{k'}i_l}(s)\, W_{l'j'}^{\mathrm{f}}(-s) \left[\left(W_{jl}^{\mathrm{f}}(s)\, W_{l'j}^{\mathrm{f}}(-s)\right)^{-}\right]^{-1}\right\}_{+} \end{aligned}$$

and

$$\begin{aligned} 0 = \left\{[\Phi_{a_{k'}a_{k''}}^{-}(s, \lambda)]^{-1}\, \Phi_{r_{k'}i_l}(s) W_{l'j'}^{\mathrm{f}}(-s) \left[\left(W_{jl}^{\mathrm{f}}(s)\, W_{l'j'}^{\mathrm{f}}(-s)\right)^{-}\right]^{-1}\right\}_{-} + \\ + [\Phi_{a_{k'}a_{k''}}^{-}(s, \lambda)]^{-1}\, F_{k'j'}^{\mathrm{ck}}(s, \lambda) \left[\left(W_{jl}^{\mathrm{f}}(s)\, W_{l'j'}^{\mathrm{f}}(-s)\right)^{-}\right]^{-1} \end{aligned} \tag{24}$$

where the lower index + (plus) denotes a matrix *component* with physically realizable elements, belonging to positive-time functions, while the lower index — (minus) denotes a matrix component with physically unrealizable elements, that is, with right-half-plane poles, and thus belonging to negative-time functions. Generally speaking, the physically realizable component can be obtained by first performing an inverse Fourier transformation and then a Laplace transformation.

Finally, from Eq. (21) the physically realizable optimum transfer-function matrix of the cascade controller in case of constraints can be expressed as

$$\begin{aligned} W_{kj}^{\mathrm{ckm}}(s, \lambda) &= [\Phi^{+}_{a_{k''}a_{k'}}(s, \lambda)]^{-1} \times \\ &\times \left\{[\Phi^{-}_{a_{k'}a_{k}}(s, \lambda)]^{-1}\, \Phi_{r_{k'}i_{l}}(s)\, W^{\mathrm{f}}_{l'j'}(-s) \left[\left(W^{\mathrm{f}}_{jl}(s)\, W^{\mathrm{f}}_{l'j'}(-s)\right)^{-}\right]^{-1}\right\} \times \\ &\times \left[\left(W^{\mathrm{f}}_{jl}(s)\, W^{\mathrm{f}}_{l'j'}(-s)\right)^{+}\right]^{-1}. \end{aligned} \tag{25}$$

The solution $W_{kj}^{\mathrm{ckm}}(s, \lambda)$ may now be substituted into the condition of constraint. This can be performed by substituting first $W_{kj}^{\mathrm{ckm}}(s, \lambda)$ and its adjoint matrix instead of $W_{kj}^{\mathrm{c}}(s)$ and its adjoint, respectively, in Eq. (9). Thus, the power-density-spectrum matrix $\Phi_{b_{h'}b_{h}}(s, \lambda)$ is obtained.

Substituting the latter matrix into Eq. (2) the parameter λ can be adjusted so that the condition of constraint, that is, unequality (2) will be satisfied.

After having determined the proper value of the parameter λ, the latter can be substituted back into Eq. (25) and finally the physically realizable optimum transfer-function matrix of the cascade controller $W_{kj}^{\mathrm{ckm}}(s)$ is obtained. It must be emphasized that after the previous procedure the parameter λ is already missing. The transfer-function matrix $W_{kj}^{\mathrm{ckm}}(s)$ is the final explicit solution of our problem for the case of the semi-free configuration with constraints.

Substituting the so obtained matrix expression of $W_{jk}^{\mathrm{ckm}}(s)$ and its adjoint into Eq. (8) instead of $W_{kj}^{\mathrm{c}}(s)$ and $W_{j'k'}^{\mathrm{c}}(-s)$, respectively, the power-density-spectrum matrix of the error can be computed. Henceforth, using Eq. (3) the minimum sum of the mean square-error components can be determined.

3. Some supplementary remarks

Let us now examine some possibilities of specializations and generalizations concerning the obtained results.

First, it is obvious that taking $\lambda = 0$, on the one hand $\Phi_{a_{k'}a_{k}}(s, \lambda)$ is immediately reduced to $\Phi_{r_{k'}r_{k}}(s)$ [see Eq. (10)], and on the other hand, Eq. (25) yields

$$\begin{aligned} W_{kj}^{\mathrm{cm}}(s) &= [\Phi_{r_{k'}r_{k'}}(s)]^{-1} \times \\ &\times \left\{[\Phi^{-}_{r_{k'}r_{k'}}(s)]^{-1}\, \Phi_{r_{k'}i_{l}}(s)\, W^{\mathrm{f}}_{l'j'}(-s)\, \left[\left(W^{\mathrm{f}}_{jl}(s)\, W^{\mathrm{f}}_{l'j'}(-s)\right)^{-}\right]^{-1} \times \right. \\ &\times \left[\left(W^{\mathrm{f}}_{jl}(s)\, W^{\mathrm{f}}_{l'j'}(-s)\right)^{+}\right]^{-1} \end{aligned} \tag{26}$$

which is the final explicit solution formula of the optimum cascade controller for the case of the semi-free configuration without constraints, as given in reference [14] as Eq. (18).

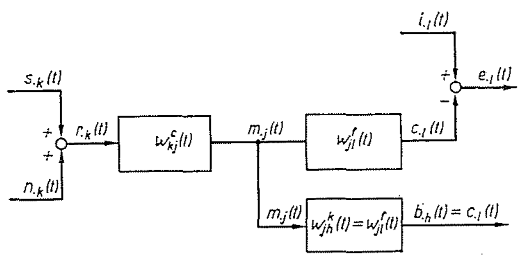

Fig. 2

Secondly, returning from the multivariable case to the single variable one the matrices become scalar quantities and Eq. (25) can be written as follows:

$$W^{\mathrm{ckm}}(s) = \frac{\left[\dfrac{W^{\mathrm{f}}(-s)\,\Phi_{ri}(s)}{[W^{\mathrm{f}}(-s)\,W^{\mathrm{f}}(s) + \lambda\,W^{\mathrm{k}}(-s)\,W^{\mathrm{k}}(s)]^{-}\,\Phi^{-}_{rr}(s)}\right]_{+}}{[W^{\mathrm{f}}(-s)\,W^{\mathrm{f}}(s) + \lambda\,W^{\mathrm{k}}(-s)\,W^{\mathrm{k}}(s)]^{+}\,\Phi^{-}_{rr}(s)} \tag{27}$$

since Eq. (10) is reduced to the form

$$W^{\mathrm{f}}(-s)\,W^{\mathrm{f}}(s)\,\Phi_{aa} = [W^{\mathrm{f}}(-s)\,W^{\mathrm{f}}(s) + \lambda W^{\mathrm{k}}(-s)\,W^{\mathrm{k}}(s)]\,\Phi_{rr}(s) \tag{28}$$

Eq. (27) has, of course, the same form as Eq. (40) in reference [6].

Now, let us concentrate our attention to the constraint matrix $w^{\mathrm{k}}_{jh}(t)$ and to its transform $W^{\mathrm{k}}_{jh}(s)$. If the manipulated variables are indirectly constrained the transfer-function matrix $W^{\mathrm{k}}_{jh}(s)$ may assume quite a general form. For example, if even the sum of the mean-square values of the controlled variables are limited, then the constraint matrix $W^{\mathrm{k}}_{jh}(s)$ (or $w^{\mathrm{k}}_{jh}(t)$) becomes the very same as the plant matrix $W^{\mathrm{f}}_{jl}(s)$ (or $w^{\mathrm{f}}_{jl}(t)$). See *Fig. 2*.

Furthermore, if the plant transfer-function matrix $W^{\mathrm{f}}_{jl}(s)$ can be expressed as the matrix multiplication of two corresponding transfer-function matrices

$$W^{\mathrm{f}}_{jl}(s) = W^{\mathrm{f}}_{jh}(s)\, W^{\mathrm{f}}_{hl}(s) \tag{29}$$

or in other words, the plant weighting-function matrix can be expressed as the convolution of two corresponding weighting-function matrices:

$$w^{\mathrm{f}}_{jl}(t) = w^{\mathrm{f}}_{jh}(t) \; * \; w^{\mathrm{f}}_{hl}(t) \tag{30}$$

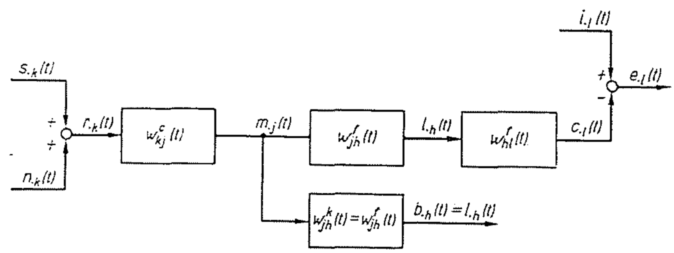

Fig. 3

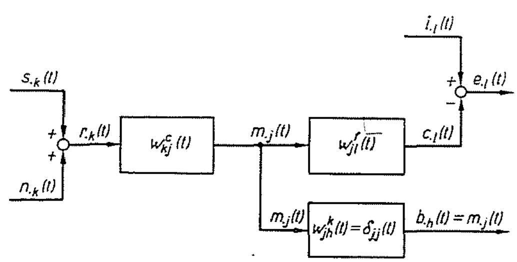

Fig. 4

and even the sum of the mean-square-value of the variables acting between the two control link mentioned above has to be limited, then $W^{\mathrm{k}}_{jh}(s)$ (or $w^{\mathrm{k}}_{jh}(t)$) must be taken as identical with $W^{\mathrm{f}}_{jh}(s)$ (or $w^{\mathrm{f}}_{jh}(t)$) as shown in *Fig.* 3.

If, on the other hand, the set of the manipulated variables is directly constrained then the constraint transfer-function matrix assumes a certain special form, namely, it becomes a diagonal matrix. For example, if even the sum of the mean-square values of the manipulated variables is limited (Fig. 4), then $W^{\mathrm{k}}_{jh}(s)$ becomes an unity (or in other words: idem) matrix: $W^{\mathrm{k}}_{jh}(s) = I_{jj}$, the latter being independent of the variable $s = j\omega$. If the mean-square values of manipulated variables must be added by taking some weights: $g_{11}, \ldots g_{jj}, \ldots g_{JJ}$ into consideration then the constraint transfer function $W_{jh}^{k}(s)$ becomes a diagonal matrix composed of the square roots of the weights as elements

$$W_{jh}^{k}(s) = \mathrm{diag}\,[\sqrt{g_{11}}, \ldots \sqrt{g_{jj}}, \ldots \sqrt{g_{JJ}}] = \begin{bmatrix} \sqrt{g_{11}}, \ddots & & \\ & \sqrt{g_{jj}}, \ddots & \\ & & \sqrt{g_{JJ}} \end{bmatrix}. \quad (31)$$

This matrix is also independent of the variable s.

A semi-direct constraint arises from the case when not the sum of the mean-square values of the manipulated variables themselves is limited but that of the first (or second) derivative of the manipulated variables has to be constrainted. In the latter cases the following choice will do

$$W_{jh}^{k} = s\, I_{jj} \quad (32)$$

or

$$W_{jh}^{k}(s) = s^2\, I_{jj} \quad (33)$$

If weights are needed then

$$W_{jh}^{k}(s) = s\, \mathrm{diag}\,[\sqrt{g_{11}}, \ldots \sqrt{g_{jj}}, \ldots \sqrt{g_{JJ}}] \quad (34)$$

or

$$W_{jh}^{k}(s) = s^2\, \mathrm{diag}\,[\sqrt{g_{11}}, \ldots \sqrt{g_{jj}}, \ldots \sqrt{g_{JJ}}] \quad (35)$$

are the proper choices.

When, for example, the first manipulated variable itself, the second manipulated variable by its first derivative, the third manipulated variable by its second derivative and so on..., must be taken into consideration in the mean-square-summing procedure with weights, then the following diagonal matrix will do:

$$W_{jh}^{k}(s) = \mathrm{diag}\,[\sqrt{g_{11}}, s\sqrt{g_{22}}, s^2\sqrt{g_{33}} \ldots]. \quad (36)$$

Similarly, some other special matrix forms can be chosen according to the special need if the sum of the mean-square values of the manipulated variables must be constrained semi-directly.

Now, the question arises, how multiple constraints can be performed. If, for example, the manipulated variables are simultaneously submitted to two or more constraints (Fig. 5), then instead of unequality (2) we have a system of unequalities:

$$\frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \mathrm{tr}\,[\Phi_{b_h' b_h(i)}(s)]\, ds < \sigma_i^2 \quad (2^*)$$

and instead of Eq. (5) we have the following relation

$$\operatorname{tr}\,[\varphi_{x_l' x_l}(0)] = \operatorname{tr}\left[\overline{x_{l.}(t)\, x_{.l}(t)}\right] = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \operatorname{tr}\left[\Phi_{e_l' e_l}(s) + \sum_{i=1}^{i=I}\lambda_i \Phi_{b_h' b_{h(i)}}(s)\right] ds \qquad (5^*)$$

where the corresponding power-density-spectrum matrices are

$$\Phi_{b_h' b_{h(i)}}(s) = W^{\mathrm{k}}_{h'j'(i)}(-s)\; W^{\mathrm{c}}_{j'k'}(-s)\; \Phi_{r_k' r_k}(s)\; W^{\mathrm{c}}_{kj}(s)\; W^{\mathrm{k}}_{jh(i)}(s).$$

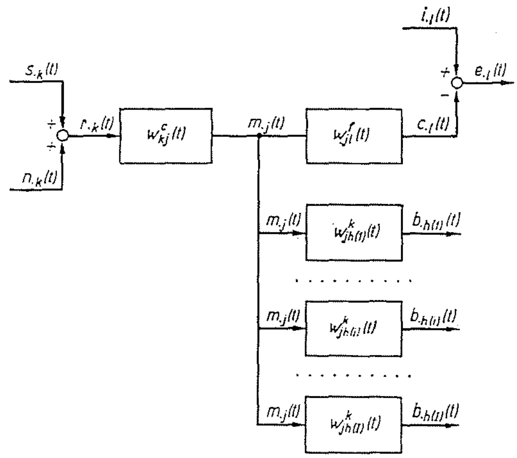

Fig. 5

Here $W^{\mathrm{k}}_{jh(i)}(s)$ $(i = 1, \ldots I)$ are the corresponding constraint matrices.

Following in the latter case the simplified derivation technique both the auxiliary power-density-spectrum matrix $\Phi_{a_k' a_k}(s, \lambda_1 \ldots \lambda_I)$ and the physically realizable optimum transfer-function matrix $W^{\mathrm{ckm}}_{kj}(s, \lambda_1 \ldots \lambda_I)$ becomes a multivariable function of the parameters λ_i.

Thus, instead of Eq. (10) we now have

$$W^{\mathrm{f}}_{l'j'}(-s)\; W^{\mathrm{c}}_{j'k'}(-s)\; \Phi_{r_k' r_k}(s)\; W^{\mathrm{c}}_{kj}(s)\; W^{\mathrm{f}}_{jl}(s) +$$

$$+ \sum_{i=1}^{i=I}\lambda_i\, W^{\mathrm{k}}_{h'j'(i)}(-s)\; W^{\mathrm{c}}_{j'k'}(-s)\; \Phi_{r_k' r_k}(s)\; W^{\mathrm{c}}_{kj}(s)\; W^{\mathrm{k}}_{jh(i)}(s) = \qquad (10^*)$$

$$= W^{\mathrm{f}}_{l'j'}(-s)\; W^{\mathrm{c}}_{j'k'}(-s)\; \Phi_{a_k' a_k}(s, \lambda_1, \ldots \lambda_I)\; W^{\mathrm{c}}_{kj}(s)\; W^{\mathrm{f}}_{jl}(s)$$

from which the auxiliary power-density-spectrum matrix can be expressed as

$$\Phi_{a_k' a_k}(s, \lambda_1, \ldots \lambda_I) = \Phi_{r_k' r_k}(s) + [W^{\mathrm{c}}_{j'k'}(-s)]^{-1}\,[W^{\mathrm{f}}_{l'j'}(-s)]^{-1} \times \qquad (11^*)$$

$$\times \left[\sum_{i=1}^{i=I}\lambda_i W^{\mathrm{k}}_{h'j'(i)}(-s)\; W^{\mathrm{c}}_{j'k'}(-s)\; \Phi_{r_k' r_k}(s)\; W^{\mathrm{c}}_{kj}(s)\; W^{\mathrm{k}}_{jh(i)}(s)\right]\,[W^{\mathrm{f}}_{jl}(s)]^{-1}\,[W^{\mathrm{c}}_{kj}(s)]^{-1}$$

Finally, the explicit solution formula now becomes:

$$W_{kj}^{\mathrm{ckm}}(s, \lambda_1, \ldots \lambda_I) = [\Phi^{+}_{a_{k''}a_{k'}}(s, \lambda_1, \ldots \lambda_I)]^{-1} \times$$
$$\times \left\{ [\Phi^{-}_{a_{k'}a_k}(s, \lambda_1, \ldots \lambda_I)]^{-1} \Phi_{r_{k'}i_l}(s) W^{\mathrm{f}}_{l'j'}(-s) [(W^{\mathrm{f}}_{jl}(s) W^{\mathrm{f}}_{l'j'}(-s))^{-}]^{-1} \right\}_{+} \times$$
$$\times [(W^{\mathrm{f}}_{jl}(s)\, W^{\mathrm{f}}_{l'j'}(-s))^{+}]^{-1} \tag{25*}$$

The adjustment procedure of the parameters λ_i must now be performed in such a manner that the most rigorous of the inequalities (2*) will be fulfilled.

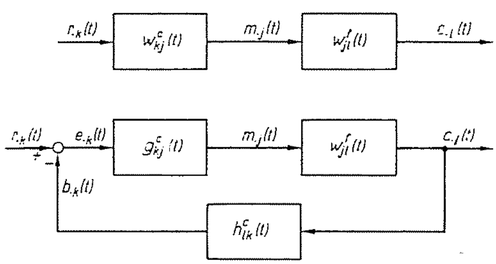

Fig. 6

Naturally, a generalized matrix notation is also possible. In this case $\sigma_i^2 = (\sigma^2)_i$ and λ_i must be considered as vectors (or one dimensional matrices), while $\Phi_{b_h b_{h(i)}}(s)$, $W^{*}_{jh(i)}(s)$, $W^{*}_{h'j'(i)}(-s)$ must be treated as three dimensional matrices. Then in Eqs. (5), (10) and (11) the summations become matrix multiplications.

Many special cases can be considered on the basis of the foregoing discussion. Here only the most interesting one will be treated. Let us assume that the mean-square value of the individual indirect manipulated variables is limited. Then the constraint transfer-function matrices become degenerated

$$W^{\mathrm{k}}_{jh(1)}(s) = \begin{bmatrix} W^{\mathrm{k}}_{11}(s) & W^{\mathrm{k}}_{12}(s) \ldots & W^{\mathrm{k}}_{1H}(s) \\ 0 & 0 & 0 \\ \ldots & \ldots & \ldots \\ 0 & 0 & 0 \end{bmatrix}; \; W^{\mathrm{k}}_{jh(2)}(s) = \begin{bmatrix} 0 & 0 & 0 \\ W^{\mathrm{k}}_{21}(s) & W^{\mathrm{k}}_{22} \ldots & W^{\mathrm{k}}_{2H}(s) \\ \ldots & \ldots & \ldots \\ 0 & 0 & 0 \end{bmatrix}$$

and so on. This case is much more simple than the general multiconstraint case when the sum of the mean-square values of some set of variables has to be constrained (assuming that in both cases the number I of the parameters λ_i is the same, for example, $I = J$). A significant variant arises when the individual manipulated variables are directly limited. Then the constraint matrices become the following:

$$W^{k}_{jh(1)}(s) = \text{diag}\,[1, 0, 0 \ldots 0]$$
$$W_{jh(2)}(s) = \text{diag}\,[0, 1, 0 \ldots 0]$$

etc. When the derivatives of the manipulated variables have to be limited then the corresponding matrices are

$$W^{k}_{jh(1)}(s) = \text{diag}\,[s, 0, 0 \ldots 0]$$
$$W_{jh(2)}(s) = \text{diag}\,[0, s, 0 \ldots 0]$$

and so on.

4. Conclusions

Unfortunately, the case of semi-free configurations with constraints is far too complicated for a simple illustrative example to be constructed. It is hoped, however, that in view of the matrix calculus and the complementary remarks the application of the final results had become clear. Furthermore, it is also hoped, that even the complicated enough case of the semi-free configuration with constraints had thrown into relief the advantages of the so-called simplified derivation technique showing the design procedure of the optimum cascade controller for multivariable systems.

5. Appendix

After having determined the optimum transfer-function matrix of the cascade controller according to the equivalence of the two configurations shown in Fig. 6 the transfer-function matrix of the series controller or that of the feed-back controller can also be ascertained. For example, if there is no feed-back controller then the transfer-function matrix of the series controller can be expressed as

$$G^{c}_{kj}(s) = [I - W^{c}_{kj}(s)\, W^{f}_{jl}(s)]^{-1}\, W^{c}_{kj}(s)$$

while, on the other hand if the series controller is missing then the transfer-function matrix of the feed-back controller is given in the following relation

$$H^{c}_{lk}(s) = [W^{c}_{kj}(s)\, W^{f}_{jl}(s)]^{-1} - [W^{f}_{jl}]^{-1}$$

$$(k = 1, \ldots K\,;\; j = 1, \ldots J\,;\; l = 1, \ldots L\,;\; K = J = L)$$

Summary

In this paper, as a continuation of the previous two papers concerning the optimum design of multivariable control systems, the case of the semi-free configuration with constraints is treated. For stationary ergodic stochastic processes taking as performance criterion the sum of the least-mean-square errors between the sets of actual and ideal outputs and considering as constraint the limitation of one or more sums of the mean-square values of

some sets of manipulated variables explicit formulas are derived for the multivariable cascade controller. The so-called simplified derivation technique is used, based only on frequency-domain notions in connection with matrix calculus. Finally some special cases are shown.

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