SOME ADDENDA TO THE THEORY OF SAMPLED-DATA CONTROL SYSTEMS WITH FINITE SETTLING TIME

By

GY. FODOR

Department for Theoretical Electricity, Polytechnical University Budapest, and Automation Research Laboratory of the Hungarian Academy of Sciences

(Received November 21, 1963)*

In two previous papers [1, 2] a method was given for the synthesis of sampled-data control systems with finite settling time, with the help of which the finite settling time and the minimum statistical error can be ensured with respect to both the reference input and the disturbing variable. In the following on the one hand, the generalization of the method will be discussed, on the other hand the effect of the uncertainty of the parameters will be examined. Concerning the symbols we refer to the above mentioned two papers.

1. Generalization of the problem

In the case of follow-up systems the requirement is that the controlled variable y should be identical at the sampling instants $t = kT$ with the determined reference input $x_D(t)$, or with the useful portion $f(t)$ of the random reference input $x_S(t)$.

$$y_D[k] = x[k],\ x(t) = x_D(t); \tag{1}$$

$$y_S[k] = f(k],\ x_S(t) = f(t) + \varphi(t). \tag{2}$$

Accordingly the desired transfer functions of the system are

$$W_D(Z) = \frac{Y_D(Z)}{X(Z)} = 1, \tag{3}$$

$$W_S(Z) = \frac{Y_S(Z)}{F(Z)} = 1. \tag{4}$$

Such a system cannot be realized on account of various reasons. By the method given earlier [1] however we are able to design an impulse-compensator in such a way that the controlled variable of the closed system changes, on

* Presented by Prof. Dr. K. SIMONYI.

the one hand in accordance with prescription (1) after a determined number of sampling moments r (settling time $T_s = rT$), if the reference input is of the order m at the maximum, i.e.

$$x_m(t) = 1(t)\,\frac{t^{m-1}}{(m-1)!}\,, \quad m = 1, 2, 3\,; \tag{5}$$

on the other hand, in the case of a reference input of determined statistical characteristics the statistical error

$$\zeta^2 = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} (y[k] - y_S[k])^2 \tag{6}$$

is minimum.

Let us now design a system the controlled variable of which depends on the reference input in a more general way than that prescribed by (1) and (2), respectively, for the case of a follow-up system.

On the one hand the output signal should be at the instants $t = nT$ some prescribed superposition of the values assumed by the determined reference input at the instants $t = nT$, $(n-1)T$, $(n-2)T, \ldots, (n-u)T$,

$$\begin{gathered} y_D[n] = h_0\,x[n] + h_1\,x\,[n-1] + h_2\,x\,[n-2] + \ldots + \\ + h_u\,x\,[n-u] = \sum_{i=0}^{u} h_i\,x\,[n-i]\,. \end{gathered} \tag{7}$$

If e.g. $h_0 = 1$, $h_1 = h_2 = \ldots = h_u = 0$, then this is a follow-up system. By the system of coefficients $h_0 = 0$, $h_1 = 1$, $h_2 = h_3 = \ldots = h_u = m$ a delay of one sampling period is prescribed. The above formula can be rewritten in a form containing the reference input $x(t)$, its first, second, ..., u-th difference. We insist that the controlled variable should change after the finite settling time $T_s = rT$ in the way given by (7). Accordingly the required transfer function for a determined reference input is given by

$$W_D\,(Z) = \sum_{i=0}^{u} h_i\,Z^i\,. \tag{8}$$

The coefficients h_i figuring in formulae (7) and (8) can be regarded as undetermined on account of a proportionality factor. It only depends on the amplifier of the reference input forming organ, that we can write the coefficients $h_i' = Kh_i$ in the place of the coefficients h_i. The correlation between the controlled variable and the reference input remains unaltered thereby. As will be seen later, it is advisable to normate the coefficients as follows.

$$\sum_{i=0}^{u} h_i = 1. \tag{9}$$

In another form, this means that

$$W_D\,(Z=1)=1;\;\; W_D\,(Z)=1-(1-Z)\,C(Z)\,, \tag{10}$$

where $C(Z)$ is a determinable polynomial. With this restriction, the follow-up system simply means the case $u = 0$.

Let us similarly generalize the required controlled variable pertaining to the random reference input. The value of this at the instants $t = nT$ should be a linear superposition of the values assumed by the useful portion of the reference input at the instants $t = iT$, accordingly

$$\begin{aligned} y_S[n] &= k_{-v} f[n+v] + k_{-v+1} f\,[n+v-1] + \ldots + \\ &+ k_0 f\,[n] + k_1 f\,[n-1] + \ldots + k_v f\,[n-v] = \sum_{i=-v}^{v} k_i f\,[n-i]\,. \end{aligned} \tag{11}$$

If $k_0 = 1$ and $v = 0$, then this is a follow-up system. If $k_1 = 1$ and $k_i = 0$ $(i \neq 1)$, then the problem is to delay the reference input. If $k_{-1} = 1$ and $k_i = 0$ $(i \neq -1)$, then the problem is a prediction performed with time T. From this it is obvious that in the case of a random reference input the shifting time may also be negative. Generalization is not restricted by the fact that in formula (11) the shifting time is changing between $-vT$ and $+vT$, since any of the values k_i may be zero.

It is a demand that the statistical error ζ^2 given by (6) should be minimum. Accordingly the expression of the transfer function to be approximated, pertaining to the random reference input is

$$W_S(Z) = \sum_{i=-v}^{v} k_i\, Z^i\,. \tag{12}$$

A normating subcondition can be prescribed for the coefficients k_i as well, e.g. $W_S(Z = 1) = 1$.

2. The solution of the generalized problem

The condition of the case that the settling time is finite, even for a reference input of m-order is

$$W(Z) - W_D(Z) = (1-Z)^m C_1(Z)\,, \tag{13}$$

where $C_1(Z)$ is temporarily an arbitrary polynomial. Namely in this case, the transformed error function $Y(Z) - Y_D(Z)$ is a polynomial, assuming that the reference input is of m-order (or of lower order). If the controlled system is stable in itself, then this is equivalent to the following m pieces of condition equations:

$$\left[\frac{d^\mu W(Z)}{dZ^\mu}\right]_{Z=1} = \left[\frac{d^\mu W_D(Z)}{dZ^\mu}\right]_{Z=1}, \quad \mu = 0, 1, \ldots (m-1). \tag{14}$$

In accordance with our previously used symbols [1], let

$$W(Z) = A(Z)B(Z) = \sum_{k=0}^{u} a_k Z^k \sum_{i=1}^{q} b_i Z^i = \sum_{i=1}^{r} w_i Z^i, \tag{15}$$

where $B(Z)$ is a polynomial known on the basis of the data and specifications, while $A(Z)$ is a polynomial to be determined. The system of equations (4) then takes the form

$$\sum_{k=0}^{u} a_k = \frac{1}{\beta_0}, \quad m = 1, 2, 3\,; \tag{16}$$

$$\sum_{k=0}^{u} k a_k = \frac{\varkappa_1 \beta_0 - \beta_1}{\beta_0^2}, \quad m = 2, 3\,; \tag{17}$$

$$\sum_{k=0}^{u} k(k-1)\, a_k = \frac{\varkappa_2 \beta_0 - 2\varkappa_1 \beta_0 \beta_1 + 2\beta_1^2 - \beta_0 \beta_2}{\beta_0^3}, \quad m = 3, \tag{18}$$

where the expression for the parameters β_j, $\varkappa_j$ is

$$\beta_j = \sum_{i=j}^{q} \frac{i!}{(i-j)!}\, b_i, \quad j = 0, 1, 2, \tag{19}$$

as in the case of designing follow-up systems, while

$$\varkappa_j = \sum_{i=j}^{u} \frac{i!}{(i-j)!}\, h_i, \; j = 1, 2\,. \tag{20}$$

In the case of a follow-up system $\varkappa_1 = \varkappa_2 = 0$.

The statistical error can be expressed by the transformed autocorrelation functions in the following way:

$$\zeta^2 = \sum_{|Z_i|<1} \operatorname*{Res}_{Z=Z_i} \frac{\bar{R}_{\psi\psi}(Z)}{Z}, \tag{21}$$

$$\begin{aligned} \bar{R}_{\psi\psi}(Z) = [\hat{W}(Z)W(Z) + \hat{W}_S(Z)W_S(Z) - \hat{W}(Z)W_S(Z) - \\ - \hat{W}_S(Z)\, W(Z)]\, \bar{R}_{ff}(Z) + \hat{W}(Z)\, W(Z)\, \bar{R}_{\varphi\varphi}(Z)\,, \end{aligned} \tag{22}$$

where $\widehat{W}(Z) = W(Z^{-1})$. We assume that the useful signal $f(t)$ and the noise $\varphi(t)$ are uncorrelated. From the here on the steps of the calculation are similar to those described or discussing the follow-up system [1], only the expression of the following parameters is somewhat more complicated on account of the more general form of $W_S(Z)$.

The expression for the statistical error is

$$\zeta^2 = \sum_{i=1}^{r} \sum_{j=1}^{r-1} w_i\, w_{i+j}\, M_j + \frac{1}{2} N \sum_{i=1}^{r} w_i^2 - \sum_{i=1}^{r} w_i\, P_i + Q. \tag{23}$$

This is formally completely identical with the formula for the follow-up system. The expression for the parameters are found to be

$$\begin{aligned} M_j = \frac{\bar{R}_{ff}^{(j)}(0) + \bar{R}_{\varphi\varphi}^{(j)}(0)}{j!} + \sum_h \frac{\mu_h^j + \mu_h^{-j}}{\mu_h} \operatorname{Res} \bar{R}_{ff}(\mu_h) + \\ + \sum_h \frac{\nu_h^j + \nu_h^{-j}}{\nu_h} \operatorname{Res} \bar{R}_{\varphi\varphi}(\nu_h)\,, \end{aligned} \tag{24}$$

$$\begin{aligned} N = 2\Big[\bar{R}_{ff}(0) + \bar{R}_{\varphi\varphi}(0) + \sum_h \frac{1}{\mu_h} \operatorname{Res} \bar{R}_{ff}(\mu_h) + \\ + \sum_h \frac{1}{\nu_h} \operatorname{Res} \bar{R}_{\varphi\varphi}(\nu_h)\Big]\,, \end{aligned} \tag{25}$$

$$\begin{aligned} P_i = \sum_{j=1}^{v-1} k_{i+j} \frac{\bar{R}_{ff}^{(j)}(0)}{j!} + \sum_{j=1}^{v+i} k_{i-j} \frac{\bar{R}_{ff}^{(j)}(0)}{j!} + 2k_i\, \bar{R}_{ff}(0) + \\ + \sum_h \sum_{j=-v-i}^{v-i} k_{i+j} \frac{\mu_h^j + \mu_h^{-j}}{\mu_h} \operatorname{Res} \bar{R}_{ff}(\mu_h)\,, \end{aligned} \tag{26}$$

$$\begin{aligned} Q = \sum_{i=-v}^{v} \sum_{j=0}^{v-i} k_i\, k_{i+j} \frac{\bar{R}_{ff}^{(j)}(0)}{j!} + \\ + \sum_h \sum_{i=-v}^{v} \sum_{j=i-v}^{i+v} k_i\, k_{i+j}\, \mu_h^{j-1} \operatorname{Res} \bar{R}_{ff}(\mu_h), \end{aligned} \tag{27}$$

where e.g.

$$\bar{R}_{ff}^{(j)}(0) \equiv \left[\frac{d^j\, \bar{R}_{ff}(Z)}{dZ^j}\right]_{Z=0},$$

$$\operatorname{Res} \bar{R}_{ff}(\mu_h) \equiv \operatorname*{Res}_{Z=\mu_h} \bar{R}_{ff}(Z)\,.$$

The parameters M_j and N are the same as in the case of a follow-up system, the expressions for the parameters P_i and Q are, however, more complicated. If $k_0 = 1$ and $k_i = 0$ $(i \neq 0)$, then P_i and Q have the simpler forms shown previously.

In consequence of the formal identities the same system of linear equations for the determination of the unknown coefficients a_k, as in the case of a follow-up system, are obtained. The practical course of the calculation is, therefore, not detailed here.

3. The effect of the disturbing variable

The output signal of the system containing a single impulse-compensator has a finite settling time with respect to the disturbing variable (acting at the output of the system), if the order of the disturbing variable is not higher than m [2]. In the more general case examined now, the transfer function pertaining to the disturbing variable, by force of (13) and (10) is

$$\begin{aligned} W_u(Z) = \frac{Y_u(Z)}{X(Z)} = 1 - W(Z) = 1 - W_D(Z) - (1-Z)^m C_1(Z) = \\ = (1-Z)\,C(Z) - (1-Z)^m C_1(Z)\,. \end{aligned} \tag{28}$$

The transfer function pertaining to the disturbing variable is now not identical with the error transfer function. In the case of a disturbing variable having the form of a unit step, the transform of the controlled variable is found to be

$$Y_u(Z) = W_u(Z)\frac{1}{1-Z} = C(Z) - (1-Z)^{m-1} C_1(Z)\,. \tag{29}$$

Since $Y_u(Z)$ is a polynomial, the effect of the step-formed disturbing variable disappears after a finite settling time. This is not valid, however, for a disturbing variable of higher order, as is evident from formula (28), even in the case that $m > 1$.

If an impulse-compensator is applied also in the feedback loop, so as to eliminate or to reduce the effect of the disturbing variable, then the finite settling time and the minimum statistical error can be ensured separately for the reference input and for the disturbing variable. If we content ourselves with the compensation of the step-formed disturbing variable, the method serving for the determination of the transfer function of impulse-compensators described in [2], can be employed accordingly.

4. The effect of the variation of the parameters.

With the knowledge of the transfer function of the controlled system, the settling time of the system designed with the help of the method described in [1] is theoretically finite. In actual practice, however, the parameters of the controlled system are not exactly known, and neither can the impulse-compensator be produced in such a way that its transfer function would be exactly identical with the prescribed value. We should also take into consideration a slight alteration of the parameters of the system in the course of time.

It is obvious that in consequence of a slight alteration of the parameters the statistical error is also modified to a small extent only, i.e. it will differ only slightly from the minimum value. The question arises however, how the transient performance of the system is altered in consequence of a slight variation of the parameters. We shall prove that the error values occurring at the sampling instants are of the order of magnitude of the variations, and generally tend to zero with growing time. Accordingly though the system is not of finite settling time, still the recovery time remains finite.

Let us examine the error sequence i.e. the difference between the actual and the required values of the controlled variable at the sampling instants.

$$\psi[k] = y[k] - y_D[k] . \tag{30}$$

From this the error transfer function is

$$W_\psi(Z) = W(Z) - W_D(Z) . \tag{31}$$

where the expression for the transfer function of the closed system is

$$W(Z) = \frac{D(Z)\,G(Z)}{1 + D(Z)\,G(Z)} . \tag{32}$$

On the other hand, by expressing in view of (13) and (10) by theoretical quantities we find that

$$W(Z) = 1 - (1 - Z)C(Z) + (1 - Z)^m C_1(Z) . \tag{33}$$

From the two formulae the nominal expression for $D(Z)G(Z)$ is

$$D(Z)\,G(Z) = \frac{W(Z)}{1 - W(Z)} = \frac{1 - (1 - Z)\,C(Z) + (1 - Z)^m\,C_1(Z)}{(1 - Z)\,C(Z) - (1 - Z)^m\,C_1(Z)} . \tag{34}$$

Let us now assume that $D(Z)$ and $G(Z)$ depend on certain parameters $\varrho_1, \varrho_2, \ldots$. Such parameters are e.g. the gain factor, the time constants, etc. Let ϱ_i denote the nominal value of the i-th parameter. Let us assume that the individual parameters are altered to a small degree. The actual values are marked by a comma.

$$\varrho_i' = \varrho_i + \Delta\varrho_i\,; \qquad |\Delta\varrho_i| \ll \varrho_i\,. \tag{35}$$

Then the altered (actual) value of $D(Z)G(Z)$ in the first degree approximation is

$$\begin{aligned} D'(Z)\,G'(Z) &\approx D(Z)\,G(Z) + \sum_i \left[\frac{\partial D(Z)\,G(Z)}{\partial \varrho_i}\right]_{\varrho_i} \Delta\varrho_i \equiv \\ &\equiv D(Z)\,G(Z) + \sum_i S_i(Z)\,\Delta\varrho_i\,, \end{aligned} \tag{36}$$

where $D(Z)G(Z)$ denotes the nominal expression and the function $S_i(Z)$ can be regarded as known. On substituting the former from (34), the expression for the error transfer function, in view of (31) and (32), will be

$$W_\varphi'(Z) = \frac{D'(Z)\,G'(Z)}{1 + D'(Z)\,G'(Z)} - W_D(Z) = \tag{37}$$

$$= \frac{(1-Z)^m C_1(Z) + [(1-Z)C(Z) - (1-Z)^m C_1(Z)](1-Z)C(Z)\sum_i S_i(Z)\,\Delta\varrho_i}{1 + [(1-Z)\,C(Z) - (1-Z)^m\,C_1(Z)]\sum_i S_i(Z)\,\Delta\varrho_i}\,.$$

On expanding in series down to the linear term,

$$\begin{aligned} W_\varphi'(Z) &= (1-Z)^m C_1(Z) + [(1-Z)C(Z) - \\ &- (1-Z)^m C_1(Z)]^2 \sum_i S_i(Z)\Delta\varrho_i\,. \end{aligned} \tag{38}$$

If the reference input is of p-order ($p \leq m$), then the transform of the error is

$$\begin{aligned} \Psi_p'(Z) = W_\varphi'(Z)\,\frac{\Phi_p(Z)}{(1-Z)^p} &= (1-Z)^{m-p}\,C_1(Z)\,\Phi_p(Z) + \\ + (1-Z)^{2-p}\,[C(Z) - (1-Z)^{m-1}\,C_1(Z)]^2\,&\Phi_p(Z)\,\Sigma\,S_i(Z)\,\Delta\varrho_i\,, \end{aligned} \tag{39}$$

where $\Phi_p(Z)$ is a polynomial of the $(p-1)$-order. The first term is a polynomial, denoting the error component which becomes zero after the finite settling time and which also arises in the ideal case. On most occasions, the second term is not a polynomial since $S_i(Z)$ is generally not a polynomial but a rational fractional function. Accordingly this term describes an error component appearing at the sampling instants also after the elapse of the settling time. This error component, however, is proportional to the small deviations $\Delta\varrho_i$, hence it is small in itself as was stated earlier.

Let us also determine the steady-state error. According to the limit value theorem of the discrete transformation,

$$\psi_p' [\infty] = \lim_{k \to \infty} \psi_p' [k] = \lim_{Z \to 1} (1 - Z)\, \Psi_p' (Z) \tag{40}$$

On employing this in formula (39),

$$\psi_1 [\infty] = 0, \quad m = 1, 2, 3; \tag{41}$$

$$\psi_2 [\infty] = 0, \quad m = 2, 3; \tag{42}$$

$$\psi_3 [\infty] = \sum_i [\lim_{Z \to 1} C^2 (Z)\, \Phi_p (Z)\, S_i (Z)]\, \Delta\varrho_i, \quad m = 3\,. \tag{43}$$

The steady-state value of the actuating error is accordingly zero in the case of a step function or of a ramp function reference input, while in the case of an acceleration step function reference input it is finite, but proportional to $\Delta\varrho_i$, hence small.

It follows from the preceding that the system with theoretically finite settling time will not in reality have a finite settling time. The errors appearing at the sampling instants however have the same order of magnitude after the elapse of the settling time as the deviation of the parameters from their nominal value. We should therefore have no fear that the system with finite settling time will perform in reality "quite badly" in consequence of the uncertainty or alteration of the parameters.

5. A numerical example

Let us assume that the transfer function of the controlled system is given by

$$G_S(s) = \frac{K_0}{s(sT_0 + 1)} = \frac{K_0\, \delta}{s(s + \delta)}\,. \tag{44}$$

The task is to design a follow-up system the settling time of which is minimum and which follows only a step function reference input without steady-state error. The nominal parameter values are

$$k_0 = 1, \quad \delta = 1\,. \tag{45}$$

By omitting the details of the calculation only the results are given here. The following cases were examined:

1) $K_0 = 1.0$, $\delta = 1$ (nominal values);
2) $K_0 = 0.9$, $\delta = 1$;
3) $K_0 = 1.0$, $\delta = 0.9$.

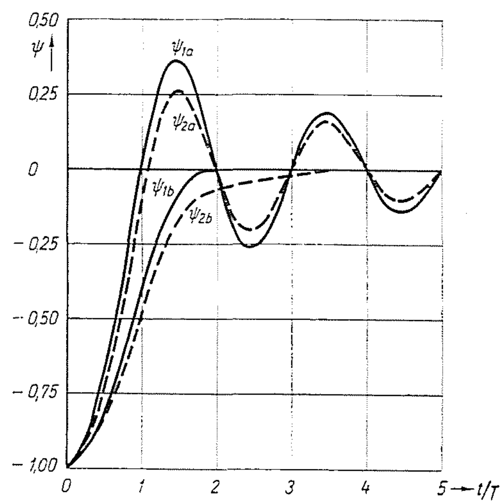

Fig. 1

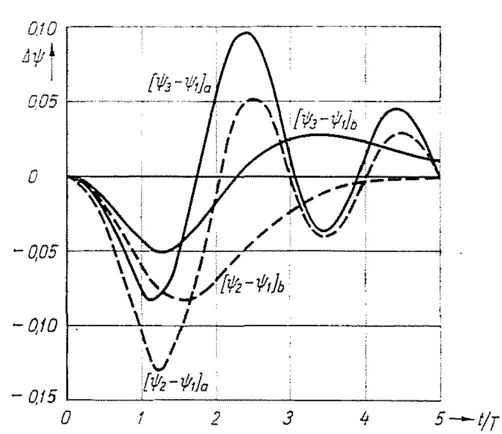

Fig. 2

The error signal was determined
a) in a not ripple-free system,
b) in a ripple-free system.

The curve $\psi_{1a} = y_{1a} - 1$ in *Fig.* 1 represents the error signal of the not ripple-free system in the nominal case, while the curve ψ_{1b} represents the same signal, if $K_0 = 0.9$. Since the overshoot is smaller, this case is even more advantageous in practice.

The corresponding curves for the ripple-free system are also illustrated in the figure. Here dynamical characteristics are already impaired by deviations from the nominal value.

For a better demonstration of the effect of parameter variations, we have charted in *Fig. 2* the difference between the actual and the nominal errors. In accordance with the parameter variations of 10 p.c., the order of magnitude of the absolute value of the error differences will be 10 p.c. of the maximum error value. It is evident from the figure that the error differences are decreasing rapidly, as we have already proved theoretically.

Summary

A method has been given to design a sampled-data control system the settling time of which is finite and the statistical error is minimum. The required transfer function pertaining to the determined reference input may be an arbitrary power series in terms of Z, with finite number of terms, while the required transfer function pertaining to the random reference input may be an arbitrary generalized power series, with finite number of terms. On determining the transfer function of the closed system and of the impulse-compensator a similar system of equations is to be solved as when designing a follow-up system, only the coefficients should be calculated in a little different way. The procedure is also applicable in the case of separate compensation for the disturbing variable.

It was proved that the error of the system having theoretically a finite settling time is small and tends to zero on most occasions, even in the case when the system parameters deviate a little from their nominal values. On the whole, the sampled-data control system with finite settling time is not more sensitive to the variation of the parameters than the system of continuous operation.

References

1. Fodor, Gy.: The Synthesis of Sampled-data Control Systems with Finite settling Time. Periodica Polytechnica, Electrical Engineering 7, 2. p. 111-126. (1963).
2. Fodor, Gy.: The Effect and the Compensation of the Disturbing Variable in Sampled-Data Control Systems with Finite Settling Time. Periodica Polytechnica, Electrical Engineering 7, 4. p. 267-279. (1963).

Dr. György Fodor, Budapest XI., Műegyetem rkp. 3, Hungary.