THE THEORY OF TRANSMISSION LINES CONSISTING OF CYLINDRICAL LEADS, ON THE BASIS OF THE ELECTROMAGNETIC FIELD

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Introduction

The propagation ratio of transmission lines is generally calculated on the basis of the Kelvin Telegraph Equations, which are equations for the voltage and current of the transmission line. The Kelvin Telegraph Equations can also be obtained on the basis of KIRCHHOFF's equations written for a section of the differential length of the transmission line, as well as directly from Maxwell's equations. In neither of these methods is the axial displacement current in the dielectric taken into consideration, arising in the case of a lossy lead. Transmission lines were examined on the basis of the electromagnetic field by *Mie* [1]. *Mie* has solved the problem of two leads having an identical cross-section and material in the bipolar coordinate system.

In the following the calculation of transmission lines on the basis of the electromagnetic field, but differing from that of *Mie*, will be presented, where known results are obtained from the boundary conditions written for the electromagnetic field. On the other hand the limitations of known methods of calculation will be pointed out. In this theory the axial displacement current is also taken into consideration and the examination of systems consisting of leads having a different cross-section or material, or of multileads systems is similarly possible. In theory voltages and currents have a role only as quantities exciting the field. Consequently reflexions can be calculated with this method only indirectly. The electromagnetic field of the transmission line is built on the theory taken from the Sommerfeld surface waves.

The Sommerfeld surface wave

The Sommerfeld surface wave is the cylindrically symmetrical electromagnetic field of current I flowing in a cylindrical lead of finite specific conductivity [2], [3]. The equation describing the electromagnetic field can be obtained by determining a solution of Maxwell's equations in cylindrical coordinates, with such a cylindrically symmetrical Tm mode, by which the time variable is sinusoidal and in which a travelling wave of propagation ratio γ is described in the axial direction, i.e. in the direction of the coordinate z. The so obtained solution is

$$E_z = C\, Z_0(gr)\, e^{j\omega t - \gamma z}$$

$$E_r = C\,\frac{\gamma}{g}\, Z_1(gr)\, e^{j\omega t - \gamma z} \tag{1}$$

$$H_\varphi = C\,\frac{\sigma + j\omega\varepsilon}{g}\, Z_1(gr)\, e^{j\omega t - \gamma z} = C\,\frac{j\omega\varepsilon_k}{g}\, Z_1(gr)\, e^{j\omega t - \gamma z}$$

$$E_\varphi = H_r = H_z = 0$$

and

$$\gamma^2 = g^2 + (j\omega\sqrt{\mu\varepsilon_k})^2 = g^2 + k^2 \tag{2}$$

where ε denotes the permittivity of the dielectric medium surrounding the lead, σ its specific conductivity, $\varepsilon_k = \varepsilon\left(1 - j\,\frac{\sigma}{\omega\varepsilon}\right)$ the complex permittivity containing the specific conductivity too. and μ the permeability. $Z_0(x)$ and $Z_1(x)$ are the zero order and first order cylindrical functions, respectively, the solutions of Bessel's differential equations of the zero and first order.

Equations (1) and (2) should be written separately for inside the lead and for the field outside the lead, while the propagation ratio γ is identical in both parts of the space. Equations written for inside the lead are discussed in detail in the literature on skin effect [3]. Consequently this subject will not be discussed here in detail, only some results will be made use of. On writing the quotient of the tangential components of the electric and magnetic field strength (E_φ and H_z respectively) for the surface of the lead, on the basis of the field inside the lead, we find that

$$\frac{E_z}{H_\varphi} = \frac{p}{\sigma_v}\,\frac{J_0(pa)}{J_1(pa)} \tag{3}$$

Here

$$p^2 = -j\omega\mu\sigma_v \tag{4}$$

In equation (3), $J_0(x)$ and $J_1(x)$ denote BESSEL's functions of the first kind, and first order of the zero respectively, σ_v, μ_v and a the specific conductivity permeability and radius of the lead, respectively. The expression for p^2 as given in equation (4) is valid if the displacement current in the lead can be neglected in comparison with the conduction current. This condition is valid with good approximation in the case of the usual lead materials even at a frequency of several 10^9 Hz. The H_f value arising at the surface of the lead can be expressed by the current in the lead, on employing the excitation law, as

$$H_\varphi = \frac{I}{2\pi a} \qquad (5)$$
$$r = a$$

Equation (5) is naturally correct, if H_φ is written either from the equations valid inside the lead, or outside of it. Upon substituting (5) in (4) we find that

$$E_z = I \frac{2\pi ap}{\sigma_v} \frac{J_0(pa)}{J_1(pa)} = I(R_b + j\omega L_b) = I \cdot Z_b \qquad (6)$$

The inside impedance Z_b of the lead on taking the skin effect into account, relating to the unit length of the lead is defined by the relationship (6). The real part of this is the internal resistance of the lead R_b, the imaginary part is the internal reactance, while L_b denotes the internal induction coefficient. These values are independent of the propagation ratio γ as long as (4) is valid. For the numerical determination of Z_b an easily manageable formula can be deduced on the basis of (6).

Formula (6) has been written on the basis of the field inside the lead. The tangential components of E and H (E_z, H_φ) are continuous on the surface of the lead. Consequently the quotient E_z/H_φ is similarly continuous. It follows from this that E_z as written in equation (6) can be made equal with the value E_z as calculated from the field outside the lead at the point $r = a$. In the later course of the calculation, the field inside the lead or leads is taken into account just because of the relationship between the value E_z arising at the surface of the lead and the current I in the lead, as given by equation (6). In the case of several leads the assumption is that the electromagnetic field inside the lead is distorted only to a negligible extent by the field of the neighbouring leads.

Outside the lead, the boundary condition for the infinitely large radius can be satisfied if

$$Z_n(gr) = H_n^{(1)}(gr) \qquad (7)$$
$$\operatorname{Im} g > 0$$

where $H_n^{(1)}(x)$ denotes Hankel's function of the n-order and of the first kind. The constant C figuring in (1) can be expressed by the current I of the lead, when using (5) and (7).

$$C = I \frac{g}{j\omega\varepsilon_k} \frac{1}{2\pi a\, H_1^{(1)}(ga)} \qquad (8)$$

Let us write equation (6) taking (1), (7) and (8) into consideration.

$$\frac{g}{j\omega\varepsilon_k} \frac{1}{2\pi a} \frac{H_0^{(1)}(ga)}{H_1^{(1)}(ga)} = Z_b \qquad (9)$$

(9) is a transcendent equation for g, from which the propagation ratio γ can be obtained on the basis of (2), when determining g^2.

In practical cases g is at least one order of magnitude smaller than the phase factor of the plane wave propagating in the given medium. Thus, if $a \ll \lambda$ (where λ denotes the wavelength of the plane wave), then $ga < \frac{a}{\lambda} \ll 1$. Consequently HANKEL's functions can be approached by their approximative small argument expressions.

$$\underset{x\to 0}{H_0^{(1)}}(x) \approx \frac{2j}{\pi} \ln(-mjx)$$

$$m = 0{,}890536\ldots \tag{10}$$

$$\underset{x\to 0}{H_1^{(1)}}(x) \approx = \frac{2j}{\pi x}$$

By approximating (10), relation (9) is simplified.

$$-\frac{g^2}{j\omega\varepsilon_k 2\pi} \ln(-mjga) = Z_b \tag{11}$$

The conduction current of the lead of the Sommerfeld wave is closed off by the displacement current outside the lead.

The electromagnetic field of Lecher's lead

Lecher's lead consists of a pair of cylindrical leads, which have identical radius and are made of the same material (Fig. 1). The radius of the leads is designated by a, the distance between the axes by d. The current I flowing in one of the leads returns through the other. The resultant field is produced as the superposition of the fields of the two leads, with the assumption that the distribution of current density in one of the leads is influenced only to a negligible degree by the field of the other lead. This is the case if $d \ll a$.

On the basis of the aforesaid, the value of E_z at the point P being at distances r_1 and r_2 from the two leads, is on considering formulae (1), (7) and (8),

$$E_z = I \frac{g}{j\omega\varepsilon_k} \frac{1}{2\pi a H_1^{(1)}(ga)} \left[H_0^{(1)}(gr_1) - H_0^{(1)}(gr_2)\right] \tag{12}$$

Let us write equation (6) for the surface of the lead designated by 1, on the basis of (12). $r_1 = a$, $r_2 \approx d$.

$$E_z = IZ_b = \frac{g}{j\omega\varepsilon_k} \frac{1}{2\pi a H_1^{(1)}(ga)} \left[H_0^{(1)}(ga) - H_0^{(1)}(gd)\right] = I(Z_{11} - Z_{12}) \tag{13}$$

where Z_{11} denotes as a definition the exact value of the self-impedance of the lead designated by l, and its approximative value, on the basis of (10), is found to be

$$Z_{11} = \frac{g}{j\omega\varepsilon_k} \frac{1}{2\pi a} \frac{H_0^{(1)}(ga)}{H_1^{(1)}(gd)} \approx \frac{-g^2}{j\omega\varepsilon_k 2\pi} \ln(-mjga) \tag{14}$$

On defining the expression for the mutual impedance Z_{12} is similarly,

$$Z_{12} = \frac{g}{j\omega\varepsilon_k} \frac{H_0^{(1)}(gd)}{2\pi a_2 H_0^{(1)}(ga_2)} \approx \frac{-g^2}{j\omega\varepsilon_k 2\pi} \ln(-mjgd) \tag{15}$$

Z_{11} and Z_{12} denote the impedances per unit length of the lead. We have written in formula (15) the subscript 2 below radius a, taking into consideration that

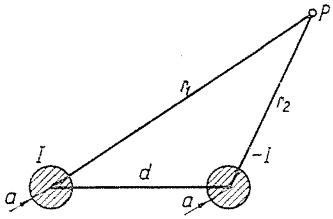

Fig. 1

this is the guiding radius of the lead designated by 2. The approximative expression for Z_{12} can be employed, if $d \ll \lambda$.

On the basis of (14) and (15), on substituting the approximative expressions for Z_{11} and Z_{12} into formula (13), we find that

$$Z_b = Z_{11} - Z_{12} = \frac{g^2}{j\omega\varepsilon_k 2\pi} \ln\frac{d}{a} \tag{16}$$

On the basis of (16) g^2 can be determined and thus we obtain γ^2 from relationship (2). As a consequence of the symmetry of the leads, when writing equation (13) for the lead designated by 2, we again obtain (16) ($Z_{12} = Z_{21}$).

With the self-impedance defined in (14), the relationship corresponding to formulae (9) and (11) written for the Sommerfeld lead, is found to be

$$Z_{11} = Z_b \tag{17}$$

The reciprocal value of the multiplier of $g^2/2$ figuring in formula (16) can be expressed by capacity C between the leads and by the conductance

$$\frac{j\omega\varepsilon_k \pi}{\ln\frac{d}{a}} = j\omega\frac{\varepsilon\pi}{\ln\frac{d}{a}} + \frac{\sigma\pi}{\ln\frac{d}{a}} = j\omega C + G \tag{18}$$

On expressing g^2 from (16) and taking formulae (5) and (7) into consideration, we obtain

$$g^2 = (R_b + j\omega L_b)(j\omega C + G) \tag{19}$$

The external inductivity of the two leads is given by

$$L_k = \frac{\mu}{\pi} \ln \frac{d}{a} \tag{20}$$

The expression for k^2 figuring in formula (2) can be written according to (18) and (20) in the form

$$k^2 = j\omega\varepsilon_k j\omega\mu = j\omega L_k (j\omega C + G) \tag{21}$$

Substituting (19) and (21) into formula (2) the value of γ^2 can be determined.

$$\gamma^2 = [2R_b + j\omega(2L_b + L_k)][G + j\omega C] \tag{22}$$

Relationship (22) is identical with the expression for γ as determined with the help of the Kelvin Telegraph Equations. The electromagnetic field of Lecher's transmission line is essentially the sum of two identical Sommerfeld surface waves advancing in opposite directions to each other. Consequently in the case of Lecher's transmission line the total displacement current flowing in the dielectric medium in the axial direction is zero. Thus, by neglecting of displacement current, as was done during the calculation on the basis of the Kelvin Telegraph Equations, it supplies by a certain approximation a correct result. The accuracy of the approximation depends on the accuracy of approximating the Hankel functions figuring in formulae (14) and (15) by the expressions given in (10). If we want to write for g^2 an equation more accurate than that given in (19), we should approximate $H_0^{(1)}(gd)$ figuring in (13) more accurately than by formula (10). The more accurate approximation, on considering the following term too, is given by

$$\underset{x \to 0}{H_0^{(1)}}(x) = \frac{2j}{\pi}\left(1 - \frac{x^2}{4}\right) \ln(-mjx) \tag{23}$$

Considering (23), we obtain from (13) that

$$2(R_b + j\omega L_b)(j\omega C + G) = g^2 \left[1 - \frac{g^2 d^2}{4 \ln \frac{d}{a}} \ln(-mjgd)\right] \tag{24}$$

Relationship (24) is a transcendent equation for g^2. Solving this and substituting the solution into (2) we obtain an expression for γ^2 more accurate than that given in (22).

Two-lead transmission line of asymmetric arrangement

Let us now examine the case when the equality of the radii or of the specific conductivity of the two leads is not set as a condition.

We shall later see that it cannot be presumed in general that the current in the two leads differs only in the sign. The Sommerfeld surface wave lead is the example that the conductance current of the leads is closed outside the lead by a displacement current. Thus, the equation of continuity is not injured even if the total current of the leads at a given cross-section is not zero.

In the followings our calculations will be limited to that case when the approximation (10) is valid for the Hankel functions. The characteristics of the two leads will be designated by the subscripts 1 and 2 respectively.

On writing the value of E_z for the surface of the lead, similarly to formulae (12) and (13), by considering what was mentioned in connection with equation (6), we find that

$$\begin{aligned} I_1 Z_{b1} &= I_1 Z_{11} + I_2 Z_{12} \\ I_2 Z_{b2} &= I_1 Z_{21} + I_2 Z_{22} \end{aligned} \tag{25}$$

where I_1 and I_2 designate the current in the leads, Z_{b1} and Z_{b2} the interna impedance of the leads, Z_{11}, Z_{12} and Z_{12}, Z_{21} the self and mutual impedances as defined by equations (14) and (15). Taking the approximation (10) of the Hankel functions into consideration, so $Z_{21} = Z_{12}$. Assuming that $I_2 = I_1$, we generally obtain two different values for g^2, i.e. the two equations are contradictory. If only the equation obtained from the difference of the two equations is satisfied, then it can already be assumed that $I_2 = -I_1$. The equation obtained in this way is, on considering (14) and (15),

$$Z_{b1} + Z_{b2} = Z_{11} + Z_{22} - 2Z_{12} = \frac{g^2}{j\omega\varepsilon_k\pi} \ln \frac{d}{\sqrt{a_1 a_2}} \tag{26}$$

where a_1, a_2 denote the radii of the respective leads and d the distance between their axes. The reciprocal value of the multiplier of g^2 as figuing in (26), similarly to (18), is

$$\frac{j\omega\varepsilon_k \pi}{\ln \dfrac{d}{\sqrt{a_1 a_2}}} = j\omega C + G \tag{27}$$

Expressing γ^2 from equations (2), (21), (26) and (27), we find that

$$\gamma^2 = [R_{b1} + R_{b2} + j\omega (L_k + L_{b1} + L_{b2})] [G + j\omega C] \tag{28}$$

Formula (28) can be obtained from the Kelvin Telegraph Equations. This means that in the case of asymmetric leads the solution supplied by the Kelvin Telegraph Equations contains an approximation, since the two equations given under (25) are not individually satisfied, only the difference between the two.

(25) is a homogeneous linear system of equations for I_1 and I_2. This has a solution different from the trivial one, if

$$(Z_{11} - Z_{b1})(Z_{22} - Z_{b2}) = Z_{12}^2 \tag{29}$$

(29) is a transcendent equation for g^2. In the case of Lecher's transmission line $Z_{b1} = Z_{b2} = Z_b$, $Z_{11} = Z_{22}$, thus

$$Z_{11} - Z_b = \pm Z_{12}$$

By putting the positive sign at the right side of the equation we obtain (16). For the interpretation of the solution valid for the case of a negative sign we shall return later.

By substituting (29) into some of the equations given under (25) we obtain the ratio of the two currents.

$$\frac{I_1}{I_2} = \mp \sqrt{\frac{Z_{22} - Z_{b2}}{Z_{11} - Z_{b1}}} \tag{31}$$

In the case of Lecher's transmission line

$$\frac{I_1}{I_2} = \mp 1 \tag{32}$$

By using the upper sign we find that $I_1 = -I_2$. This is just the case discussed in the previous chapter.

By taking the lower sign $I_1 = I_2$. In this case the pair of leads transmits a Sommerfeld surface wave by essentially forming a single lead. Different propagation ratios belong to the two current ratios.

If the leads are asymmetric, then we obtain two different current ratios again by taking the two different signs in (31), but the magnitude of the currents is generally not equal. By taking the upper sign, as in the case of Lecher's transmission line $I_1 = -I_2$, the examined mode will be called the L-mode, while the mode corresponding to the lower sign is called the S-mode.

The g^2 values pertaining to both the L and S-mode can be determined from (30) (g_L^2, g_S^2). Substituting these values into (2), the two γ^2 values can be obtained.

$$\gamma_L^2 = k^2 + g^2$$
$$\gamma_s^2 = k^2 + g_s^2$$

Naturally different Z_{11}, Z_{22}, Z_{12} values belong to different g^2 values. From these the ratio of the currents can be determined for the two modes.

$$\frac{I_{1L}}{I_{2L}} = -\sqrt{\frac{Z_{22L} - Z_{b2}}{Z_{11L} - Z_{b1}}} = A_L \qquad \frac{I_{1s}}{I_{2s}} = \sqrt{\frac{Z_{22s} - Z_{b2}}{Z_{11s} - Z_{b1}}} = A_s \tag{34}$$

On the basis of formula (34), the equations of the waves expanding in direction z are given by

$$\begin{aligned} i_1(z,t) &= I_{1L}\, e^{-\gamma_L Z + j\omega t} + I_{1s}\, e^{-\gamma_s Z + j\omega t} \\ i_2(z,t) &= I_{1L}\, A_L e^{-\gamma_L + j\omega t} + I_{1s}\, A_s\, e^{-\gamma_s Z + j\omega t} \end{aligned} \tag{36}$$

The degree of the establishment of the L and S-mode in a given case depends on the kind of excitation. If the transmission line is excited from its voltage sources and it is not specified that the currents in the two leads at the feeding point should be equal whereby the problem would become redundant, then only the L-mode comes into existence. If only one of the leads is excited by a current generator, while no excitation is acting on the other lead, then only the S-mode comes into existence. If, in turn, both leads are excited by a current generator and the values of currents flowing in the leads at a given place are specified as I_1 and I_2, then generally the L and S-modes jointly come into existence. Let the given place be $z = 0$. Then in view of formula (36),

$$\begin{aligned} I_1 &= I_{1L} + I_{1s} \\ I_2 &= I_{1L}\, A_L + I_{1s}\, A_s \end{aligned} \tag{37}$$

From formula (37) we obtain

$$I_{1L} = \frac{I_1 A_s - I_L}{A_s - A_L} \qquad I_{1s} = \frac{I_2 - A_s I_1}{A_s - A_L} \tag{38}$$

In the matrix equations (37) and (38)

$$\mathbf{I} = A\,\mathbf{I}_1 \quad \mathbf{I}_1 = A^{-1}\,\mathbf{I} \tag{39}$$

where

$$\mathbf{I} = \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \quad \mathbf{I}_1 = \begin{bmatrix} I_{1L} \\ I_{1s} \end{bmatrix} \qquad 40$$

and

$$A = \begin{bmatrix} 1 & 1 \\ A_L & A_s \end{bmatrix} \quad A^{-1} = \frac{1}{A_s - A_L} \begin{bmatrix} A_s & -1 \\ -A_L & 1 \end{bmatrix} \tag{41}$$

Multi-lead systems

In the case of multi-lead systems the value E_z can be written for the surface of each lead as the sum of the E_z values produced by all the leads, that is equal to the current in the lead multiplied by the internal impedance of the lead (similarly to equation (25). If the number of leads is designated by n, the equation which can be written for the k-th lead is given by

$$I_k Z_{bk} = I_1 Z_{1k} + I_2 Z_{2k} + \ldots + I_k Z_{kk} + \ldots + I_n Z_{nk} \tag{42}$$

The equations written for the surface of each lead supply a homogeneous system of equations for the I_k values, consisting of n equations. This system has a solution different from the trivial one if

$$D = \begin{vmatrix} Z_{11} - Z_{b1} & Z_{12} & \dots Z_{1n} \\ Z_{21} & Z_{22} - Z_{b2} & \dots Z_{2n} \\ \cdot & \cdot & \dots \cdot \\ Z_{n1} & Z_{n2} & \dots Z_{nn} - Z_{1n} \end{vmatrix} = 0 \qquad (43)$$

where Z_{bk} denotes the internal impedance of the leads as calculated by equation (6), Z_{kk} the self-impedance defined by equation (14) for the k-th lead, while Z_{kj} the mutual impedance between the k-th and j-th leads as determined with the aid of (15),

$$Z_{kj} = \frac{-g^2}{j\omega\varepsilon_k 2\pi} \ln(-mjgd_{kj}) \qquad (44)$$

where d_{kj} denotes the distance between the axes of the k-th and j-th leads. The g^2 values can be determined from (43). The numerical calculation is cumbersome since (43) is a transcendent equation. The equation generally has n roots. Thus, we generally obtain n values for γ^2 too. This means that in a system consisting of n leads generally n modes come into existence, all having different expanding ratios. A determined current ratio belongs to every mode. Let the Greek letters $\alpha, \beta, \dots, \nu$ designate the different modes. The current ratios pertaining to the k-th mode can be determined from the expression D_k arising in the case of the k-th mode for D. The quotient of the currents of the 1-th and k-th leads is the quotient of the minor pertaining to the 1-th and k-th elements in the last row of D_k,

$$\frac{I_{1k}}{I_{kk}} = \frac{D^n_{k1}}{D^n_{kk}} A_{kk} \qquad (45)$$

This means that in the case of a given mode, by prescribing the value of some of the leads, in the present case that of the lead designated by 1, all the other currents have a determined value.

Equation (39) can be generalized for the case of several leads. The currents of the individual leads can be summarized in column matrix $\mathbf{I}$ in the case of a given value z. Similarly the currents pertaining to different modes of the lead designated by 1 this can be summarized in a column matrix $\mathbf{I}_1$. From $\mathbf{A}$ a square matrix can be formed

$$\mathbf{I} = \begin{bmatrix} I_1 \\ I_2 \\ \cdot \\ \cdot \\ I_n \end{bmatrix} \qquad \mathbf{I}_1 = \begin{bmatrix} I_{1\alpha} \\ I_{1\beta} \\ \cdot \\ \cdot \\ I_{1\nu} \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} 1 & 1 & \dots 1 \\ A_{2\alpha} & A_{2\beta} & \dots A_{2\nu} \\ \cdot & \cdot & \dots \cdot \\ A_{n\alpha} & A_{n\beta} & \dots A_{n\nu} \end{bmatrix} \qquad (46)$$

The correlation between I and I_1 is given by

$$\mathbf{I} = A\,\mathbf{I}_1 \quad \mathbf{I}_1 = A^{-1}\,\mathbf{I} \tag{47}$$

In the case of a given I the values of I_1 belonging to the various modes can be determined with the help of (47). These are all expanding with different γ expansion ratios. Consequently the ratio of the currents in the leads depends on the z coordinate. At a given place z, the $\mathbf{i}(z)$ values can be determined on the basis of (47) from the $\mathbf{i}_1(z)$ values.

$$\mathbf{i}_1(z) = \begin{bmatrix} I_{1\alpha}\, e^{-\gamma_\alpha Z} \\ I_{1\beta}\, e^{-\gamma_\beta Z} \\ \vdots \\ I_{1\nu}\, e^{-\gamma_\nu Z} \end{bmatrix} \quad \mathbf{i}(z) = A\,\mathbf{i}_1(z) \tag{48}$$

The case of multiple roots, ideal leads

Our considerations so far were concerned with the case when equation (43) has n pieces of different roots. If the equation has identical roots, the number of expansion ratios is less than n. At the mode belonging to a double root, the value of the current in the given lead does not determine the other currents on the basis of (45). On this occasion the given mode is possible in the case of as many arbitrarily chosen current values as the number specifying the multiplicity of the root indicates. Actual current values are determined by the excitations. The case of the multiple roots will however not be discussed in detail, only a special case, the conditions of the ideal lead will be examined.

In the case of ideal leads the internal impedance Z_b is zero. The Sommerfeld surface wave does not come into existence in this case. Consequently the conditions of ideal leads can only be discussed as limit cases.

In the case of $Z_b = 0$ and of Lecher's lead we obtain for g^2 zero from (19). In the general case $g^2 = 0$ is also a solution of (43) where 0 is an n-fold root. By substituting the value $g^2 = 0$ into equation (21) we obtain for the expansion ratio that

$$\gamma = \pm k = \pm j\omega \sqrt{\mu\varepsilon_k} \tag{49}$$

In the case of an ideal lead the expansion ratio is identical with that occurring in the given dielectric medium for plane waves. In the dielectric medium no displacement current flows in the axial direction, consequently the total current of the leads should supply zero in a given cross-section. Actually arising current values are determined by the excitation.

Approximative calculation of pairs of leads

The numerical solution of equation (43) is cumbersome. If several parallel pairs of Lecher's leads are arranged in the space, a simpler not transcendent equation can be deduced from (43) with the same approximation as in the case of asymmetric pairs of leads in equation (26) an approximation in comparison with (25) and (29) respectively.

Even when the individual pairs of leads are of symmetrical arrangement, even in that case, the electric conditions will be asymmetric depending on the arrangement of the other pairs of leads. Let us first examine a system consisting

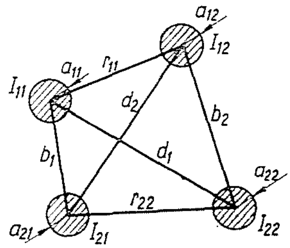

Fig. 2

of two pairs of leads (Fig. 2). The currents in the first pair of leads are designated by I_{11}, I_{12}, while in the second pair by I_{21}, I_{22}. Radii a_{11}, a_{22}, distances r_{11}, r_{22}, b_1, b_2, d_1, d_2 are indicated in Fig. 2.

Let us write the equation (42) for the E_z values arising at the surface of the individual leads, by considering (14) and (44).

$$I_{11} Z_{b1} = \frac{-g^2}{j\omega\varepsilon_k 2\pi} [I_{11} \ln(-jmga_{11}) + I_{12} \ln(-jmgr_{11}) + + I_{21} \ln(-jmgb_1) + I_{22} \ln(-jmgd_1)]$$

$$I_{12} Z_{b1} = \frac{-g^2}{j\omega\varepsilon_k 2\pi} [I_{11} \ln(-jmgr_{11}) + I_{12} \ln(-jmga_{11}) + + I_{21} \ln(-jmgd_2) + I_{22} \ln(-mjgb_2)]$$

$$I_{21} Z_{b2} = \frac{-g^2}{j\omega\varepsilon_k 2\pi} [I_{11} \ln(-jmgb_1) + I_{12} \ln(-jmgd_2) + + I_{21} \ln(-jmga_{22}) + I_{22} \ln(-jmr_{22})]$$

$$I_{22} Z_{b2} = \frac{-g^2}{j\omega\varepsilon_k 2\pi} [I_{11} \ln(-jmgd_1) + I_{12} \ln(-mjgb_2) + + I_{21} \ln(-jmgr_{22}) + I_{22} \ln(-jmga_{22})] \tag{50}$$

In the following the approximation will be employed, in which $I_{11} = -I_{12} = I_1$ and $I_{21} = -I_{22} = I_2$. In this case the 1st and 2nd equations given under (50), and similarly the 3rd and 4th equations, generally become contradictory. As to avoid contradictions in the employed equations, all the four equations under (50) will not be satisfied separately. On deducing the 1st and 2nd equation, and the 3rd and 4th equation under (50) one from the other, we obtain two equations. The satisfaction of these two will not be contradictory with the above approximation.

$$I_1 2Z_{b1} = \frac{g^2}{j\omega\varepsilon_k \pi}\left[I_1 \ln \frac{r_{11}}{a_{11}} + I_2 \ln \frac{r_{22}}{a_{22}}\right] \tag{51}$$

$$I_2 2Z_{b2} = \frac{g^2}{j\omega\varepsilon_k \pi}\left[I_1 \ln \frac{r_{12}}{a_{12}} + I_2 \ln \frac{r_{22}}{a_{22}}\right]$$

where

$$r_{12} = \sqrt{d_1 d_2} \qquad a_{12} = \sqrt{b_1 b_2} \tag{52}$$

By generalizing formula (51) for a system consisting of n pairs of leads, the equation written for the k-th pair of leads is given by

$$I_k 2Z_{bk} = \frac{g^2}{j\omega\varepsilon_k \pi}\left[I_1 \ln \frac{r_{1k}}{a_{1k}} + I_2 \ln \frac{r_{2k}}{a_{2k}} + \ldots + I_n \ln \frac{r_{nk}}{a_{nk}}\right] \tag{53}$$

Writing equation (53) in the form of a matrix equation,

$$\left(\frac{g^2}{j\omega\varepsilon_k \pi}\boldsymbol{M} - \boldsymbol{Z}_b\right)\mathbf{I} = 0 \tag{54}$$

where

$$\mathbf{I} = \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_n \end{bmatrix} \qquad \boldsymbol{Z}_b = 2\begin{bmatrix} Z_{b1} & 0 & \ldots 0 \\ 0 & Z_{b2} & \ldots 0 \\ \cdot & \cdot & \cdots \\ 0 & 0 & \ldots Z_{bn} \end{bmatrix}$$

$$\boldsymbol{M} = \begin{bmatrix} \ln \frac{r_{11}}{a_{11}} & \ln \frac{r_{12}}{a_{12}} & \ldots & \ln \frac{r_{1n}}{a_{1n}} \\ \ln \frac{r_{21}}{a_{21}} & \ln \frac{r_{22}}{a_{22}} & \ldots & \ln \frac{r_{2n}}{a_{2n}} \\ \cdot & \cdot & \cdots & \cdot \\ \ln \frac{r_{n1}}{a_{n1}} & \ln \frac{r_{n2}}{a_{n2}} & \ldots & \ln \frac{r_{nn}}{a_{nn}} \end{bmatrix} \tag{55}$$

(54) is a homogeneous linear system of equations for the I values. This has a solution differing from the trivial one, if the value of the determinant formed from the coefficients of the equations is zero, i.e.

$$\left|\frac{g^2}{j\omega\varepsilon_k \pi}\boldsymbol{M} - \boldsymbol{Z}_b\right| = 0\,. \tag{56}$$

Equation (56), contrary to (43), is not a transcendent algebraic equation of n-order for g^2. The equation generally supplies n pieces of roots. That is to say, by the approximation described here the number of roots has been reduced from $2n$ to n. On the basis of (2), to each root of g^2 a γ^2 value and a mode pertains. Similarly as in the case of relationship (47), given currents can be decomposed in accordance with their mode and thus the expansion of the waves can be calculated.

As we shall show in a paper to be published later, equation (56) can be deduced from the Telegraph Equations generalized for systems of leads, just the same as equation (26) can be deduced from the Telegraph Equations. The deduction from the Telegraph Equations has the advantage compared to the calculation method presented here, that there the voltages are also figuring. Thus if voltages at the beginning of the transmission line are given, further if we want to calculate the reflexion of transmission lines, the calculation on the basis of the Telegraph Equations is more advisable. The deduction of equation (56), however, has shown the degree of approximation when using the calculation based on the Telegraph Equations instead of the previously discussed more precise calculation.

Summary

The theory of transmission lines consisting of cylindrical leads can be discussed on the basis of the electromagnetic field, as the superposition of the Sommerfeld surfaces waves of the individual leads. In the case of Lecher's pair leads of, if the radius and the distance of the leads are negligible in comparison with the wavelength, the approximative expression for the expansion ratio is identical with the result obtained from the telegraph equations. In the course of calculations performed with the help of the Kelvin Telegraph Equations the axia displacement current in the dielectric medium is neglected. In the case of leads of different radii or materials, if the axial displacement currents are taken into consideration, it becomes evident that the currents in the leads are generally not identical. In the case of a multi-lead system the number of modes with different propagation ratios pertaining to waves expanding either in the positive or negative direction is identical with the number of the leads. At each mode the ratio of the currents in the leads is a determined value. Accordingly in the case of given currents, when calculating a system of leads, currents should be decomposed to modes and if necessary, at a given place, modes should be composed. In the case of ideal leads values of different expansion ratios coincide and are identical with the expansion ratio arising in the case of a plane wave in the dielectric medium.

The described method has two difficulties in the course of practical calculations. One is that transcendent equations are to be solved by numerical calculations. The other is that voltages are not figuring directly in the equations, consequently the calculation of the reflexion, the input impedance in the case of a given termination is cumbersome.

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