METHODS OF MATHEMATICAL STATISTICS FOR EVALUATING ELECTRIC BREAKDOWN MEASURING SERIES

By

E. NÉMETH and E. CSÁKI

Department for High Voltage Technics and Equipments of the Technical University of Budapest and Research Institute for Mathematics of the Hungarian Academy of Sciences

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1. Introduction

The aim of measuring is to determine the value of a quantity. The unavoidable error impedes the obtaining by only one measurement either the value to be measured or the error committed. The simplest method to reduce the error, i.e. to increase accuracy is to several times repeat the measurement. Evaluating the results of this series of measurements it becomes possible to determine the value of the quantity and the value of the error, respectively.

From the point of view of evaluation the measurements can be placed into two categories. The first is the category of the so-called reproducible measurements. Here the aim of measuring is to determine by repeated measurements the value of a quantity which does not change during the series of measurements (e.g. a resistor, a capacitor, etc.). Several times repeating the measurements is to assure that the unavoidable errors made with each measurements should level off and thus to obtain the value of the quantity investigated with the highest possible accuracy. The errors committed are mainly due to the inaccuracies of the employed equipments (sub-standards, instruments, etc.) but the personal error may be important, too. With this type of measurements the error committed may be greatly reduced by increasing the accuracy of the equipment used and by appropriately training the persons making the measurements, as the errors are not characteristic for the quantity and material, respectively, to be measured.

To the second category belong the non-reproducible measurements, those where the quantity to be measured is not the same when the measurings are repeated. So are those measurings where the quantity to be measured casually varies around a mean value (e.g. the number of ionized particles in the interelectrode space, that is to say the flash-over voltage between the electrodes) or measurings where the sample investigated is ruined. In this latter case the measurement can only be reproduced on another, similar specimen (e.g. measuring breakdown strength).

The divergences between the single measuring results are caused by other factors in the two categories of measurements. Thus the evaluation of the measurements from a certain point of view is different, too, in the two categories.

In the category of reproducible measurements only the actual errors due to the imperfection of the equipment appear. Assuming that no systematic errors are committed, the results of a measuring series will assemble around the true value of the quantity to be measured. In this case the aim of evaluation is to eliminate the error and thus to determine the true value. With reproducible measurements it is possible to reduce the error and to approach with discretion the accuracy of the true value by increasing the number of measurements. As a result of measuring the true value is guessed and the error limits characteristic for the equipment are given.

In the second category are the strictly speaking non-reproducible measurements. Here the divergences of the results are not only caused by the errors explained above, but by the random deviations of the quantity to be measured, too. These deviations appearing while measuring are due to unknown factors, thus it is preferable to discern them from the error and to call them "deviation". The deviation is not characteristic of the measuring equipment but of the quantity to be measured and of the investigated material, respectively. (E.g. by measuring the breakdown strength the deviation may be 10 to 20 fold the value of the error and is characteristic for the material investigated.) When evaluating the results of measurements of this category it is not possible to speak about a true value, as such a value is not constant, it fluctuates around the most probable value. Thus the results of the single measurements assemble around this value. With non-reproducible measurements the true values are different. On the other hand the rate of divergence of the true values, the deviation is characteristic for the material investigated.

In this way e.g. by determining the dielectric strength of an insulating material it is not enough to give the mean of the measured results (the most probable value of breakdown strength) in order to characterize the insulator, for dimensioning a security factor should be used, too. This factor covers among others the possibility that the material investigated may show breakdown strengths greatly inferior to the given value.

On the other hand the value of the deviation may greatly vary between insulating materials of different qualities. With uniform material of good quality the deviation is small. A bad quality material shows higher deviations, i.e. the difference between the lowest breakdown strength and mean strength is higher. There may be two materials, having the same mean value, but of different qualities. The one showing higher deviations will only match the other by considering the identity of mean values, but will have a far smaller minimum breakdown strength, i.e. a much lower security factor when fitted into an equipment (Fig. 1). Using a good quality insulator it may happen to have a too high security factor, i.e. the construction is oversized. With low quality insulators and a low security factor too high stresses may occur.

Therefore, it would be advisable to take as a basis for economic dimensioning the lowest breakdown strength, as a puncture cannot happen at a lower field strength. Thus, there should be no need, when determining the security factor, to take the deviation of breakdown strength into consideration. Its value may be lower and better advantage could be taken of the insulator.

Evaluating measuring series of the second category the mean value (most probable value) is not the only characteristic value. It is advisable to complete it by a value characterizing the deviation (value of deviation, the prospective lowest or highest value, etc.).

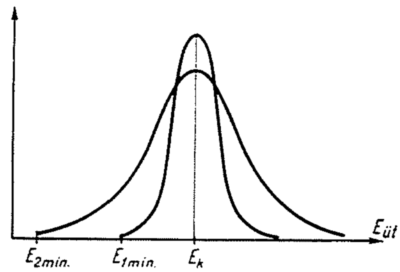

Fig. 1

The above mentioned characteristic values may be directly obtained by evaluating a series of numerous measurements. The methods of mathematical statistics afford the possibility of concluding on the characteristic values even on the basis of shorter series, by making certain assumptions. The present treatise aims to propose a simple and easy way of using methods to evaluate short measuring series belonging to the category of non-reproducible (mainly destructive) tests.

2. Evaluation of measuring series

As mentioned above it is possible to evaluate shorter series of measurements with the methods of mathematical statistics and in this way to determine the values of the quantity investigated. For this evaluation it is necessary to know the distribution of measuring results in their quality as random variables. As the majority of measuring results follow the Gauss distribution, the present treatise only deals with the evaluation of measuring results following this distribution.

In paragraph 2.1 methods are given to determine whether the members of a measuring series may be considered as following the Gauss distribution.

Paragraph 2.2 shows how to establish the prospective minimum and maximum values of a measuring series by means of the methods of mathematical statistics, i.e. how to evaluate a measuring series.

However evaluating a measuring series of normal distribution there is a preliminary condition: a sufficient number of measurements must be made. Generally no criterion is given as to the adequate number of measurements. It is only stressed to make as many measurements as possible. With technical measurements and chiefly with destructive tests, belonging to the second category, it is economical to increase the number of measurements until the accuracy of evaluation substantially increases with the number of measurements.

Therefore, it could be taken as a criterion of the sufficient number of measurements how far the accuracy of evaluation is increased by the augmentation of the number of measurements. Paragraph 2.3 gives a criterion and a simple method to determine the necessary number of measurements in order to allow their reliable evaluation, both based on this train of thoughts.

Paragraph 2.4 shows measuring series can be compared among themselves by using methods of mathematical statistics. This is the problem when puncture voltages are measured and on the basis of two measuring series is to be decided whether the investigated materials can be considered as being identical or not. A similar problem is whether an external factor (e.g. the shape of an electrode) exerts a substantial influence on dielectric strength or not.

Paragraph 3 gives some numerical examples of the evaluation methods.

Finally in paragraph 4 the bases of mathematical statistics are summarized.

2.1 Investigating distribution

Methods are shown how to discern whether a random variable may be considered of normal distribution or not. First of all some mention should be made about statistical tests in general.

A statistical test always serves to decide whether an assumption (hypothesis) is right or wrong. The result of the test is always a verdict whether the original assumption was right or not.

For the sake of this — possibly two or more — numerical values, so called statistics, are determined on the basis of the measured data. The statistic to be calculated is determined by the test employed.

This statistic is in itself a random variable, too, as it was calculated from samples, i.e. random quantities. Therefore, in its quality of random variable it has a distribution. Knowing this distribution it is possible to determine the probability limits between which this random variable has to be with a pre-determined probability (e.g. 90 or 95 per cents). Is the statistic to be calculated from samples actually within these limits, the assumption is accepted, otherwise it is to be rejected.

The decision is not entirely certain. There are two possibilities for mistakes.

1. The verdict can be "not true" and actually the assumption is true.
2. The verdict can be "true" and actually the assumption is not true.

Each test's aim is to reduce the probability of deficient decisions.

2.11 The Kolmogorov test

It is known that of a sample containing n elements the mean is estimated from the relation

$$\bar{x} = \frac{x_1 + x_2 + \ldots + x_n}{n}$$

and the standard deviation from

$$s = \sqrt{\frac{\Sigma (x_i - \bar{x})^2}{n - 1}} .$$

The Kolmogorov test is a simple way to decide whether a random variable can be considered as having normal distribution.

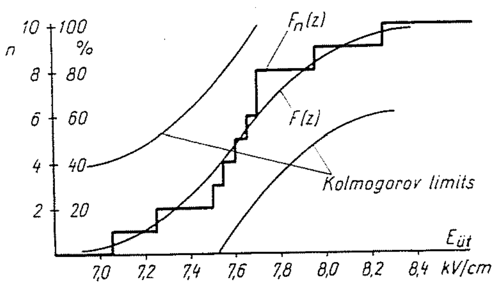

Fig. 2

Let us take the normal distribution, the mean value of which is $\bar{x}$ and the standard deviation is s, i.e. having the distribution function:

$$F(z) = \frac{1}{2 \pi s} \int_{-\infty}^{z} e^{-\frac{(x_i - \bar{x})^2}{2s^2}} \cdot dx .$$

The empirical distribution function of the sample $F_n(z)$ is formed (Fig. 2). The following quantity (trial statistic) has to be considered

$$\max_{-\infty < z < +\infty} |F_n(z) - F(z)|.$$

That is, the maximum discrepancy between the empirical distribution function and the theoretically assumed distribution function. If the assumption made is true the above quantity will be less by the prescribed probability as the quantity given by Table I. With a measuring series, if the above quantity is less than the numerical value given by Table I, the assumption of normal distribution is right, otherwise it can not be accepted.

Table I

Kolmogorov limits to examine distribution of measuring series

Number of measurements n	Kolmogorov limit h_n
5	0.51
6	0.47
7	0.44
8	0.41
9	0.39
10	0.37
12	0.34
14	0.31
15	0.30
16	0.29
18	0.28
20	0.26
25	0.24
30	0.22
35	0.20
40	0.19
45	0.18
50	0.17
55	0.16
60	0.16
70	0.15
80	0.14
90	0.13
100	0.12

A measuring series may be considered of normal distribution if the function values of step distribution fulfil the condition

$$F(z) - h_n < F_n(z) < F(z) + h_n$$

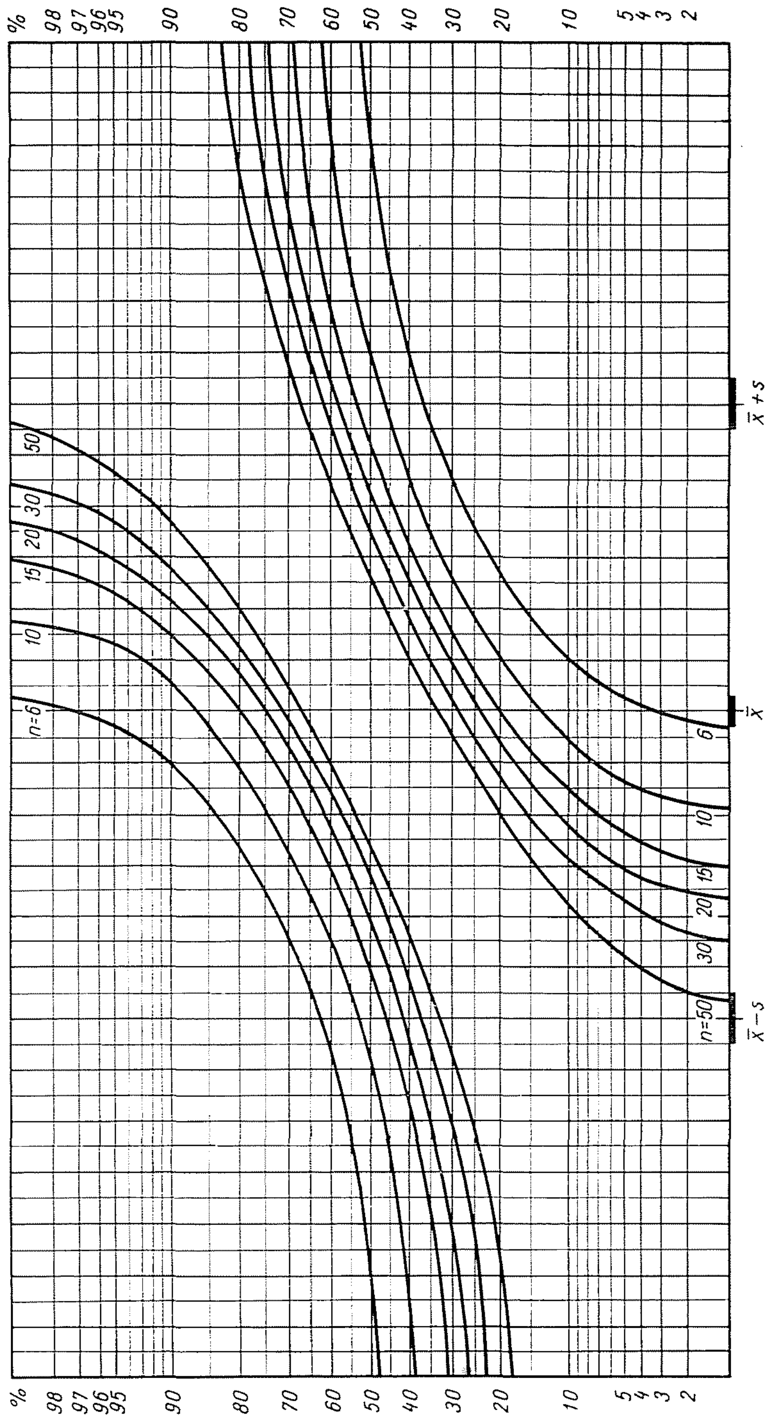
%
98
97
96
95
90
80
70
60
50
40
30
20
10
5
4
3
2
n=6
10
15
20
30
50
n=50
x̄ + s
x̄
x̄ − s

Fig. 3

The diagram in Fig. 3 gives a graphical method for easily doing the Kolmogorov test without determining the theoretical distribution function. The function $F(z)$ appears on the so-called Gauss paper shown in the figure, as a straight line. The data of the sample are put on the horizontal axis by choosing the scale as to $\bar{x}$, $\bar{x} + s$ and $\bar{x} - s$ in the assigned small interval of numbers. Then the distribution step-function of the measuring series is plotted. If these points are within the previously plotted curves belonging to the given number of elements, the assumption of normal distribution is accepted, otherwise it is rejected. (See Fig. 6.)

2.12 Other tests

Another possible method is the so-called Geary test. The following statistic has to be calculated:

$$a = \frac{\Sigma |x_i - \bar{x}|}{\sqrt{n \Sigma (x_i - \bar{x})^2}} = \frac{\text{mean deviation}}{\text{standard deviation}} .$$

With normal distribution and $n = \infty$ the theoretical value is

$$a_{\text{theor}} = \sqrt{\frac{2}{\pi}} = 0.7979 .$$

Table II gives the upper, respectively, lower limits surrounding the normal distribution with 90 per cent probability. Thus, if the above quantity is between the two, the assumption of normal distribution is accepted, otherwise it is rejected.

The test of goodness of fit χ^2 should be mentioned, too. With this test the scale of data is divided into intervals of numbers, but so that at least 5 data be in an interval. Now it is possible to calculate with the assumed distribution function $F(z)$ with which p_i probability the measuring data are within the single intervals. The number of data in the interval i are marked ν_i and the following quantity is formed

$$v = \sum_{1}^{N} \frac{(\nu_i - np_i)^2}{np_i}$$

where

N is the number of intervals

n is the total number of elements.

This statistic follows with fair approximation an χ^2 distribution of $N-1$ degree of freedom.

The lower and upper limits are given in tables [3, 4, 5.].

Table II

Geary factors to examine distribution of measuring series

Number of measurements n	Lower limit a_n	Upper limit b_n
11	0.7409	0.8899
16	0.7452	0.8733
21	0.7495	0.8631
26	0.7530	0.8570
31	0.7559	0.8511
36	0.7583	0.8468
41	0.7604	0.8436
46	0.7621	0.8409
51	0.7636	0.8385
61	0.7662	0.8349
71	0.7683	0.8321
81	0.7700	0.8298
91	0.7714	0.8279
101	0.7726	0.8264

A measuring series may be considered of normal distribution if the condition below is fulfilled:

$$a_n < \frac{\Sigma |x_i - \bar{x}|}{\sqrt{n \, \Sigma (x_i - \bar{x})^2}} < b_n .$$

This test may only be employed with a high number of sample elements. As the present treatise deals with the evaluation of samples composed of a small number of elements, a further analysis does not seem to be indicated.

In the following chapters it is always assumed that the investigation on distribution had a positive result, i.e. the data may be considered having normal distribution. In the majority of practical cases this condition is covered.

2.2 Determination of the expected minimum and maximum values

With destructive tests, e.g. determining the breakdown strength of insulators, it is very important to know the smallest expected value, of which a smaller can only occur with an insignificant probability. If the exact distribution of the measured data, as a random variable were known, there would be no difficulty in determining such a lower limit with the aid of the probability limits of this distribution. The exact distribution is but unknown, the mean and the standard deviation have been guessed on the basis of a sample composed of a relatively small number of elements.

Among some possibilities the method exposed by HALD in his book to determine the limits of tolerance, should be retained [2].

The lower limit should be determined against which smaller results could be only obtained with a probability of $(1 - P_0)$.

If the theoretical mean of the distribution m and the theoretical standard deviation σ were known, the quantity $(m - u_p\, \sigma)$ would give the desired lower limit, where u_p is the appropriate probability limit of the normal distribution.

It would be possible to estimate it by the quantity $(\bar{x} - u_p s)$, but this would not give sufficient security. Therefore, first a fairly great P_1 significance level is chosen and then a limit is determined, which is overpassed by the limit belonging to the probability P_0 by the probability P_1.

Table III

Factors to determine expected minimum and maximum values based on a measuring series

Number of measurements n	t_0	Number of measurements n	t_0
5	5.12	25	2.80
6	4.48	26	2.78
7	4.10	28	2.74
8	3.84	30	2.71
9	3.65	32	2.69
10	3.51	34	2.66
11	3.40	35	2.65
12	3.31	36	2.64
13	3.23	38	2.62
14	3.17	40	2.60
15	3.11	45	2.57
16	3.06	50	2.54
17	3.02	55	2.51
18	2.98	60	2.48
19	2.95	65	2.46
20	2.91	70	2.45
21	2.89	75	2.43
22	2.86	80	2.42
23	2.83	90	2.40
24	2.81	100	2.38

Expected minimum and maximum values:

$$x_{\min} = x - t_0 \cdot s$$

$$x_{\max} = x + t_0 \cdot s$$

This limit is given by the relation

$$x_{\min} = \bar{x} - st_0$$

where

$$t_0 = \frac{u_{p0} + u_{p1}\sqrt{\frac{1}{n}\left(1 - \frac{u_{p1}^2}{2(n-1)}\right) + \frac{u_{p0}^2}{2(n-1)}}}{1 - \frac{u_{p1}^2}{2(n-1)}}$$

and

u_{P0} resp. u_{P1} are the adequate lower probability limits of the normal distribution and

n is the size of the sample.

In our investigations $P_1 = 95$ per cent and $P_0 = 98$ per cent were chosen. The values of t_0 are given in Table III and in the diagram of Fig. 4.

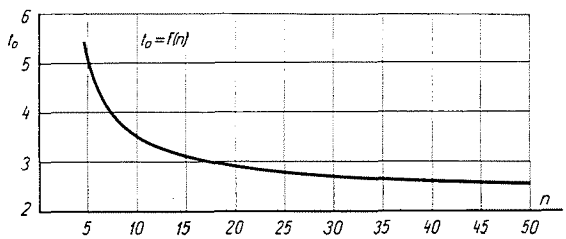

Fig. 4

Other values of t_0 may be calculated from the above formula. The probability limits u_P of the normal distribution are given in Table IV.

Table IV

Probability limits of normal distribution

Probability	P	90	95	98	99	99.5%
Probability limit	u_p	1.28	1.64	2.05	2.33	2.58

The members of the distribution will be between the limits $x \pm u_p \cdot s$ with the probability of P per cent

The lower limit determined in this way has the following meaning. When $x_{\min}$ has been determined on the basis of a measuring series, then in P_1 part of the cases the lower limit belonging to the probability P_0 will be above $x_{\min}$.

The same train of thoughts is substantially valid to determine the upper limit, the value of which is

$$x_{\max} = \bar{x} + t_0\, s.$$

2.3 *Determination of the necessary number of measurements*

The lower limit determined according to the previous chapter, first rapidly changes when the number of measurements is increased, later the change becomes less and less. The more measurements done, the more accurate will the ascertainments be. On the other hand, it is not practical to increase the number of measurements beyond a certain limit, as the accuracy of evaluation does not vary substantially in this range. A number of measurements are necessary and at the same time sufficient to obtain the accuracy wanted. The following part deals with specifying the sufficient number of measurements in order to determine the lower limit.

Table V

Factors to check the sufficient number of measurings

Number of measurements n	Standard deviation in per cent r_{max}%	Number of measurements n	Standard deviation in per cent r_{max}%
5	1.34	25	5.55
6	1.70	26	5.70
7	2.01	28	5.98
8	2.30	30	6.25
9	2.57	32	6.50
10	2.82	34	6.75
11	3.06	35	6.88
12	3.28	36	7.00
13	3.50	38	7.24
14	3.70	40	7.47
15	3.90	45	8.02
16	4.08	50	8.55
17	4.27	55	9.04
18	4.44	60	9.53
19	4.61	65	9.97
20	4.78	70	10.40
21	4.94	75	10.82
22	5.10	80	11.23
23	5.26	90	12.01
24	5.41	100	12.74

The number of measurements are sufficient if

$$r = \frac{s}{\bar{x}} \cdot 100 < [r_{max}]_n$$

The lower limit, as established in paragraph 2.2, is also a random variable, thus has a deviation, too. This deviation decreases with an increasing number of measurements. There is no use of increasing the number of measurements beyond a certain limit, as then the standard deviation does not vary substantially any more. Practice has shown that with measurements of breakdown voltages the number of measurements are sufficient if the standard deviation is smaller than 0.025 fold the mean value of the measuring series.

As shown in paragraph 2.2, the lower limit is

$$x_{\min} = \bar{x} - t_0 s$$

having a standard deviation

$$D(x_{\min}) = \sigma \sqrt{\frac{1}{n} + \frac{t_0^2}{2(n-1)}}$$

where

σ is the standard deviation of the original assembly. Now, as previously mentioned the condition of a sufficient number of measurements is

$$D(x_{\min}) \leqq 0.025 \cdot \bar{x}.$$

Putting instead of σ its estimation, it becomes possible to write

$$r = \frac{s}{\bar{x}} \leqq \frac{0.025}{\sqrt{\frac{1}{n} + \frac{t_0^2}{2(n-1)}}} = r_{\max}.$$

Now $r = \frac{s}{\bar{x}}$ is the relative deviation. Thus the criterion gives an upper limit to the relative deviation as a function of the number of measurements, because the quantity at the right side, $r_{\max}$ only depends upon "n" the number of measurements. Its values are given in Table V.

2.4 Comparison of the measuring series

It often occurs that two measuring series have to be compared between themselves to establish whether they originate from the same assembly or not.

As normal distribution was assumed the two measuring series may differ between themselves in standard deviation or in mean value or in both. This chapter gives means to compare the standard deviations first. Should they prove to be identical, then means are given to compare the mean values.

2.41 Comparing standard deviation of measuring series

2.411 *Comparison of measuring series having about the same sample size.* The investigation may be done by the so-called F test (Fisher test). The

Table VI

Factors to compare the standard deviation of two measuring series, if the standard deviations of both series are known

Sample size of series having the smaller deviation n_1	Sample size of series having the greater deviation n_2: 2	3	4	5	6	7	8	9	10	11	15	20	30	50	100	200	500	∞
2	39.9	49.5	53.6	55.8	57.2	58.2	58.9	59.4	59.9	60.2	61.2	61.7	62.3	62.7	63.0	63.2	63.3	63.3
3	8.53	9.00	9.16	9.24	9.29	9.33	9.35	9.37	9.38	9.39	9.42	9.44	9.46	9.47	9.48	9.49	9.49	9.49
4	5.54	5.46	5.39	5.34	5.31	5.28	5.27	5.25	5.24	5.23	5.20	5.18	5.17	5.15	5.14	5.14	5.14	5.13
5	4.54	4.32	4.19	4.11	4.05	4.01	3.98	3.95	3.94	3.92	3.87	3.84	3.82	3.80	3.78	3.77	3.76	3.76
6	4.06	3.78	3.62	3.52	3.45	3.40	3.37	3.34	3.32	3.30	3.24	3.21	3.17	3.15	3.13	3.12	3.11	3.10
7	3.78	3.46	3.29	3.18	3.11	3.05	3.01	2.98	2.96	2.94	2.87	2.84	2.80	2.77	2.75	2.73	2.73	2.72
8	3.59	3.26	3.07	2.96	2.88	2.83	2.78	2.75	2.72	2.70	2.63	2.59	2.56	2.52	2.50	2.48	2.48	2.47
9	3.46	3.11	2.92	2.81	2.73	2.67	2.62	2.59	2.56	2.54	2.46	2.42	2.38	2.35	2.32	2.31	2.30	2.29
10	3.36	3.01	2.81	2.69	2.61	2.55	2.51	2.47	2.44	2.42	2.34	2.30	2.25	2.22	2.19	2.17	2.17	2.16
11	3.28	2.92	2.73	2.61	2.52	2.46	2.41	2.38	2.35	2.32	2.24	2.20	2.16	2.12	2.09	2.07	2.06	2.06
12	3.23	2.86	2.66	2.54	2.45	2.39	2.34	2.30	2.27	2.25	2.17	2.12	2.08	2.04	2.00	1.99	1.98	1.97
13	3.18	2.81	2.61	2.48	2.39	2.33	2.28	2.24	2.21	2.19	2.10	2.06	2.01	1.97	1.94	1.92	1.91	1.90
14	3.14	2.76	2.56	2.43	2.35	2.28	2.23	2.20	2.16	2.14	2.05	2.01	1.96	1.92	1.88	1.86	1.85	1.85
15	3.10	2.73	2.52	2.39	2.31	2.24	2.19	2.15	2.12	2.10	2.01	1.96	1.91	1.87	1.83	1.82	1.80	1.80
16	3.07	2.70	2.49	2.36	2.27	2.21	2.16	2.12	2.09	2.06	1.97	1.92	1.87	1.83	1.79	1.77	1.76	1.76
17	3.05	2.67	2.46	2.33	2.24	2.18	2.13	2.09	2.06	2.03	1.94	1.89	1.84	1.79	1.76	1.74	1.73	1.72
18	3.03	2.64	2.44	2.31	2.22	2.15	2.10	2.06	2.03	2.00	1.91	1.86	1.81	1.76	1.73	1.71	1.69	1.69
19	3.01	2.62	2.42	2.29	2.20	2.13	2.08	2.04	2.00	1.98	1.89	1.84	1.78	1.74	1.70	1.68	1.67	1.66
20	2.99	2.61	2.40	2.27	2.18	2.11	2.06	2.02	1.98	1.96	1.86	1.81	1.76	1.71	1.67	1.65	1.64	1.63
21	2.97	2.59	2.38	2.25	2.16	2.09	2.04	2.00	1.96	1.94	1.84	1.79	1.74	1.69	1.65	1.63	1.62	1.61

Sample size of series																		
22	2.95	2.56	2.35	2.22	2.13	2.06	2.01	1.97	1.93	1.90	1.81	1.76	1.70	1.65	1.61	1.59	1.58	1.57
25	2.93	2.54	2.33	2.19	2.10	2.04	1.98	1.94	1.91	1.88	1.78	1.73	1.67	1.62	1.58	1.56	1.54	1.53
27	2.91	2.52	2.31	2.17	2.08	2.01	1.96	1.92	1.88	1.86	1.76	1.71	1.65	1.59	1.55	1.53	1.51	1.50
29	2.89	2.50	2.29	2.16	2.06	2.00	1.94	1.90	1.87	1.84	1.74	1.69	1.63	1.57	1.53	1.50	1.49	1.48
30	2.88	2.49	2.28	2.14	2.05	1.98	1.93	1.88	1.85	1.82	1.72	1.67	1.61	1.55	1.51	1.48	1.47	1.46
40	2.84	2.44	2.23	2.09	2.00	1.93	1.87	1.83	1.79	1.76	1.66	1.61	1.54	1.48	1.43	1.41	1.39	1.38
50	2.81	2.41	2.20	2.06	1.97	1.90	1.84	1.80	1.76	1.73	1.63	1.57	1.50	1.44	1.39	1.36	1.34	1.33
60	2.79	2.39	2.18	2.04	1.95	1.87	1.82	1.77	1.74	1.71	1.60	1.54	1.48	1.41	1.36	1.33	1.31	1.29
80	2.77	2.37	2.15	2.02	1.92	1.85	1.79	1.75	1.71	1.68	1.57	1.51	1.44	1.38	1.32	1.28	1.26	1.24
100	2.76	2.36	2.14	2.00	1.91	1.83	1.78	1.73	1.70	1.66	1.56	1.49	1.42	1.35	1.29	1.26	1.23	1.21
200	2.73	2.33	2.11	1.97	1.88	1.80	1.75	1.70	1.66	1.63	1.52	1.46	1.38	1.31	1.24	1.20	1.17	1.14
500	2.72	2.31	2.10	1.96	1.86	1.79	1.73	1.68	1.64	1.61	1.50	1.44	1.36	1.28	1.21	1.16	1.12	1.09
∞	1.71	2.30	2.08	1.94	1.85	1.77	1.72	1.67	1.63	1.60	1.49	1.42	1.34	1.26	1.18	1.13	1.08	1.00

The standard deviations of the two measuring series may be considered identical, if

$$\frac{s_2^2}{s_1^2} < F$$

members of the first series should be designated by $x_1, x_2 \ldots x_n$, their arithmetical mean by $\bar{x}$ and by s_1 the empirical standard deviation. With the second series these notations are $y_1, y_2 \ldots y_{n2}$ and $\bar{y}$ and s_2, then

$$F = \frac{s_2^2}{s_1^2}$$

should be formed. Its distribution is the so-called F distribution, the 95 per cent limits of which (as a function of the element numbers of the two samples) are given in Table VI. The notations have to be made so as to have $s_2 > s_1$. If the above F quantity is smaller than the value contained in the table, the two standard deviations may be considered as being identical.

2.412 Comparison of a short measuring series with one having a much bigger sample size. If the sample size of one measuring series were much bigger than that of the other, then the F test of paragraph 2.411 may still be employed, but the measuring series having a big sample size should be considered as being infinite, thus $n_2 = \infty$.

Table VII

Factors to compare the standard deviation of two measuring series, if the standard deviations of one series and the range of the other are known

Sample size of smaller series n_2	w_1	w_2
5	1.03	3.86
6	1.25	4.03
7	1.44	4.17
8	1.60	4.29
9	1.74	4.39
10	1.86	4.47
11	1.97	4.55
12	2.07	4.62
13	2.16	4.69
14	2.24	4.74
15	2.32	4.80
16	2.39	4.85
17	2.45	4.89
18	2.51	4.93
19	2.57	4.97
20	2.63	5.01

The standard deviation may be considered identical, if

$$w_1 < \frac{R}{s} < w_2$$

Another method should be introduced here which may be used without calculating the standard deviation of the short series, only using its range. Be R the range, i.e. the difference of the biggest and smallest element, of the short measuring series containing the members $x_1 \ldots x_n$. Be σ the standard deviation of the long series. Now the quantity $\frac{R}{\sigma}$ should be calculated. If this value falls between the two appropriate values given by Table VII, then the standard deviations of the two series shall be identical on the 95 per cent probability level.

2.42 Comparing the mean values of measuring series

If by using the method given in paragraph 2.41 there was no considerable difference between the standard deviations, a further investigation should be made to see whether there is a departure between the mean values or not. This is done with the t test (so-called Student test).

The elements of the two measuring series should be designated, as formerly,with $x_1 \ldots x_{n1}$, and $y_1 \ldots y_{n2}$. The respective mean value should be $\bar{x}$ and $\bar{y}$. Then the following quantity has to be calculated

Table VIII

Factors to compare the means of two measuring series

f	t_f
10	1.81
15	1.75
20	1.72
25	1.70
30	1.69
50	1.68
60	1.67
100	1.66
∞	1.64

The mean values of two series may be considered to be identical, if

$$\left| \frac{\bar{x} - \bar{y}}{s_e \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \right| < t_f$$

where

$$f = n_2 + n_1 - 2, \quad \text{and} \quad s_e = \sqrt{\frac{\sum^{n_1} (x_i - \bar{x})^2 + \sum^{n_2} (y_i - \bar{y})^2}{n_1 + n_2 - 2}}$$

$$t = \frac{\bar{x} - \bar{y}}{s_e \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

where

$$S_e = \sqrt{\frac{\sum_1^{n_1} (x_i - \bar{x})^2 + \sum_1^{n_2} (y_i - \bar{y})^2}{n_1 + n_2 - 2}}$$

If the means of the two distributions are conformable, then the above quantity has to be of the so-called t-distribution. The 95 per cent limits of this distribution are given in Table VIII as a function of sample size.

The decision is: If the absolute value of the above t quantity is smaller than the value in Table VIII, the two mean values do not sensibly differ.

3. Numerical example

3.1 Evaluation of a breakdown measuring series

10 punctures were made on an insulating material and the results contained in the second column of Table IX were obtained.

Table IX

n	E_n kV/mm	Δ_n	Δ_n^2
1	7.65	0.03	0.0009
2	8.25	0.63	0.3969
3	7.60	0.02	0.0004
4	7.05	0.57	0.3249
5	7.25	0.37	0.1369
6	7.50	0.12	0.0144
7	7.70	0.08	0.0064
8	7.95	0.33	0.1089
9	7.55	0.07	0.0049
10	7.70	0.08	0.0064
		$\Sigma \Delta_n^2 =$	1.0010

The mean of the measurements

$$\bar{E}_{10} = \frac{\sum_1^{10} E_n}{10} = 7.62 \text{ kV/mm}.$$

Knowing the mean, the error squares of the single measurements are determined (see column 3 and 4 in Table IX).

$$\Delta_n^2 = (E_n - \bar{E}_{10})^2$$

and the standard deviation is calculated:

$$s_{10} = \sqrt{\frac{\sum_{1}^{10} \Delta_n^2}{n-1}} = \sqrt{\frac{1.001}{9}} = 0.348 \text{ kV/mm}.$$

The first check is, whether the measured results may be considered as being of normal distribution. In Fig. 2 are plotted the distribution step-function of the measuring series and the Kolmogorov limits belonging to $n = 10$ measurements, according to Table I. The distribution function always falls between the Kolmogorov limits, therefore, the measuring series may be considered of normal distribution.

The same check may be made in an easier way by using the diagram on Figure 3. In Fig. 6 the limits belonging to $n = 10$ measurements were plotted only.

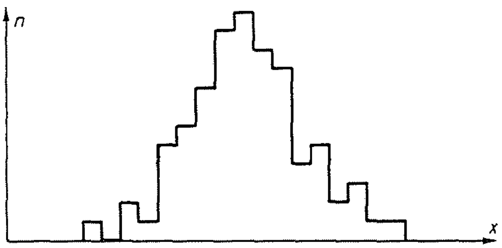

Fig. 5

For checking, the scale is so chosen that the mean and the standard deviation should fall into the assigned small interval. (See 2.11.) This condition is fulfilled if a scale of 0.1 kV = 16 mm is chosen.

The distribution function of the measurements are plotted, taking into consideration that each measurement corresponds on the axis of ordinates to

$$\frac{1}{n} = \frac{1}{10} = 10 \text{ per cent.}$$

The distribution function lays within the adequate limits everywhere, thus the series may be considered to be of normal distribution.

The next check is to see whether the number of measurements is sufficient for evaluation or not. Table V gives for $n = 10$ measurements $r_{max} =$ $= 2.82$ per cent.

The relative standard deviation of the measuring series is

$$r_{10} = \frac{s}{E_{10}} = \frac{0.348}{7.62} = 4.61\ \%$$

so

$$r_{10} > r_{max}$$

the number of measurements is not sufficient.

It can be seen from the table that invariable standard deviation being assumed with increasing number of measurements the condition of $r < r_{max}$

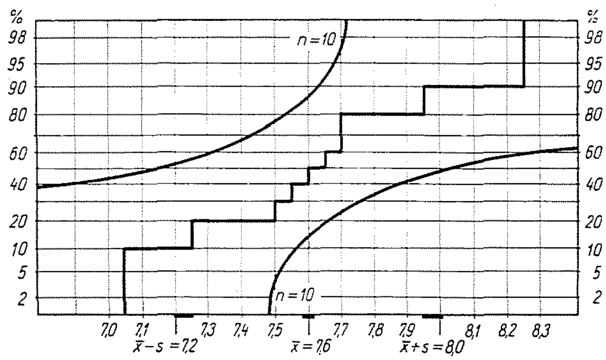

Fig. 6

is fulfilled with $n > 19$ measurements. (Here it has not been taken into consideration that the relative standard deviation generally decreases the number of measurements increasing.)

Table X

n	E_n kV/mm	Δ_n	Δ_n^2
11	7.50	0.11	0.0121
12	7.15	0.46	0.2116
13	7.65	0.04	0.0016
14	7.75	0.14	0.0196
15	7.70	0.09	0.0081
16	7.80	0.19	0.0361
17	7.60	0.01	0.0001
18	7.60	0.01	0.0001
19	7.40	0.21	0.0441
20	7.85	0.24	0.0576
			$\Sigma \Delta_n^2 = 0.3910$

After a further 10 measurements the values in column 2 of Table X were obtained.

The mean of measurements 1 to 20 is

$$\bar{E}_{20} = \frac{\sum_{1}^{20} E_n}{20} = 7.61 \text{ kV/mm}.$$

The error squares correlated with this mean are in column 4 of Table X.

With the theorem well-known from mathematics, the sum of error squares of measurements 1 to 10 correlated with $\bar{E}_{10}$ are reduced to the mean E_{20} of measurements 1 to 20 and the error squares of measurements 11 to 20 contained in Table X are added. So the sum of error squares 1 to 20 correlated with $\bar{E}_{20}$ are obtained

$$\sum_{1}^{20} \Delta_n^2 = \sum_{1}^{10} \Delta_n^2 + n(\bar{E}_{10} - \bar{E}_{20})^2 + \sum_{11}^{20} \Delta_n^2$$

$$\sum_{1}^{20} \Delta_n^2 = 1.001 + 10 \cdot 0.1 + 0.3910 = 1.4920.$$

So the standard deviation of measuring series 1 to 20 is

$$s_{20} = \sqrt{\frac{1.4920}{19}} = 0.28 \text{ kV/mm}$$

respectively

$$r_{20} = \frac{0.28}{7.61} = 3.7\%.$$

Now

$$r_{20} < [r_{\max}]_{n=20} = 4.78\%$$

i.e. the number of measurements is sufficient.

The smallest breakdown strength to be expected is now calculated according to paragraph 2.2:

$$E_{\min} = \bar{E}_{20} - t_0 \cdot s_{20}.$$

In Table III, $t_0 = 2.91$ with $n = 20$ and so the smallest breakdown voltage to be expected:

$$E_{\min} = 7.16 - 2.91 \cdot 0.28 = 6.8 \text{ kV/cm}.$$

3.2 Comparison of the measuring series

It is to be decided whether another insulator may be considered identical as to the dielectric strength or not, with the insulating material investigated according to paragraph 3.1. With this new insulating material the following data were measured:

n	1	2	3	4	5	6	
E_n	8.4	8.0	7.9	8.2	7.8	8.1	kV/mm

The mean of the measurements is

$$\bar{E} = 8.07 \text{ kV/mm}.$$

It should be checked according to paragraph 2.41 whether the two series may be considered identical as to standard deviations.

The range of the measuring series is

$$R = E_{\max} - E_{\min} = 0.6 \text{ kV/mm}.$$

The standard deviation of the previous measuring series

$$s = 0.28 \text{ kV/mm}.$$

So, $\frac{R}{s} = \frac{0.6}{0.29} = 2.14$ falls between the two values given in Table VII

$$1.25 < 2.14 < 4.03.$$

The standard deviation of the two series may be considered identical.

Based on paragraph 2.42 it was checked whether the two series may be considered identical as to mean values.

$$n_1 = 20 \qquad \bar{E}_1 = 7.61 \text{ kV/mm} \quad \sum_1^{20} \Delta_n^2 = 1.49$$

$$n_2 = 6 \qquad \bar{E}_2 = 8.07 \text{ kV/mm} \quad \sum_1^{6} \Delta_n^2 = 0.321$$

$$s_e = \sqrt{\frac{1.49 + 0.321}{20 + 6 - 2}} = 0.275$$

$$t = \frac{0.46}{0.275\sqrt{\frac{1}{20} + \frac{1}{6}}} = 3.7.$$

This value is higher than $t_f = 1.7$ to be found in Table VIII with $f = 20 + 6 - 2 = 24$. Thus the means of the two series cannot be considered identical and the material investigated is not identical with the previous one.

4. Fundamental notions on the theory of probability

From the point of view of probability calculation and of mathematical statistics, respectively, measured data are considered to be variables depending upon chance, so-called random variables.

Concerning random variables it is not possible to draw positive conclusions. It is only possible to make probability statements, e.g. that a variable may take certain values or value collectivities or e.g. that the mean of variables will with great probability vary around a certain value, etc.

Random variables may be various according to their character, their distribution may be different. The distribution of a random variable is determined by its distribution function and by its density function.

The *distribution function* is a function, having a meaning with real numbers, which gives with what a probability a variable will take smaller values than a certain z value. As a formula

$$F(z) = P(x < z)$$

where

P denots the probability of the phenomenon put in brackets,

x is the random variable,

z is a discretional real number.

There are random variables, which may take a finite number of different values and to each of these values belongs a distinct positive probability (such may be e.g. the result of a throw of dice).

These have a discrete distribution, their distribution function is a so-called step-function. (Such a distribution function is given in Fig. 2, the $F_n(z)$ curve.)

On the other hand, there are random variables, which have a continuous stock of values i.e. they can take every value in an interval. In such a case there is no use to say with what probability the random variable will take a certain value, as this probability is in general zero. Nevertheless, it is possible to speak about with what probability the random variable will take the values falling within a certain small interval. This is given by the derivative of the distribution function, the so-called density function.

The *density function* is a function, having a meaning with real numbers, the integral of which (the surface beneath the density function) gives the probability with which the random variable will fall in this interval. The symbol of density function is: $f(z)$, with $F'(z) = f(z)$.

It has to be stressed that not all random variables have a density function (e.g. those with discrete distribution have not). It is always necessary that the stock of values of the random variable should be continuous. That is why random variables of such a distribution are called continuous distribution.

Within discrete and continuous distributions the latter may be of all sorts. Naturally the distribution is entirely determined by the distribution function and the density function. Though there are some quantities which do not define unequivocally a distribution, nevertheless they well characterize a certain feature of the distribution of the random variable. Such quantities are:

The *expected value*, or *theoretical mean*, or mean is the quantity around which the random variable varies. More exactly: if many observations are made concerning a random variable, their arithmetical mean shall with high probability fairly well approach the value to be expected. Stating not quite precisely: the mean calculated of infinite observations. Its symbol: $m = M(x)$.

Knowing the density function the value to be expected may be calculated as follows

$$M(x) = \int z\, f(z)\; dz.$$

The *standard deviation* is the measure of the deviation of the random variable. The variance (square of standard deviation) is equal to the value to be expected for the square of the difference between random variable and its mean. Its formula is

$$\sigma^2 = D^2\,(x) = M\,(x—M\,(x))^2 = \int\,(z—m)^2 \cdot f(z) \cdot\, dz.$$

Its root is the standard deviation.

The mean deviation is also a measure of the deviation, but different from the standard deviation. The mean deviation is equal to the expected absolute value of the difference between the variable and the mean. As a formula

$$d = M\,|\,x — M(x)\,| = \int\,|\,z — M(x)\,|\,\cdot\, f(z)\,\cdot\, dz.$$

Mode is the place where the density function has its maximum value. The greatest number of observations fall into its surroundings. It should be remarked that there are distributions having more modes. In practice mostly distributions with only one mode occur.

Median is a value in comparison with which the random variable takes lower and higher values with equal probability, i.e. a place where the distribution function has the value ½. With random variables having symmetrical distribution, i.e. where the density function is symmetrical to one point, the median corresponds to the value to be expected. Nevertheless there are random variables with "skew" distribution, where the median is lower or higher than the value to be expected.

The *limit(s) of probability* is (are) the quantity(ies) in comparison with which the random variable may take lower values or higher values (or values between them) with a given probability.

If the distribution function or the density function is known, the above definitions may be used to determine certain values. Still in practice the distribution function is mostly unknown (at most its type is known). These and the characteristics of distribution have to be estimated from the data.

The *empirical distribution function* is determined by the following relations

$$F_n(z) = \begin{cases} 0 & \text{when } z < x^* \\ \dfrac{k}{n} & x_k^* < z < x_{k+1}^* \\ 1 & x_n^* < z \end{cases}$$

where $x_1^* \ldots x_n^*$ are the elements of the series, ranged according to their magnitude.

That is a step-function. With a series of "n" observation elements it leaps at the place x_1^* as many times as the value $\frac{1}{n}$, as many observations as belong to the place x_i^*. If the number of measurements is sufficiently high this step-function will fit the theoretical distribution function well.

More information is given by the *histogram* of probability frequency, which corresponds to the density function.

The frequency histogram is established by dividing the straight of numbers into intervals, the frequency of measured data falling to each interval is plotted against the interval. Such a histogram is shown in Fig. 5.

Experience has shown that in the great majority of practical cases the envelope curve of the histogram is the so-called bell-curve, which is the density function of normal or Gauss distribution. Its equation is

$$f(z) = \frac{1}{2\,\pi\sigma}\, e^{-\frac{(z-m)^2}{2\sigma^2}}$$

where m and σ are real numbers (parameters) with

m the expected value and

σ the standard deviation.

The normal distribution is symmetrical to "m" the value to be expected, which is at the same time also mode and median.

Generally the expected value m and the standard deviation σ are unknown. They have to be estimated from the measured data.

The expected value may be estimated by the formula

$$\bar{x} = \frac{\sum_{1}^{n} x_i}{n}$$

of the arithmetical mean. The variance may be estimated by the empirical variance

$$s_0^2 = \frac{\sum_{1}^{n} (x_i - \bar{x})^2}{n}$$

respectively, by the corrected empirical variance

$$s^2 = \frac{\sum_{1}^{n} (x_i - \bar{x})^2}{n - 1} .$$

As mentioned above in the majority of practical cases the distribution of measured data may be considered to be of normal distribution.

Still it has to be stressed that normal distribution is not always present. This question has to be investigated in every case. If it is established that the measured data do not follow the normal distribution, then the methods elaborated for normal distribution may not be used any longer. (E.g. this happen if the data are not symmetrical.) What is to be done in these cases? Sometimes it is possible to find a transformation after which the data may be considered as being of normal distribution. (E.g. often the logarithm of the data may be considered to have normal distribution.) If there is no possibility of finding such a transformation, another distribution must be chosen, though it is possible that methods have not yet been elaborated for this method. Still there are methods, the so-called distribution free methods with which it becomes possible to work entirely independently of distribution, even without knowing it. These methods may be found in handbooks of statistics [6, 7].

Summary

The treatise exposes methods to evaluate shorter breakdown measuring series. These methods make it possible to check

a) whether the measuring results may be considered to have normal distribution,

b) whether the number of measuring made is sufficient to afford reliable evaluation, and enables to determine

c) the value of the minimum dielectric breakdown strength to be expected and

d) whether two measuring series may be considered to be identical.

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E. NÉMETH, E. CSÁKI — Budapest XI., Egry József u. 18. Hungary.