THE SYNTHESIS OF SAMPLED-DATA CONTROL SYSTEMS WITH FINITE SETTLING TIME

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1. Survey of designing methods

Two methods have been evolved for designing impulse-compensated sampled-data control systems. One of the methods can be called deterministic, its main point is the following: The change in time of the input signal is regarded as given and the change of the ouput signal with time is examined. Typical input signales are

$$x_1(t) = 1(t), \quad x_2(t) = 1(t) \cdot t, \quad x_3(t) = \frac{1}{2} 1(t) t^2, \tag{1}$$

i.e. in the general case

$$x_m(t) = 1(t) \frac{1}{(m-1)!} t^{m-1} \quad m = 1,2,3, \tag{2}$$

where m denotes, also in the following, the ordinal number of the determined input signal. The $y(t)$ output signal of the system is examined in respect of transient performance. To do this it is advisable to introduce the difference of the actual output signal $y(t)$ and of the desired output signal $y_0(t)$, the error-signal

$$\psi(t) = y(t) - y_0(t). \tag{3}$$

In the case of a follow-up system $y_0(t) = x(t)$. The data characterizing the transient performance are e.g. the following: settling time T_s; maximum overshoot $[\psi(t)]_{max}$; maximum overshoot in respect to sampling time $[\psi(kT)]_{max}$, where T is the period time of sampling: the square integral of the error

$$\Theta^2 = \int_0^\infty \psi^2(t)\, dt, \tag{4}$$

or the quadratic sum of the error in the sampling moments

$$\vartheta^2 = \sum_{k=0}^\infty \psi^2(kT). \tag{5}$$

A recurrent requirement is that the settling time should be finite. This may pertain either to the sampling moments only, when

$$\psi[k] \equiv \psi(kT) = 0, \quad k \geqq T_s/T, \tag{6a}$$

or it can be prescribed for the continuous actuating error, too:

$$\psi(t) = 0, \quad t \geqq T_s\,. \tag{6b}$$

This requirement means a restriction in respect to the form of the transfer function of the closed system [8, 11, 12].

The second designing method can be called stochastic when the input signal is regarded as random, having known statistical characteristics. Be the input signal the sum of the control input $f(t)$ and of the noise $\varphi(t)$

$$x(t) = f(t) + \varphi(t) \qquad -\infty < t < \infty. \tag{7}$$

The required $y_0(t)$ output signal is a function of the control input $f(t)$. In the case of a follow-up system $y_0(t) = f(t)$. By defining the error again in the form

$$\psi(t) = y(t) - y_0(t) \tag{8}$$

and by assuming a stationary control input, the problem is the evaluation of the stationary expression for the error. That system is usually regarded as optimum for which ζ^2, the quadratic mean value of the error series is minimum, where

$$\zeta^2 = \psi^2(kT) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} \psi^2[k]\,. \tag{9}$$

Various methods are known for determining the transfer function of the closed system, satisfying the above condition [3, 12].

2. Formulation of the problem

The performance of a system designed according to the deterministic method is naturally not good in the case of a random input signal and the effect of the noise cannot be taken into account at all. Beyond these a troublesome effect arises which does not occur in the case of continuous systems. The output signal of a system tuned for an input signal of higher order ($m = 2, 3$) in the case of an input signal of lower order contains strong overshoots. This effect can be reduced at the price of increasing the settling time in such a way that additional parameters are adopted. Afterwards these parameters are determined *e.g.* in such a way that the square integral of the error Θ^2 or the quadratic sum of the error ϑ^2 should be minimum [2, 5]. Other conditions can also be prescribed, these are, however, more difficult to handle mathematically [10, 13].

Systems designed by statitical methods, in turn, have not the favourable characteristic which in case of a defined (*e.g.* constant) input signal that the steady-state error should become zero within a finite time. We can state that systems designed by this method have bad transient characteristics.

In the following the question will be examined, how the two designing methods could be coupled. Hence, the problem can be formulated as follows: For the closed system a transfer function should be determined which characterizes a system having the following features:

1. The complete system is stable
2. The compensating elements can be realized
3. In the case of an input signal $x_m(t)$ of determined order, the steady-state error becomes zero after the elapse of a finite settling time T_s, namely *a)* only in the sampling moments, *b)* at every moment (ripple-free system).
4. In the case of a control input and noise of determined statistical characteristics the quadratic mean error ζ^2 should be minimum.
5. In the case of a high-order input signal tuning, no excessive overshoot should occur in the case of an input signal of lower order.

From the above requirements the fifth one cannot be defined unequivocally in mathematics. In the given cases it should be individually decided whether the arising overshoot is permissible or it should be reduced at the price of increasing the settling time.

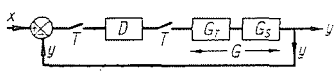

Fig. 1

The build-up of the examined control system is shown in Fig. 1. G_s is the controlled system, G_T is the hold circuit and D is the symbol and transfer function of the impulse-compensator, respectively. From the point of view

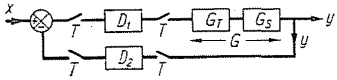

Fig. 2

of the transfer function, our results can be applied for the system shown in Fig. 2 as well which contains two impulse-compensators. From the point of view of eliminating the sustained effect of the disturbing signal, this may be more advantageous [7], the effect of the disturbing signal, however, will not be examined.

To ensure perspicuity, the question will not be discussed generally, but it will be assumed that the control input $f(t)$ and the noise $\varphi(t)$ are independent of one an other; more exactly they are not correlated. For the sake of simplicity only the follow-up systems will be examined.

3. Calculation procedure

The most convenient method of calculating impulse-compensated sampled-data control systems is the one or two-sided discrete Laplace transformation. Be the variable of the Laplace transformation s, then the variable of the discrete Laplace transformation will be $z = e^{sT}$, where T is the period time of sampling. As series expansion is generally carried out in respect to the powers of z^{-1}, in the following the variable

$$Z = z^{-1} = e^{-sT} \tag{10}$$

will be used. The basic correlation of the one-sided discrete Laplace transformation, or $\mathcal{Z}$-transformation is

$$\mathcal{Z} f(t) \equiv F(Z) = \sum_{k=0}^{\infty} f(kT)\, Z^k \equiv \sum_{k=0}^{\infty} f[k]\, Z^k, \tag{11}$$

while that of the two-sided transformation

$$\overline{\mathcal{Z}} f(t) \equiv \overline{F}(Z) = \sum_{k=-\infty}^{\infty} f[k]\, Z^k =$$

$$= \sum_{k=0}^{\infty} f[k]\, Z^k + \sum_{k=0}^{\infty} f[-k]\, Z^{-k} - f[0]. \tag{12}$$

For statistical calculations the autocorrelation and cross correlation series are defined:

$$r_{xx}[n] = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} x[k]\, x[k+n], \tag{13}$$

$$r_{xy}[n] = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} x[k]\, x[k+n]. \tag{14}$$

and the two-sided $\mathcal{Z}$-transforms of the same, as formed on the basis of equation (12):

$$\overline{R}_{xx}(Z) = \overline{\mathcal{Z}}\, r_{xx}, \qquad \overline{R}_{yy}(Z) = \overline{\mathcal{Z}}\, r_{xy}. \tag{15}$$

The expression of the transfer function of the closed system shown in Fig. 1 is

$$W(Z) = \frac{Y(Z)}{Z(Z)} = \frac{D(Z)\,G(Z)}{1 - D(Z)\,G(Z)}\,, \tag{16}$$

where in the case of a hold circuit of zero order

$$G(s) = G_T(s)\,G_s(s) = \frac{1 - e^{-sT}}{s}\,G_s(s). \tag{17}$$

If $W(Z)$ is already known, the expression for the transfer function of the impulse-compensator will be

$$D(Z) = \frac{1}{G(Z)}\;\frac{W(Z)}{1 - W(Z)}\,. \tag{18}$$

To the statistical synthesis, the prescribed transfer function $W_0(Z)$ is introduced:

$$\overline{Y}_0(Z) = W_0(Z)\,\overline{F}(Z), \tag{19}$$

where $\overline{Y}_0(Z) = \overline{\mathcal{Z}}y_0(t)$, $\overline{F}(Z) = \overline{\mathcal{Z}}f(t)$. The system having the transfer function $W_0(Z)$ would, therefore, produce the required output signal from the control inputs in the sampling moments. The actual output signal will be different as, on the one hand $W(Z) \neq W_0(Z)$, on account of the other assumptions, and on the other hand at the input side noise is also acting beside the control input. It can be proved [12], that the quadratic mean of the error series as defined under (9) will in the general case be

$$\zeta^2 = \frac{1}{2\pi j}\oint_C \overline{R}_{\varphi\varphi}(Z)\,\frac{dZ}{Z}\,, \tag{20}$$

where curve C is the unit circle of the Z-plane, further by using the designation

$$\hat{W}(Z) \equiv W(Z^{-1}) \tag{21}$$

$$\begin{aligned} \overline{R}_{\varphi\varphi}(Z) = {} & [W(Z)\hat{W}(Z) + W_0(Z)\hat{W}_0(Z) - W_0(Z)\hat{W}(Z) - \\ & - W(Z)\hat{W}_0(Z)]\;\overline{R}_{ff}(Z) + [W(Z)\hat{W}(Z) - W(Z)\hat{W}_0(Z)]\overline{R}_{f\varphi}(Z) + \\ & + [W(Z)\hat{W}(Z) - W_0(Z)\hat{W}(Z)]\overline{R}_{\varphi f}(Z) + W(Z)\hat{W}(Z)\overline{R}_{\varphi\varphi}(Z). \end{aligned} \tag{22}$$

On the basis of our simplifying preconditions $\overline{R}_{f\varphi}(Z) = \overline{R}_{\varphi f}(Z) = 0$, further for the case of follow-up systems $W_0(Z) = 1$, consequently equation

(22) is reduced to the following form:

$$\bar{R}_{\varphi\varphi}(Z) = [W(Z)\hat{W}(Z) + 1 - \hat{W}(Z) - W(Z)]\bar{R}_{ff}(Z) + W(Z)\hat{W}(Z)\bar{R}_{\varphi\varphi}(Z). \quad (23)$$

This expression should be substituted into equation (20).

4. Satisfying the deterministic prescriptions

From the point of view of mathematics our problem can be formulated in this way: The transfer function $W(Z)$ of the closed system should be sought for which determination satisfies prescriptions 1—4 as laid down in Chapter 2 and eventually satisfies point 5 as well.

To ensure a finite settling time, $W(Z)$ should be a polynome:

$$W(Z) = \sum_{i=1}^{r} w_i Z^i. \quad (24)$$

We write the transfer function of the controlled system and of the hold circuit in the following form:

$$G(Z) = \frac{G_1(Z)\, G_2(Z)}{G_3(Z)\, G_4(Z)}, \quad (25)$$

where all the $G_i(Z)$ are polynomes and the zeros of $G_1(Z)$ and $G_3(Z)$ fall outside the unit circle, while the zeros of $G_2(Z)$ and $G_4(Z)$ are inside the unit circle. If the controlled system in itself is stable, then $G_4(Z) = 1$.

To ensure stability, $W(Z)$ should contain the instable zeros of $G(Z)$, *i.e.* the factor $G_2(Z)$. If a ripple-free system should be designed in accordance with condition 3b, then $W(Z)$ should contain all the zeros of $G(Z)$, *i.e.* the factor $G_1(Z)G_2(Z)$ [11]. Consequently as a final result

$$W(Z) = A(Z)B(Z) \quad (26)$$

$$A(Z) = \sum_{k=0}^{n} a_k Z^k \quad (27)$$

$$B(Z) = \sum_{i=1}^{q} b_i Z^i \quad (28)$$

where the a_k coefficients, as well as the degree n of the polynome are unknown values, while the b_i coefficients are known:

$$B(Z) = G_2(Z) \tag{29a}$$

$$B(Z) = G_1(Z)G_2(Z). \tag{29b}$$

By comparing equations (24)—(28) it is evident that

$$w_i = \sum_{k=0}^{i} a_k b_{i-k}, \qquad r = n + q \cdot \tag{30}$$

The Z-transform of the error defined by equation (3) is

$$\Psi(Z) = Y(Z) - Y_0(Z) = [W(Z) - 1]\ X(Z). \tag{31}$$

If the examined input signal is of the m-order, then by transformation

$$X_m(Z) = \frac{\Phi_m(Z)}{(1-Z)^m}, \tag{32}$$

where $\Phi_m(Z)$ is a polynome of $(m - 1)$-order. The steady-state error becomes zero, *i.e.*

$$\Psi_m(Z) = [W(Z) - 1]X_m(Z) \tag{33}$$

is a polynome, if $[W(Z) - 1]$ contains a factor of the form $(1 - Z)^m$. Further, to ensure stability, $[W(Z) - 1]$ should contain the instable poles of $G(Z)$, that is

$$W(Z) - 1 = (1 - Z)^m G_4(Z)C(Z), \tag{34}$$

where $C(Z)$ is a polynome. If $G_4(Z) = 1$, then this is equivalent to the following $(m - 1)$ equiponderates of condition [7]:

$$\lim_{z\to 1} \frac{d^\mu W(Z)}{dZ^\mu} = \begin{cases} 1 & \mu = 0 \\ 0 & \mu = 1, 2, \ldots (m - 1). \end{cases} \tag{35}$$

From this it follows that we should have $n \geqq m - 1$.
After carrying out the operations we obtain the following eqations:

$$\sum_{k=0}^{n} a_k = \frac{1}{\beta_0}, \quad \beta_0 = \sum_{i=1}^{c} b_i, \ m = 1, 2, 3 \tag{36}$$

$$\sum_{k=0}^{n} k a_k = -\frac{\beta_1}{\beta_0^2}, \quad \beta_1 = \sum_{i=1}^{q} i\, b_i, m = 2, 3 \tag{37}$$

$$\sum_{k=0}^{n} k(k-1)\,a_k = \frac{2\,\beta_1^2 - \beta_2\,\beta_0}{\beta_0^3}\,, \quad \beta_2 = \sum_{i=1}^{q} i\,(i-1)\,b_i, m = 3. \tag{38}$$

If the a_k coefficients satisfy the above equations then conditions 1, 2 and 3 are satisfied.

5. Satisfying the statistical prescriptions

The other unknown coefficients in the transfer function can be determined on the basis of condition 4, that is, the quadratic mean error ζ^2 should be minimalized by using equations (20) and (23). Equations (36), (37) and (38) previously obtained can be taken into account by the Lagrange method. By applying the residuum theorem, as the final result the minimum of the function

$$V = \sum_{|Z_i|<1} \text{Res}\, \frac{\bar{R}_{\gamma\gamma}(Z)}{Z} + \lambda_0 \left[\sum_{k=1}^{n} a_k - \frac{1}{\beta_0}\right] +$$

$$+ \lambda_1 \left[\sum_{k=0}^{n} k a_k + \frac{\beta_1}{\beta_0^2}\right] + \lambda_2 \left[\sum_{k=0}^{n} k(k-1)\,a_k - \frac{2\beta_1^2 - \beta_2\,\beta_0}{\beta_0^3}\right] \tag{39}$$

is to be determined. If $m = 1$, then $\lambda_1 = \lambda_2 = 0$ and if $m = 2$, then $\lambda_2 = 0$ should be taken.

As $W(Z)$ is a polynome, it is evident from equation (23) that the poles of $\bar{R}_{\gamma\gamma}(Z)/Z$ which should be taken into account are: $Z = 0$ (this is a multiple pole), the poles of $\bar{R}_{ff}(Z)$ inside the unit circle ($Z = \mu_l$) and the poles of $\bar{R}_{\varphi\varphi}(Z)$ inside the unit circle ($Z = \nu_l$).

By differentiating the expression for V in respect to the coefficient a_r we obtain the following system of equations:

$$\sum_{k=0}^{n} A_{rk}\,a_k + \lambda_0 + \lambda_1 r + \lambda_2 r(r-1) = B_r \qquad r = 0, 1 \ldots n. \tag{40}$$

Coefficient A_r in the system of equations depends only on $|r - k|$, hence

$$A_{rk} = A_p,\; p = |r - k| \tag{41}$$

$$A_p = \sum_{j=1}^{n+q} \sum_{i=1}^{q} b_i\,(b_{i+j+p} + b_{i+j-p})\,M_j + + N \sum_{i=1}^{q} b_i\,b_{i+p}\,, \tag{42}$$

further the expression for B_r at the right side of the equation is:

$$B_r = \sum_{i=1}^{n+q} b_{i-r} P_i . \tag{43}$$

The expressions for M_j, N and P_i can be determined from the available data:

$$M_j = \frac{1}{j!} [\bar{R}_{ff}^{(j)}(0) + \bar{R}_{\varphi\varphi}^{(j)}(0)] + \sum_l \frac{\mu_l^j + \mu_l^{-j}}{\mu_l} \operatorname{Res} \bar{R}_{ff}(\mu_l) + \\ + \sum_l \frac{\nu_l^j + \nu_l^{-j}}{\nu_l} \operatorname{Res} \bar{R}_{\varphi\varphi}(\nu_l) \tag{44}$$

$$N = 2 \left[\bar{R}_{ff}(0) + \bar{R}_{\varphi\varphi}(0) + \sum_i \frac{1}{\mu_l} \operatorname{Res} \bar{R}_{ff}(\mu_l) + \\ + \sum_l \frac{1}{\nu_l} \operatorname{Res} \bar{R}_{\varphi\varphi}(\nu_l) \right], \tag{45}$$

$$P_i = \frac{1}{i!} \bar{R}_{ff}^{(i)}(0) + \sum_i \frac{\mu_l^i + \mu_l^{-i}}{\mu_l} \operatorname{Res} \bar{R}_{ff}(\mu_l), \tag{46}$$

where *e.g.*

$$\bar{R}_{ff}^{(j)}(0) = \left[\frac{d^j \bar{R}_{ff}(Z)}{d Z^j} \right]_{Z=0}$$

$$\operatorname{Res} \bar{R}_{ff}(\mu_l) \equiv [\operatorname{Res} \bar{R}_{ff}(Z)]_{Z=\mu_c} . \tag{47}$$

The transfer function formed with the aid of coefficients a_k obtained by the simultaneous solution of the system of equations under (40) and of equations (36), (37), (38) satisfies prescriptions 1—4 as laid down in Chapter 2.

6. The characteristics of the system

From the aspect of transient performance, one of the basic characteristics of the system is the settling time T_s which depends on the order of the transfer function:

$$T_s = rT = (n + q)T . \tag{48}$$

Other characteristics are the error function $\psi_p(t)$ or the error series $\psi_p[k]$ arising as a result of the input signal. It is evident that in the case of $p = 1$

$$\psi_1[k] = -1 + \sum_{i=1}^{k} w_i, \tag{49}$$

$$\psi_1[\infty] = \lim_{k\to\infty} \psi_1[k] = 0. \tag{50}$$

Similarly in the case of $p = 2$

$$\psi_2[k] = \sum_{i=0}^{k-1} \psi_1[i] = \psi_2[k-1] + \psi_1[k-1], \tag{51}$$

$$\psi_2[\infty] = \begin{cases} W'(1) = \sum_{i=1}^{n+q} i\, w_i, & m = 1 \\ 0, & m \geq 2 \end{cases}. \tag{52}$$

Finally in the case of $p = 3$

$$\psi_3[k] = \sum_{i=1}^{k-1} \psi_2[i] + \frac{1}{2}\psi_2[k] = \psi_3[k-1] + $$

$$+ \frac{1}{2}\psi_2[k-1] + \frac{1}{2}\psi_2[k], \tag{53}$$

$$\psi_3[\infty] = \begin{cases} \infty & m = 1 \\ \frac{1}{2} W''(Z) = \frac{1}{2}\sum_{i=1}^{n+q} i(i-1)\, w_i, & m = 2\,. \\ 0, & m = 3 \end{cases} \tag{54}$$

In the knowledge of these equations, $[\psi_p[k]]_{\max}$ can be formed which is important especially in the case of $p = m - 1$, further the quadratic sum of the error:

$$\vartheta_p^2 = \sum_{k=0}^{n+q} \psi_p^2[k], \quad p \leq m. \tag{55}$$

For the continuous error function the characteristic $[\psi_p(t)]_{\max}$ and the square integral of the error Θ^2 cannot be determined by the above calculation, for this purpose the modified $\mathcal{Z}$-transformation should be adopted. These characteristics, however, do not in general contain considerably more information than the previous ones, consequently it is usually not worth carrying out the complicated calculation [6, 8].

From the point of view of statistics, the system can be characterized by the quadratic mean error ζ^2. From equation (20)

$$\zeta^2 = \frac{1}{2}\sum_{k=0}^{n}\sum_{r=0}^{n} A_{rk}\, a_r\, a_k - \sum_{k=0}^{n} B_k\, a_k + \sum_{l} \frac{1}{\mu_l} \operatorname{Res} \bar{R}_{ff}(\mu_l) + \bar{R}_{ff}(0). \tag{56}$$

where all the denotations are already known from earlier equations.

The question of the order n of the polynome $A(Z)$ should still be examined. Thus to be able to influence characteristic ζ^2 at all, it is necessary to have $n \geq m$. The higher the order n is chosen the lower will naturally ζ^2 be; consequently the better will the performance of the system be statistically. On the other hand, by increasing n the settling time will be longer, hence the transient performance of the system will be worse. On evaluating the transient performance, naturally the other characteristics should be also taken into consideration.

At the price of increasing the settling time, overshoots arising in the case of input signals of lower order can also be reduced. For this purpose the polynome $A(Z)$ is chosen with the order $n_1 = n + n_0$. The n_0 pieces of still undetermined coefficients a_k can be taken intuitively [10, 13], but it is more advisable to prescribe that either ϑ_p^2 or Θ_p^2 should be minimum [2, 5]. This last condition can be adopted only in the case of a ripple-free system, as otherwise Θ_p^2 is infinite.

On choosing the order n, another aspect should also be taken into account. The expression for the transfer function of the impulse-compensator on the basis of equations (18), (26) and (34) is

$$D(Z) = \frac{G_3(Z)\,A(Z)}{G_1(Z)\,(1-Z)^m\,C(Z)}, \tag{57a}$$

$$D(Z) = \frac{G_3(Z)\,A(Z)}{(1-Z)^m\,C(Z)}, \tag{57b}$$

where the second expression is valid for the case of a ripple-free system. As the order of $C(Z)$ depends on the order of $A(Z)$, hence on n, therefore the order of $D(Z)$ also depends on n. The higher n is the more complicated the build-up of the impulse-compensator is, moreover depending on the method of realization and on the numerical values, the impulse-compensator should eventually contain more active elements.

7. Example

To illustrate the procedure, let us examine a concrete case. In the system shown in Fig. 1 let the continuous transfer function of the controlled system be

$$G_s(s) = \frac{K_1}{s(s+1)^2}.$$

If the period time of sampling is $T = 1$, then

$$G(Z) = K\,\frac{Z(1+2.33\,Z)\,(1+0.16\,Z)}{(1-Z)\,(1-0.37\,Z)^2}.$$

We intend to eliminate the steady-state error only in the sampling moments. Hence, in accordance with (29b): $B(Z) = Z(1 + 2.33\,Z)$, that is $b_1 = 1$, $b_2 = 2.33$. Further it is evident that $G_1(Z) = 1$.

The stochastic control input of the following for which the control system is being designed, is shown in Fig. 3. If the reversals occurring in the

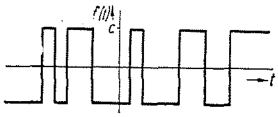

Fig. 3

time unit follow Poisson's distribution and their average value is γ, then the autocorrelation function will be [1]

$$r_{ff}(\tau) = c^2 e^{-2\gamma\tau}.$$

This autocorrelation function characterizes other stochastic signals, too, which are important in practice. The spectral density of the autocorrelation function [1]

$$\bar{S}_{ff}(\omega) = \frac{4}{\pi}\,\frac{\gamma c^2}{4\gamma^2 + \omega^2}\,.$$

Accordingly the bandwidth is $\omega_f = 2\,\gamma$. From equation $2\,\pi/T = \omega^f$ we obtain

$$T \geqq \frac{\pi}{2\gamma}\,.$$

The two-sided Z-transform of the autocorrelation function is

$$\bar{R}_{ff}(Z) = \frac{c^2}{1 - aZ} + \frac{c^2 \cdot a}{Z - a}\,, \quad a \equiv e^{-2\gamma T} > e^{-\pi} = 0.043.$$

Let us take the noise to be completely irregular (white noise), hence

$$r_{\varphi\varphi}(\tau) = kc^2\,\delta(\tau), \quad \bar{R}_{\varphi\varphi}(Z) = kc^2,$$

where k characterizes the relation of the average powers of the control input and of the noise, as

$$\frac{P_f}{P_\varphi} = \frac{\pi}{k\omega_f}\,.$$

Here ω_φ is the bandwidth of the white noise which was taken as infinite from the point of view of the autocorrelation function.

Evidently the only pole of $\bar{R}_{ff}(Z)$ within the unit circle is $\mu = a$, while $\bar{R}_{\varphi\varphi}(Z)$ has no poles. It can easily be controlled that

$$\text{Res}\ \bar{R}_{ff}(a) = c^2 a, \qquad \bar{R}_{ff}^{(i)}(0) = i\,!c^2(a^i - a^{-i}).$$

By using these values the expression of parameters (44), (45) and (46) will be

$$M_j = 2\,c^2 a^j,\ N = 2\,c^2(1 + k),\ P_i = 2\,c^2 a^i.$$

Thereupon the writing of the system of equations necessary to determine the coefficients affords no more difficulties. The solution of the system of equations will not be described in details, only the results are given.

The calculation was carried out for the following cases:

a) $m = 1,\ n = 2$
b) $m = 2,\ n = 2$

For the purpose of comparison, the calculation was carried out also for that case when only the deterministic characteristics were taken into account. The following cases were examined [5]:

c) $m = 1,\ n = 0$, minimum settling time
d) $m = 2,\ n = 1$, minimum settling time
e) $m = 2,\ n = 2$, minimum ϑ_1^2.

In all five cases the characteristics were examined for $\gamma T = 1$ and $\gamma T = 0.5$, further the values $k = 0$ (noiseless case), $k = 0.2$ and $k = 0.4$ were taken.

Table 1

Characteristic values for the case $m = 1,\ n = 2$
$T_s = 4T$

$\gamma T =$	1			0.5		
$k =$	0	0.2	0.4	0	0.2	0.4
$a_0 =$	0.154	0.148	0.144	0.220	0.200	0.187
$a_1 =$	0.020	0.026	0.030	−0.046	−0.022	−0.070
$a_2 =$	0.127	0.126	0.125	0.126	0.122	0.120
$\psi_1[1] =$	−0.846	−0.852	−0.856	−0.780	−0.800	−0.813
$\psi_1[2] =$	−0.469	−0.480	−0.499	−0.314	−0.356	−0.388
$\psi_1[3] =$	−0.296	−0.294	−0.292	−0.294	−0.284	−0.280
$\vartheta_1^2 =$	2.024	2.042	2.066	1.794	1.848	1.890
$\psi_2[\infty] =$	−2.612	−2.626	−2.647	−2.388	−2.440	−2.482
$\psi_3[\infty] =$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$
$\zeta^2/c^2 =$	1.280	1.336	1.393	1.183	1.250	1.333

Table 2

Characteristic values for the case $m = 2$, $n = 2$
$T_s = 4T$

$\gamma T =$	1			0.5		
$k =$	0	0.2	0.4	0	0.2	0.4
$a_0 =$	0.546	0.542	0.540	0.578	0.566	0.559
$a_1 =$	0.020	0.026	0.030	−0.046	−0.022	−0.007
$a_2 =$	−0.265	−0.268	−0.270	−0.232	−0.244	−0.252
$\psi_1[1] =$	−0.454	−0.458	−0.460	−0.422	−0.434	−0.441
$\psi_1[2] =$	0.837	0.832	0.830	0.880	0.864	0.855
$\psi_1[3] =$	0.618	0.625	0.630	0.542	0.570	0.587
$\vartheta_1^2 =$	2.288	2.293	2.296	2.246	2.248	2.269
$\psi_2[1] =$	−1.000	−1.000	−1.000	−1.000	−1.000	−1.000
$\psi_2[2] =$	−1.454	−1.458	−1.460	−1.422	−1.434	−1.441
$\psi_2[3] =$	−0.618	−0.625	−0.630	−0.542	−0.569	−0.586
$\psi_3[\infty] =$	−3.072	−3.082	−3.090	−2.963	−3.003	−3.027
$\zeta^2/c^2 =$	3.314	3.794	4.271	2.795	3.288	3.771

Table 3

Characteristic values of the system designed only by the deterministic method

Case	*c)*	*d)*	*e)*
$m =$	1	1	1
$n =$	0	1	2
$T_s/T =$	2	3	4
$a_0 =$	0.300	0.811	0.641
$a_1 =$	0	−0.510	−0.171
$a_2 =$	0	0	−0.170
$\psi_1[1] =$	−0.700	−0.189	−0.359
$\psi_1[2] =$	0	1.189	0.963
$\psi_1[3] =$	0	0	0.395
$\vartheta_1^2 =$	1.489	2.450	2.206
$\psi_2[\infty] =$	−1.700	0	0
$\psi_3[\infty] =$	∞	−2.189	−2.755

With the help of the tables the results of the designing methods can be compared. In case *a)* (Table 1) there is a steady-state error even for a linear input signal ($m = 2$), while for a quadratic input signal ($m = 3$) the steady-

Table 4

The statistical characteristic ζ^2/c^2 of the system designed only by the deterministic method

$\gamma T =$	1			0.5		
$k =$	0	0.2	0.4	0	0.2	0.4
c) case	1.529	1.645	1.760	1.320	1.446	1.554
d) case	4.551	5.345	6.132	3.495	4.290	5.278
e) case	3.468	3.996	4.528	2.850	3.378	3.906

state error is infinite. The statistical characteristic ζ^2/c^2 in turn, is very good that is the sonsequence of the fact that the deterministic condition can already be satisfied in the case of $n = 0$, hence, on choosing $n = 2$ we have two free parameters to minimalize ζ^2/c^2. In case *b)* (Table 2) there is already no steady-state error if $m = 2$, $\psi_1[k]$ in turn may have considerably high values. Case *c)* can be compared to case *a)* (Tables 3 and 4). Typical are the better transient and the worse statistical characteristics, as well as the more simple transfer function of the impulse-compensator. Cases *d)* and *e)* can be compared to case *b)*, the conclusions being largely the same. Attention should be called, however, to the fact that the maximum of $\psi_1[k]$ in case *a)* is lower than that arising even in case *e)*, where one parameter was determined by minimalizing ϑ_1^2.

Summary

A method has been given to directly determine the transfer function of sampled-data control systems, on condition that the steady-state error should disappear in the case of a determined input signal and the quadratic mean error should be minimum in the case of a random control input and noise of determined statistical characteristics. Examinations were carried out only on follow-up systems, on the condition that the control input and the noise are not correlated.

The procedure was illustrated by an example. Relying on the results of the example the quality characteristics of systems designed by different methods can be compared.

The method can be generalized for systems containing several impulse-compensators, as well as for the case of not incorrelated control input and noise.

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