SIMPLIFIED DERIVATION OF OPTIMUM TRANSFER FUNCTIONS FOR MULTIVARIABLE SYSTEMS

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1. Introduction

In most cases the statistical design of time-invariant continuous linear control systems is performed usually on the basis of the following simplifying conditions: the stochastic signals are assumed to be stationary, the ergodic hypothesis is adopted, finally, as a criterion of the optimum synthesis the least mean square error is taken.

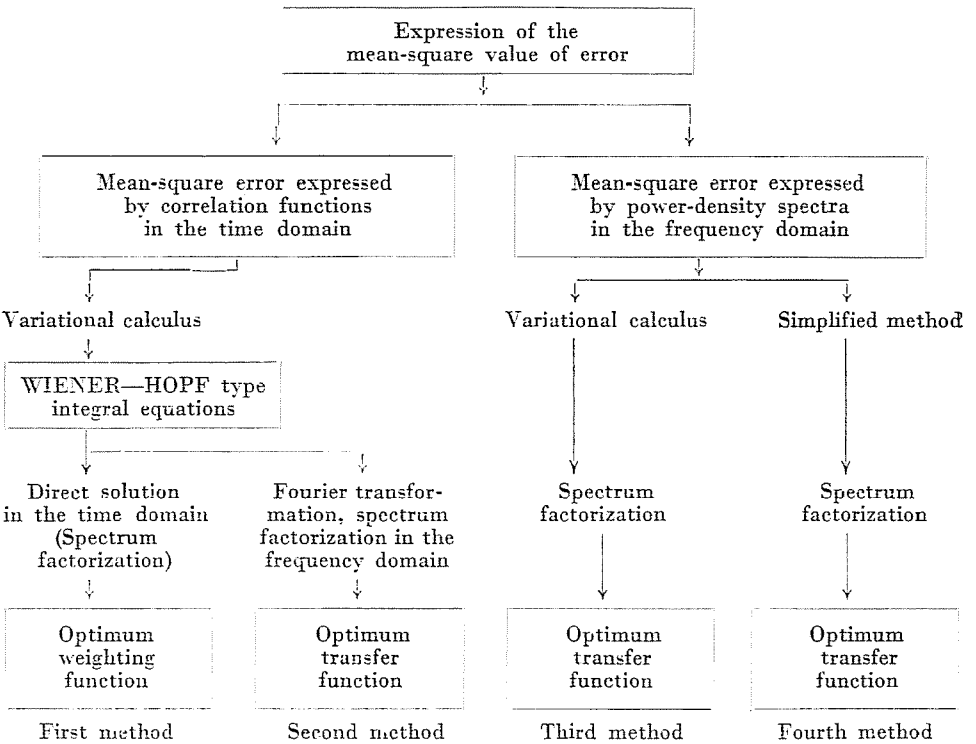
The optimization problem can be solved by one of the methods specified in Table 1. The first two methods were proposed by Wiener [1, 2] and nowadays may undoubtedly be considered as classical ones, the third method can be found in the book of Tsien [3] while the simplified method was first demonstrated by Bode and Shannon [4] but only for the case of uncorrelated signal and noise components. Recently, the author of this paper proposed a much simpler and direct method [5, 6].

The first method is somewhat awkward, because time-domain notions are only used. This difficulty will be alleviated by the second and third methods, which take advantage of the simplicity of the transform techniques and give the result in the form of transfer functions in the frequency domain. The fourth method completely relies on the frequency domain technique and avoids convolution integrals as well as the variational calculus.

A paper published previously [5, 6] has shown how the simplified method can be applied for the cases of the completely free configuration, semi-free configuration and semi-free configuration with constraints. This paper is only concerned with the completely free configuration, but instead of single-variable systems the multivariable (multipole) systems are investigated. (The semi-free configurations will be treated in a later paper.)

The optimum synthesis of multivariable systems was first solved by Amara [7] in a general way, applying the second method combined with matrix calculus. This paper could still not give a satisfactory answer to the question: is the spectrum factorization of matrices performable in every case or not. This problem was positively solved by Youla [8]. Unfortunately, the proposed procedure is very difficult and cumbersome. In some more simple cases the factorization procedure proposed by Kavanagh [9] could be

Table 1



applied. Remarkable is the optimization procedure proposed by HSIEH and LEONDES [10] which is also based on the second method, but essentially without consistent matrix calculus, it makes use of the undetermined coefficients to obtain results. The present paper completely solves the problem of optimum synthesis in the frequency domain without variational calculus taking to a great extent advantage of the matrix calculus.

2. The proposed method

In the following treatise double-index notation will be used. If both indices are variable we have a matrix, if one index is fixed or is missing and only the other index is variable we have a vector, finally, if both indices are fixed this notation refers to a scalar quantity. A column vector has a variable first index, while a row vector has a variable second index (the others being fixed).

The block diagram of a multivariable system is shown in Fig. 1. The input signals are represented by a row vector $r_{.k}$ (where $k = 1 \ldots K$), the weight-

ing functions are represented by a matrix w_{kl} (where $k = 1 \ldots K$, $l = 1 \ldots L$), finally the output signals are represented by a row vector $c_{.l}$ (where $l = 1 \ldots L$).

The row vector of the output signals can be expressed by the following convolution integral:

$$c_{.l}(t) = \int_{-\infty}^{\infty} r_{.k}(t - \vartheta)\, w_{kl}(\vartheta)\, d\vartheta = \int_{-\infty}^{\infty} r_{.k}(\zeta)\, w_{kl}(t - \zeta)\, d\zeta \qquad (1)$$

where the integrand is a row vector originating from the matrix multiplication of a row matrix and a $K \times L$ matrix.

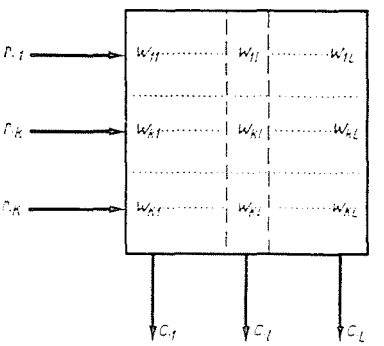


Fig. 1

The problem of optimum synthesis is demonstrated in Fig. 2. Here $s_{.k}$ is the row vector of the useful signal components, while $n_{.k}$ is the row vector of the noise components, both belonging to the whole input row vector $r_{.k}$. The ideal or desired outputs are represented by $i_{.l}$ ($l = 1 \ldots L$). This row vector can be determined from the signal-component vector $s_{.k}$ by matrix multiplication with weighting-function matrix y_{kl} and by one of the following convolution integrals:

$$i_{.l} = \int_{-\infty}^{\infty} s_{.k}(t - \vartheta)\, y_{kl}(\vartheta)\, d\vartheta = \int_{-\infty}^{\infty} s_{.k}(\zeta)\, y_{kl}(t - \zeta)\, d\zeta \qquad (2)$$

It is worth mentioning that the weighting function matrix y_{kl} is in general not physically realizable (that is, the elements of matrix y_{kl} are not physically realizable in every case).

The row vector of the error is the difference of the row vectors of the ideal and real output signals:

$$e_{.l}(t) = i_{.l}(t) - c_{.l}(t)\,. \qquad (3)$$

In case of completely free configuration as a criterion of the optimum synthesis one of the following two assumptions may be adopted: the sum of the mean square errors or the mean square errors separately should be considered as being minimum. In the first case the least-mean-square error-figure

can be expressed by the trace of a matrix, i.e. by the sum of the diagonal elements of a matrix, while in the second case by a simple scalar quantity, not speaking of the averaging process*:

$$\operatorname{tr} \overline{e_{l\cdot}(t)\, e_{\cdot l}(t)} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \operatorname{tr}\left[e_{l\cdot}(t)\, e_{\cdot l}(t)\right] dt = \operatorname{tr} \varphi_{e_l e_l}(0) = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \operatorname{tr} \Phi_{e_l \cdot e_l}(s) ds \quad (4)$$

where

$$\varphi_{e_l \cdot e_l}(\tau) = \overline{e_{l\cdot}(t)\, e_{\cdot l}(t+\tau)} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e_{l\cdot}(t)\, e_{\cdot l}(t+\tau)\, dt \quad (5)$$

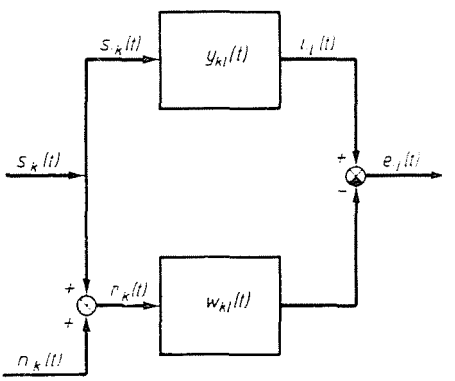


Fig. 2

is the correlation-function matrix of the errors. The power-density-spectrum matrix of the error signals can be obtained by the Fourier-transformation as follows (with $s = j\omega$):

$$\Phi_{e_l \cdot e_l}(s) = \mathcal{F}\left[\varphi_{e_l \cdot e_l}(\tau)\right] = \int_{-j\infty}^{j\infty} \Phi_{e_l \cdot e_l}(\tau)\, e^{-s\tau}\, d\tau \quad (6)$$

while the inverse relation is:

$$\varphi_{e_l \cdot e_l}(\tau) = \mathcal{F}^{-1}\left[\Phi_{e_l \cdot e_l}(s)\right] = \frac{1}{2\pi j} \int_{-\infty}^{j\infty} \Phi_{e_l \cdot e_l}(s)\, e^{\tau s}\, ds \quad (7)$$

Evidently:

$$\overline{e_{l\cdot}(t)\, e_{\cdot l}(t)} = \overline{\left[i_{l\cdot}(t) - c_{l\cdot}(t)\right]\left[i_{\cdot l}(t+\tau) - c_{\cdot l}(t+\tau)\right]} \quad (8)$$

thus,

$$\varphi_{e_l \cdot e_l}(\tau) = \varphi_{i_l i_l}(\tau) - \varphi_{i_l c_l}(\tau) - \varphi_{c_l i_l}(\tau) + \varphi_{c_l c_l}(\tau) \quad (9)$$

or

$$\Phi_{e_l e_l}(s) = \Phi_{i_l i_l}(s) - \Phi_{i_l \cdot c_l}(s) - \Phi_{c_l i_l}(s) + \Phi_{c_l \cdot c_l}(s). \quad (10)$$

* The column vector $e_{l\cdot}$ is the transposed of row vector $e_{\cdot l}$.

The task is just to minimize the trace of the power-density-spectrum matrix $\Phi_{e_i e_l}(s)$ expressed by the above mentioned matrix equation (10). The expression of the mean-square error becomes, namely, after substituting $s = j\omega$:

$$\operatorname{tr} \overline{e_{l\cdot}(t)\, e_{\cdot l}(t)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \operatorname{tr} \Phi_{e_i e_l}(j\omega)\, d\omega \tag{11}$$

which is a real-variable integral, the trace of the power-density-spectrum matrix being function of s^2 or ω^2, i. e. it is a real variable function. Thus, the above mentioned integral (11) evidently becomes minimum, if and only if the integrand is the least possible. Generalizing the index-changing rule [11], the power-density-spectrum matrix can be expressed as:

$$\begin{gathered} \Phi_{e_i,e_l}(s) = \Phi_{i_i,i_l}(s) - \Phi_{i_i,r_k}(s)\, W_{kl}(s) - W_{l'k'}(-s)\, \Phi_{r_k i_l}(s) + \\ + W_{l'k'}(-s)\, \Phi_{r_k,r_k}(s)\, W_{kl}(s) \\ (k,\ k' = 1 \ldots K), \quad (l, l' = 1 \ldots L) \end{gathered} \tag{12}$$

where $\Phi_{e_i,e_l}(s)$ and $\Phi_{i_i,i_l}(s)$ are matrix functions (or scalars), while in case of variable (or fixed) indices l and l' $\Phi_{i_i,r_k}(s)$ is a $L \times K$ matrix (or a row vector), $\Phi_{r_k,i_l}(s)$ is a $K \times L$ matrix (or a column vector), further on $W_{kl}(s)$ is a $K \times L$ matrix (or a column vector) $W_{l\,k'}(s)$ is a $L \times K$ matrix (or a row vector) and $\Phi_{r_k r_k}(s)$ is a $K \times K$ matrix. It is assumed, that the elements of the latter matrix are rational fractional functions of s and $\Phi_{r_k,r_k}(s) = \Phi_{r_k r_k}(-s)$.

Let us introduce a $K \times L$ matrix (or a column vector) $G_{kl}(s)$ and a $L \times K$ matrix (or a row vector) $G_{l'k'}(-s)$ this latter being the adj oint, that is, the complex conjugate transposed matrix of the former one:

$$\begin{aligned} \Phi_{r_k,r_k}(s)\, G_{kl}(s) &= \Phi_{r_k,i_l}(s) \\ G_{l\,k'}(-s)\, \Phi_{r_k r_k}(s) &= \Phi_{i_l r_k}(s) \end{aligned} \tag{13}$$

where the auxiliary-function matrix (or vector) $G_{kl}(s)$ is, in general, physically unrealizable. As $\Phi_{r_k r_k}(s)$ and $\Phi_{r_k i_l}(s)$ are from the beginning given $G_{kl}(s)$ and $G_{l'k'}(-s)$ are also given. With the aid of the auxiliary matrices (or vectors) introduced in definition (13) the power-density spectrum of the errors becomes:

$$\begin{gathered} \Phi_{e_i e}(s) = \Phi_{i_i i_l}(s) - G_{l'k'}(s)\, \Phi_{r_k r_k}(s)\, W_{kl}(s) - \\ - W_{l'k'}(-s)\, \Phi_{r_k,r_k}(s)\, G_{kl}(s) + \\ + W_{l'k'}(-s)\, \Phi_{r_k,r_k}(s)\, W_{kl}(s) \end{gathered} \tag{14}$$

or

$$\begin{gathered} \Phi_{e_i,e_l}(s) = \Phi_{i_i i_l}(s) - G_{l'k'}(-s)\, \Phi_{r_k r_k}(s)\, G_{kl}(s) + \\ + \left[G_{l'k'}(-s) - W_{l'k'}(-s)\right] \Phi_{r_k r_k}(s) \left[G_{kl}(s) - W_{kl}(s)\right]. \end{gathered} \tag{15}$$

It must be emphasized that matrix $W_{kl}(s)$ as well as the adjoint (or complex conjugate transposed) matrix $W_{l'k'}(-s)$ are only contained in the last term. The power-density spectrum of the errors is the least, when the last term becomes zero. As the last term is also a quadratic form, it can be zero, if and only if

$$\begin{aligned} &G_{kl}(s) - W^o_{kl}(s) = 0 \\ \text{or} \quad &G_{lk'}(-s) - W^o_{lk'}(-s) = 0 \end{aligned} \tag{16}$$

By substituting the pair of equations (16) into equations (13) yields:

$$\begin{aligned} &\Phi_{r_k r_k}(s)\, W^o_{kl}(s) - \Phi_{r_k i_l}(s) = 0 \\ &W^o_{l'k'}(-s)\, \Phi_{r_k r_k}(s) - \Phi_{i_l r_k}(s) = 0 \end{aligned} \tag{17}$$

Thus, the physically unrealizable optimum transfer functions are:

$$\begin{aligned} &W^o_{kl}(s) = G_{kl}(s) = \left[\Phi_{r_k r_k}(s)\right]^{-1} \Phi_{r_k i_l}(s) \\ \text{or} \quad &W^o_{l'k'}(-s) = G_{l'k'}(-s) = \Phi_{i_l r_k}(s) \left[\Phi_{r_k r_k}(s)\right]^{-1} \end{aligned} \tag{18}$$

It must be emphasized that the determination of the optimum transfer function has a meaning at all, if and only if the trace of the remainder on the right side of Eq. (15):

$$\Phi_{i_l i_l}(s) - G_{l'k'}(-s)\, \Phi_{r_k r_k}(s)\, G_{kl}(s) \tag{19}$$

also being a function of s^2 or ω^2, becomes negative nowhere. This follows from the physical notion of the power-density spectra belonging to auto-correlation functions and the mean square value respectively.

Taking the restriction of the physical realizability of the transfer function $W_{kl}(s)$ into consideration, and this must be done in every case, then instead of the first relation of Eq. (17) at most the following relation is valid:

$$\Phi_{r_k r_k}(s)\, W^m_{kl}(s) - \Phi_{r_k i_l}(s) = F_{k'l}(s) \tag{20}$$

where $W^m_{kl}(s)$ is the matrix (or in the case of fixed index l, the column vector) of the physically realizable transfer function, while $F_{k'l}(s)$ is a yet unknown matrix (or column vector), which contains no transfer function component with left-half-plane poles. In this case the last term (i. e. the last quadratic form) on the right side of Eq. (15) is, as a general rule not zero, but with respect to physical realizability it is the least practicable value, as in Eq. (20) only the physically unrealizable component $W^n_{kl}(s)$ of the physically unrealizable optimum transfer function $W^o_{kl}(s)$ figurs, while the physically realizable component $W^m_{kl}(s)$ is completely missing. Namely:

$$W^o_{kl}(s) = W^m_{kl}(s) + W^n_{kl}(s) \tag{21}$$

Incidentally, taking Eqs. (17), (20) and (21) into consideration:

$$\Phi_{r_{k'}r_k}(s)\, W^n_{kl}(s) = -F_{k'l}(s). \tag{22}$$

The power-density-spectrum matrix is assumed to be a real paraconjugate hermetian matrix:

$$\Phi_{r_{k'}r_k}(s) = \Phi_{r_k r_{k'}}(-s) \tag{23}$$

i. e. it is in conformity with its transposed complex conjugate (or adjoint) matrix. According to [8] those matrices can be factorized as:

$$\Phi^-_{r_{k'}r_{k''}}(s)\, \Phi^+_{r_{k''}r_k}(s) = \Phi_{r_{k'}r_k}(s) \tag{24}$$

where the second matrix factor

$$\Phi^+_{r_{k''}r_k}(s) \tag{25}$$

and also the inverse matrix

$$[\Phi^+_{r_{k''}r_k}(s)]^{-1} \tag{26}$$

contain elements with exclusively *left-half-plane* poles and zeros, while the first matrix factor

$$\Phi^-_{r_{k'}r_{k''}}(s) \tag{27}$$

and also its inverse matrix

$$[\Phi^-_{r_{k'}r_{k''}}(s)]^{-1} \tag{28}$$

have only elements with *right-half-plane* poles and zeros. Incidentally, the adjoint of matrix (27) is just the same as matrix (25):

$$\Phi^-_{r_{k''}r_{k'}}(-s) = \Phi^+_{r_{k''}r_k}(s) \tag{29}$$

Thus, from Eq. (20) on the basis of Eq. (24) we have:

$$\Phi^-_{r_{k'}r_{k''}}(s)\, \Phi^+_{r_{k''}r_k}(s)\, W^m_{kl}(s) = \Phi_{r_{k'}i_l}(s) + F_{k'l}(s) \tag{30}$$

or after multiplying with expression (28) from the left side

$$\Phi^+_{r_{k''}r_k}(s)\, W^m_{kl}(s) = [\Phi^-_{r_{k'}r_{k''}}(s)]^{-1}\, \Phi_{r_{k'}i_l}(s) + [\Phi^-_{r_{k'}r_{k''}}(s)]^{-1}\, F_{k'l}(s). \tag{31}$$

Decomposing the matrices (or vectors) figuring on both side of relation (31) into physically realizable and unrealizable matrix (or vector) components, it

becomes evident that the second term on the right side of Eq. (31) can have right-half-plane poles only and so it can not supply any physically realizable matrix (or vector) component, while the term on the left side of Eq. (31) purely consists of a physically realizable matrix (or vector) component. Thus, Eq. (31) can be separated into two parts:

$$\Phi^{+}_{r_k r_k}(s)\, W^{m}_{kl}(s) = \left\{[\Phi^{-}_{r_k r_{k'}}(s)]^{-1}\, \Phi_{r_{k'} i_l}(s)\right\}_{+} \tag{32}$$

and

$$0 = \left\{[\Phi^{-}_{r_k r_{k'}}(s)]^{-1}\, \Phi_{r_{k'} i_l}(s)\right\}_{-} + \left\{[\Phi^{-}_{r_k r_{k'}}(s)]^{-1}\, F_{k'l}(s)\right\}_{-}. \tag{33}$$

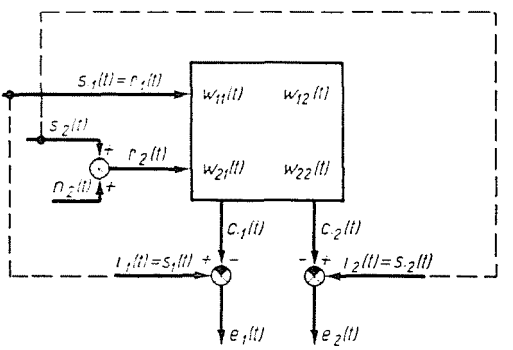


Fig. 3

Finally, from Eq. (32) after multiplying with (26) from the left, the physically realizable optimum transfer-function matrix (or column vector) is obtained:

$$W^{m}_{kl}(s) = [\Phi^{-}_{r_{k'} r_k}(s)]^{-1} \left\{[\Phi^{-}_{r_k r_{k'}}(s)]^{-1}\, \Phi_{r_{k'} i_l}(s)\right\}_{+}. \tag{34}$$

It must be emphasized, that expression (34) is the direct matrix generalization of the well-known closed formula of single-variable systems for the case of multiport systems.

3. Illustrative example

For the sake of perspicuity only a very simple two-variable system is treated (Fig. 3). Of the two inputs each contains a useful signal component, while the second input is corrupted by a white noise component, too. The useful signal components and the noise components are not correlated. For the sake of simplicity let the ideal or desired outputs be identical with the signal components of the inputs. The task of the investigation is to determine the effect of the various cross-correlation degrees on the sum of the least-mean-square errors, the two useful input signal components being correlated.

The starting point is formed by the following data: (k''; k'; $k = 1,2$; $l = 1, 2$)

$$\Phi_{s_k' s_k}(s) = \begin{bmatrix} \dfrac{4}{1-s^2} & \dfrac{2A}{(1-s)(2+s)} \\ \dfrac{2A}{(2-s)(1+s)} & \dfrac{1}{4-s^2} \end{bmatrix}$$

$$\Phi_{n_k n_k}(s) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$\Phi_{s_k' n_k}(s) = \Phi_{n_k s_k}(s) = 0 .$$

Thus, the power-density matrix of the inputs:

$$\Phi_{r_k' r_k}(s) = \Phi_{s_k' s_k}(s) + \Phi_{n_k' n_k}(s)$$

therefore,

$$\Phi_{r_k' r_k}(s) = \begin{bmatrix} \dfrac{4}{1-s^2} & \dfrac{2A}{(1-s)(2+s)} \\ \dfrac{2A}{(2-s)(1+s)} & \dfrac{5-s^2}{4-s^2} \end{bmatrix}$$

On the other hand:

$$\Phi_{r_k' i_l}(s) = \Phi_{s_k' i_l}(s) + \Phi_{n_k' i_l}(s) = \left[\Phi_{s_k' s_k}(s) + \Phi_{n_k' s_k}(s)\right] Y_{kl}(s) = \Phi_{s_k' s_k}(s)$$

as in the present case:

$$Y_{kl}(s) = I .$$

First, let us determine on the basis of Eq. (18) the auxiliary-function matrix, which, according to Eq. (16) is also the optimum transfer function matrix:

$$G_{kl}(s) = W_{kl}^{o} = \begin{bmatrix} 1 & \dfrac{A(2-s)(1+s)}{2(5-A^2-s^2)} \\ 0 & \dfrac{1-A^2}{5-A^2-s^2} \end{bmatrix}$$

as

$$\left[\Phi_{r_k r_k}(s)\right]^{-1} = \begin{bmatrix} \dfrac{(1-s^2)(5-s^2)}{5-A^2-s^2} & -\dfrac{A}{2}\dfrac{(2-s)(1+s)}{5-A^2-s^2} \\ -\dfrac{A}{2}\dfrac{(1-s)(2+s)}{5-A^2-s^2} & \dfrac{4-s^2}{5-A^2-s^2} \end{bmatrix}$$

The transfer function matrix $W^0_{kl}(s)$, as depicted in one of the previous matrices, is unrealizable physically.

The essence of the spectrum factorization is the determination of factor matrices figuring in Eq. (24). In the present case, the spectrum factorization procedure is relatively simple, giving:

$$\Phi^{-}_{r_k r_{k'}}(s) = \begin{bmatrix} \dfrac{2}{1-s} & \\ \dfrac{A}{2-s} & \dfrac{\sqrt{5-A^2}-s}{2-s} \end{bmatrix}$$

$$\Phi^{+}_{r_{k'} r_k}(s) = \begin{bmatrix} \dfrac{2}{1+s} & \dfrac{A}{2+s} \\ 0 & \dfrac{\sqrt{5-A^2}+s}{2+s} \end{bmatrix}$$

and

$$[\Phi^{-}_{r_k r_{k'}}(s)]^{-1} = \begin{bmatrix} \dfrac{1-s}{2} & 0 \\ -\dfrac{A}{2}\dfrac{1-s}{\sqrt{5-A^2}-s} & \dfrac{2-s}{\sqrt{5-A^2}-s} \end{bmatrix}$$

$$[\Phi^{+}_{r_{k'} r_k}(s)]^{-1} = \begin{bmatrix} \dfrac{1+s}{2} & -\dfrac{A}{2}\dfrac{1+s}{\sqrt{5-A^2}+s} \\ 0 & \dfrac{2+s}{\sqrt{5-A^2}+s} \end{bmatrix}$$

Thus,

$$[\Phi^{-}_{r_k r_{k'}}(s)]^{-1}\Phi_{r_k i_l}(s) = \begin{bmatrix} \dfrac{2}{1+s} & \dfrac{A}{2+s} \\ 0 & \dfrac{1-A^2}{(\sqrt{5-A^2}-s)(2+s)} \end{bmatrix}$$

Separating the physically realizable matrix component:

$$\{[\Phi^{-}_{r_k r_{k'}}(s)]^{-1}\Phi_{r_k i_l}(s)\}_{+} = \begin{bmatrix} \dfrac{2}{1+s} & \dfrac{A}{2+s} \\ 0 & \dfrac{1-A^2}{(\sqrt{5-A^2}+2)(2+s)} \end{bmatrix}$$

Finally, on the basis of Eq. (34) the physically realizable transfer-function matrix is:

$$W_{kl}^{m}(s)=\begin{bmatrix} 1 & \dfrac{A}{2}\dfrac{1+s}{2+s}\left[1-\dfrac{1-A^2}{(\sqrt{5-A^2}+2)(\sqrt{5-A^2}+s)}\right] \\ 0 & \dfrac{1-A^2}{(\sqrt{5-A^2}+2)(\sqrt{5-A^2}+s)} \end{bmatrix}$$

From the latter matrix the desired column-vector components can readily be determined. On the other hand, it is worth mentioning that clearly $W_{kl}^{m}(s) \neq [W_{kl}^{o}(s)]_{+}$.

Let us now determine expression (19), that is, the difference of the first two matrices figuring on the right side of Eq. (15).

$$\Phi_{e_l e_i}(s)\Big|_{W_{kl}=W_{kl}^{o}} = \begin{bmatrix} 0 & 0 \\ 0 & \dfrac{1-A^2}{5-A^2-s^2} \end{bmatrix}$$

It becomes clear that the possible values of A must be in the domain

$$0 \leqq A \leqq 1 .$$

The third matrix on the right side of Eq. (15) is:

$$\begin{bmatrix} 0 & 0 \\ 0 & \dfrac{(1-A^2)^2}{(\sqrt{5-A^2}+2)^2(5-A^2-s^2)} \end{bmatrix}$$

Thus, the addition of the two latter matrices yield:

$$\Phi_{e_l e_i}(s)\Big|_{W_{kl}=W_{kl}^{m}} = \begin{bmatrix} 0 & 0 \\ 0 & \left[1+\dfrac{1-A^2}{(\sqrt{5-A^2}+2)^2}\right]\dfrac{1-A^2}{5-A^2-s^2} \end{bmatrix}.$$

With the aid of formula (4) and the theorem of residues the mean-square error of the first output is:

$$\min\ \overline{e_1^2(t)} = 0$$

While the mean-square error of the second output:

$$\min_{W_{k1}=W_{k1}^{o}} \overline{e_2^2(t)} = \frac{1-A^2}{2\sqrt{5-A^2}} .$$

This would be the least-mean-square error, if $W_{12}(s) = W_{12}^{o}(s)$ and $W_{22}(s) = W_{22}^{o}(s)$ were valid. With the restriction of physical realizability

$$\min_{W_{k1}=W_{k1}} \overline{e_2^2(t)} = \left[1 + \frac{\sqrt{1-A^2}}{(\sqrt{5-A^2}+2)^2}\right] \frac{1-A^2}{2\sqrt{5-A^2}}$$

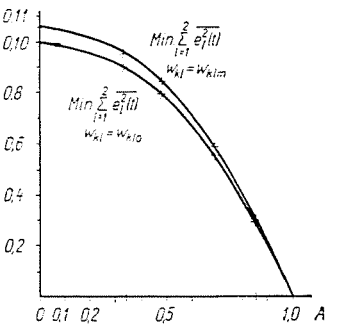


Fig. 4

This is the physically realizable least value of the mean-square error. The sum of the least-mean-square errors is maximum when $A = 0$ (i. e. the input signal components are uncorrelated) and is minimum (that is zero in the present case) when $A = 1$ (i. e. the correlation is maximum).

The sums of the least-mean-square errors as function of A are demonstrated in Fig. 4.

Summary

After reviewing the synthesis methods (based on the least-mean-square-error criterion of optimum time-invariant continuous linear control systems, the present paper gives a simplified derivation for the completely-free-configuration multivariable systems, applying, in general, the matrix calculus.

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