GENERALIZED COMPLEX-PLANE STABILITY CRITERIA

By

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To judge the behaviour of control systems, there are several complex-plane mapping methods. On the one hand, the open-loop frequency-response $G(s)\,H(s)$ or the inverse one $1/G(s)\,H(s)$ may be mapped, and, on the other hand, the closed-loop frequency-response $M(s)$ or the inverse $N(s)$ may be plotted, while the variable s delineates a distinct curve in the complex plane, for example $s = j\omega$. The transition from one of the mapping methods to the other may be facilitated by the so-called complex-plane charts. The eight mapping possibilities have been described in a previous paper [1]. Here the rules for reading the main qualitative control characteristics, as well as the stability criteria for the case, the open-loop transfer function has no right-half-plane poles and zeros may be found. The aim of the present study is to give and to summarize in a uniform treatment the stability criteria for the cases of the open-loop transfer functions $G(s)\,H(s)$ these being unstable in themselves and not being minimum phase systems, that is, for cases, when some of the poles and zeros of the open-loop transfer function are located straight to the right from the imaginary axis in the complex plane.

Among the stability criteria given in the following some generalizations referring to the closed-loop frequency response and its inverse, respectively, are described here — according to our knowledge — for the first time, while others, that is, the NYQUIST diagrams and criteria concerning the open-loop frequency response and its inverse, are generally known *e. g.* [2, 3], and are introduced merely for the sake of completeness.

1. Symbols, designations

In the following only single-loop control systems will be discussed. (This limitation does not influence the validity of theorems. Results may easily be generalized for multi-loop control systems too).

The index z and p, respectively, refer to the numerators and denominators, that is, to functions permitting the determination of zeros and poles, respectively. The transfer function of the forward branch is

$$G(s) = \frac{G_z(s)}{G_p(s)}, \tag{1.1}$$

while the transfer function of the feedback branch has the form

$$H(s) = \frac{H_z(s)}{H_p(s)}, \tag{1.2}$$

where the functions appearing in the numerator and in the denominator may contain factors as K, s, $1 + sT$, $1 + 2\zeta sT + s^2T^2$ (exceptionally also a factor e^{-sT}).

The whole open-loop transfer function is expressed as

$$G(s)\,H(s) = \frac{G_z(s)\,H_z(s)}{G_p(s)\,H_p(s)}. \tag{1.3}$$

The closed-loop transfer function is denoted by

$$M(s) = \frac{M_z(s)}{M_p(s)}, \tag{1.4}$$

while the inverse closed-loop transfer function has the form

$$N(s) = \frac{N_z(s)}{N_p(s)}. \tag{1.5}$$

As is well-known (see *e. g.* [2])

$$M(s) = \frac{G(s)}{1 + G(s)\,H(s)} = \frac{1}{N(s)}. \tag{1.6}$$

Thus, considering the aforementioned

$$N_z(s) = M_p(s) = G_p(s)\,H_p(s) + G_z(s)\,H_z(s) \tag{1.7}$$

and

$$N_p(s) = M_z(s) = G_z(s)\,H_p(s) \tag{1.8}$$

the expression $N_z(s) = M_p(s)$ figuring here set equal to zero is the so-called characteristic equation.

Incidentally it is to be mentioned, that the closed-loop transfer functions may also have the following form

$$M(s) = M_1(s)\,\frac{1}{H(s)} = M_2(s)\,G(s) \tag{1.9}$$

and

$$N(s) = N_1(s)\,H(s) = N_2(s)\,\frac{1}{G(s)}, \tag{1.10}$$

where

$$M_1(s) = \frac{G(s)\,H(s)}{1 + G(s)\,H(s)} = \frac{1}{N_1(s)} \tag{1.11}$$

and

$$N_2(s) = \frac{1 + \dfrac{1}{G(s)\,H(s)}}{\dfrac{1}{G(s)\,H(s)}} = \frac{1}{M_2(s)}. \tag{1.12}$$

By the above-mentioned complex-plane charts [1] the frequency-responses $M_1(s)$, $N_1(s)$ and $M_2(s)$, $N_2(s)$, respectively, may directly be determined.

Stability may already be decided by this reduced form. To obtain, however, the qualitative characteristics, the multiplications assigned in formulae (1.9), (1.10) must be carried out and the complete transfer functions $M(s)$ or $N(s)$ must be determined.

Finally, be the number of the corresponding right-half-plane poles and zeros, that is, poles or zeros with positive real parts of $G(s)$, $H(s)$, $M(s)$, $N(s)$ consecutively P_G, P_H, P_M, P_N and Z_G, Z_H, Z_M, Z_N, respectively, where always $P_M = Z_N$ and $P_N = Z_M$.

While P_G, P_H, Z_G, Z_H and $Z_M = P_N = Z_G + P_H$ are known, stability criteria just require the determination of the number of the right-half-plane roots

$$Z = Z_N = P_M$$

of the characteristic equation. Namely, in the denominator $M_p(s)$ of the closed-loop transfer function (in the numerator $N_z(s)$ of its inverse), *i. e.* in the characteristic equation, the *sum* of functions is figuring, consequently the roots cannot be directly read, as they do not appear in a factored form.

The stability criterion may, in general, be expressed as follows:

The closed-loop control system is stable, if, and only if $Z = 0$, that is, the characteristic equation has no right-half-plane root.

Naturally, the system is unstable even if some of the poles of the closed-loop transfer function (some of the zeros of the inverse closed-loop transfer function) get onto the imaginary axis. This limited case may easily be recognized on the basis of the stability criteria to be given.

2. Auxiliary theorems

To determine the stability criteria, following auxiliary theorems will be uniformly adopted. As they may be regarded as more or less known, their detailed verification will be omitted.

Theorem 1. Starting from the argument principle of complex variables (see *e. g.* [5, 6]), and based on the logarithmic integral theorem, one finds the following relation:

$$\frac{1}{2\pi j} \oint_{C} \frac{F'(s}{F(s} ds = P_F - Z_F = R_0\{F(s)\}. \tag{2.1}$$

Thus, for an arbitrary, simple closed path C the above integral furnishes, on the one hand, the difference between the number of poles P_F and the number of zeros Z_F of the investigated function $F(s)$—situated inside the curve C and, on the other hand, it gives the number of the net, positive (counter-clockwise) rotations of curve $F(s)$ around the origin: the value $R_0\{F(s)\}$. ($F(s)$ must be a single-valued function on and within a simple closed contour C and furthermore, $F(s)$ must be analytic and different from zero on the path C).

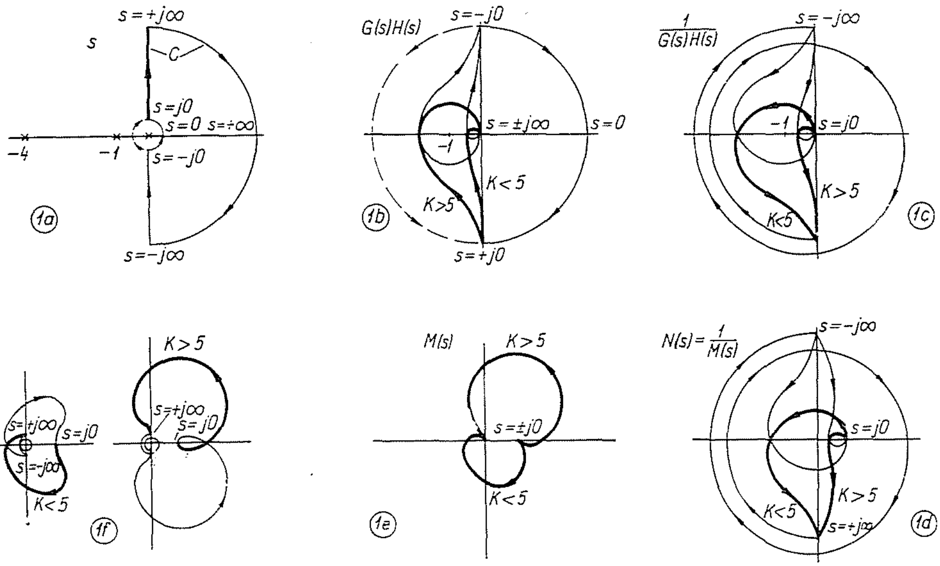
It is to be remarked, that curve C has to be plotted contrary to the custom, in *negative* (clockwise) direction, while the number of the complete encirclements R_0 is read in *positive* (counter-clockwise) direction.

Theorem 2. Convention : In order to determine the net difference between the number of the right-half-plane poles and zeros, curve C must be composed of the total imaginary axis and the joining half-plane circle of infinite radius, that is, the contour C covers the entire right-half of the plane s.

If there is a pole or a zero on the imaginary axis, then it must be bypassed from the right (in counter-clockwise direction), or from the left (in clockwise direction) by a small semicircle of a radius tending to zero.

Naturally, in the latter case the number of right-half-plane poles or zeros increase by one (see Figs. 1/a and 2/a).

Theorem 3. Contour C in the complex-plane s is mapped by the function $F(s)$ onto the complex-plane $F(s)$ also in form of a closed curve $C_{F(s)}$. From the curve obtained in this way (to be called simply $F(s)$), the number of the

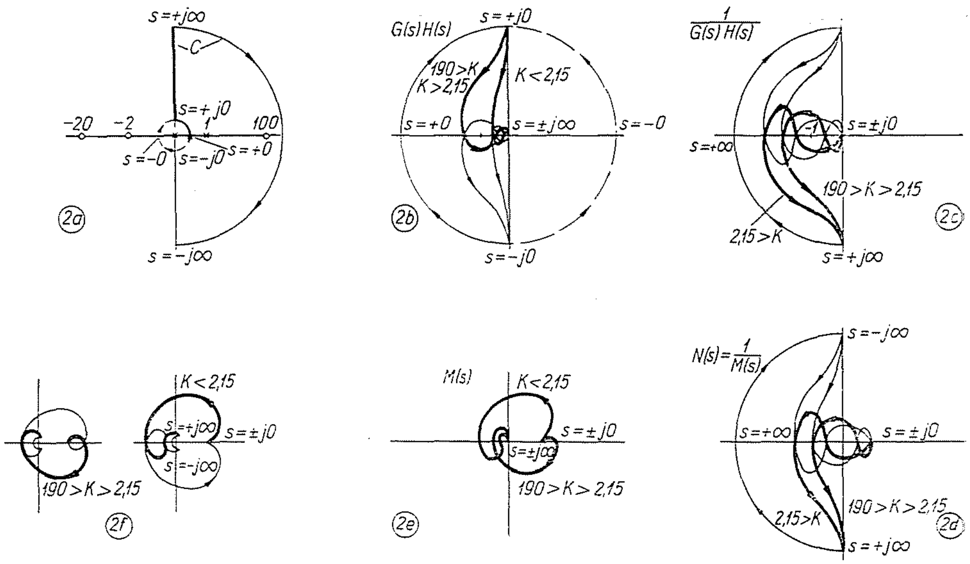


$G(s) = K \frac{1}{s(1+s)(1+0{,}25s)}$, $H(s) = 1$		
	$P_G = \begin{cases}0\\1\end{cases}$ $P_H = 0$ $Z_G = 0$ $Z_H = 0$	
	$K < 5$ $Z = 0$ stable	$K > 5$ $Z = 2$ unstable
$R_{-1}\{G(s)H(s)\}$	$\begin{cases}0\\+1\end{cases}$	$\begin{cases}2\\-1\end{cases}$
$R_{-1}\left\{\frac{1}{G(s)H(s)}\right\}$	0	−2
$R_0\{N(s)\}$ / Q_N D_N	0 / +3 +3	−2 / −1 +3
$R_0\{M(s)\}$ / Q_M D_M	0 / −3 −3	+2 / +1 −3

*Fig. 1*a. Choice of the path C and location of the poles and zeros of $G(s)\ H(s)$
*Fig. 1*b. Complex-plane plots of the open-loop transfer function
*Fig. 1*c. Complex-plane plots of the inverse open-loop transfer function
*Fig. 1*d. Complex-plane plots of the inverse closed-loop transfer function
*Fig. 1*e. Complex-plane plots of the closed-loop transfer function
*Fig. 1*f. Main branches in the complex-plane plots of the closed-loop transfer function

net encirclements $R_0\{F(s)\}$ can easily be determined. Incidentally, it must be noted, that to the small semicircle of a radius decreasing beyond all limits and enclosing on the imaginary axis *e. g.* a pole of multiplicity i, belongs a curve section consisting of semicircles in number i of radius increasing beyond all limits and being oppositely directed in the plane $F(s)$ (see *e. g.* Figs. 1/a and 2/a).

Frequently it is sufficient to restrict ourselves to the *main branch.* (Latter is established by mapping the positive imaginary axis). If the *difference* of the highest powers of the polynomials figuring in the *numerator* and *denominator*, respectively, of $F(s)$ is denoted by D_F, further the main branch



$G(s) = -K\frac{1}{s}\cdot\frac{1-0{,}01s}{1-s}\cdot\frac{1+0{,}5s}{1+0{,}05s}$ $H(s) = 1$		
	$P_G = \begin{cases}1\\2\end{cases}$ $P_H = 0$ $Z_G = 1$ $Z_H = 0$	
	$190 > K > 2{,}15$ $Z = 0$ stable	$2{,}15 > K$ $Z = 2$ unstable
$R_{-1}\{G(s)H(s)\}$	$\begin{cases}+1\\+2\end{cases}$	$\begin{cases}-1\\0\end{cases}$
$R_{-1}\left\{\frac{1}{G(s)H(s)}\right\}$	$+1$	-1
$R_0\{N(s)\}$ Q_N D_N	$+1$ $+3$ $+1$	-1 -1 $+1$
$R_0\{M(s)\}$ Q_M D_M	-1 -3 -1	1 $+1$ -1

Fig. 2a. Choice of the path C and location of the poles and zeros of $G(s)\,H(s)$
Fig. 2b. Complex-plane plots of the open-loop transfer function
Fig. 2c. Complex-plane plots of the inverse open-loop transfer function
Fig. 2d. Complex-plane of the inverse closed-loop transfer function
Fig. 2e. Complex-plane plots of the closed-loop transfer function
Fig. 2f. Main branches in the complex-plane plots of the closed-loop transfer function

of the curve $F(s)$ encircles the origin through *quadrants of number* Q_F in counter-clockwise, positive direction and considering the symmetry of curve $F(s)$ relative to the real axis, then

$$R_0\{F(s)\} = \frac{Q_F}{2} - \frac{D_F}{2} \tag{2.2}$$

(Namely, the semicircle of infinite radius of the curve C in the complex-plane s is mapped by function $F(s)$ into negatively directed semicircles of infinite radius and in number D_F, if $D_F > 0$, i. e., if the numerator of $F(s)$ is of higher power, while if $D_F < 0$, that is, if the denominator is of higher power, then positively directed semicircles are formed around the origin with infinitely small radius in number $-D_F$). In the following this theorem will be called *quadrant theorem.*

To determine the quadrants number, the following recognition may be of use: The number of the quadrants equals to the angular difference of the vectors belonging to the terminal point, and to the starting point of the curve, respectively, *expressed in right angles.*

Theorem 4. Instead of the origin, the net number of rotations around the point $-1 + j0$ may be examined, if this leads to simple results, based on the following relation

$$R_0\{ F(s) \} = R_{-1}\{ F(s) -1 \} \tag{2.3}$$

as the general course of curve $F(s)$ with respect to the origin is the same as that of curve $F(s)-1$ with respect to point $-1 + j0$.

3. Generalized stability criteria

On the basis of the above four theorems and the formulae (2.1), (2.2) (2.3), respectively, the unknown number of the right-half-plane roots of the characteristic equations may be determined for the four main cases, as follows:

a) On the basis of the *open-loop frequency response* in the complex plane and starting from function $1 + G(s)\,H(s)$ the following relation can be written:

$$Z = P_G + P_H - R_{-1}\{G(s)\ H(s)\}. \tag{3.1}$$

b) Considering the *inverse open-loop frequency response* and starting from function

$$1 + \frac{1}{G(s)\,H(s)}$$

we have the expression

$$Z = Z_G + Z_H - R_{-1}\left\{\frac{1}{G(s)\,H(s)}\right\}. \tag{3.2}$$

c) Taking into account the *inverse closed-loop frequency response,* for function $N(s)$ the following equation can be obtained

$$Z = Z_G + P_H - R_0\{N(s)\}, \tag{3.3}$$

where

$$R_0\{N(s)\} = \frac{Q_N}{2} - \frac{D_N}{2}. \tag{3.4}$$

d) Examining the *closed-loop frequency response*, for function $M(s)$ the following equation may be written:

$$Z = Z_G + P_H + R_0\{M(s)\}, \tag{3.5}$$

where

$$R_0\{M(s)\} = \frac{Q_M}{2} - \frac{D_M}{2}. \tag{3.6}$$

Naturally, always

$$R_0\{M(s)\} = -R_0\{N(s)\}. \tag{3.7}$$

The methods serving to obtain the number to be determined of the right-half-plane roots of the characteristic equation, or the stability criteria, may be summerized as follows:

a) The unknown number Z of the roots with positive real parts in the characteristic equation (the number P_M of the right-half-plane poles in the closed-loop transfer function $M(s)$ or the number of the right-half-plane zeros Z_N in the inverse closed-loop transfer function $N(s)$) is given by the difference between the number of the whole *open-loop* right-half-plane poles $P_G + P_H$ and the number of net encirclements of the plot $G(s)\,H(s)$ about the point $-1 + j0$.

Then, and only then is the closed-loop control system stable ($Z = 0$), if the open loop frequency response curve $G(s)\,H(s)$ encircles the point $-1 + j0$ exactly as many times as the number of the right-half-plane poles of the open-loop transfer function is. (Generalized NYQUIST criterion.)

b) The questionable number Z of the right-half-plane roots in the characteristic equation is given by the difference between the number of the right-half-plane zeros $Z_G + Z_H$ of the open-loop transfer function and the number of net encirclements of the *inverse* plot $1/G(s)\,H(s)$ about the point $-1 + j0$.

Then, and only then is the closed-loop control system stable, if the inverse plot $1/G(s)\,H(s)$ encircles the point $-1 + j0$ in positive direction as many times as the number of the right-half-plane zeros of the open-loop transfer function is. (Inverse NYQUIST criterion).

An example of the well-known rules not always being of common knowledge is, that in a voluminous handbook [3] the formula for determining Z, as well as the formulation of the criterion is erroneous.

c) The desired number Z of the right-half-plane roots may be obtained by subtracting the number of (positive directed) net encirclements of curve $N(s)$ about the origin from the sum of the right-half-plane zeros number Z_G of the forward branch and the right-half-plane poles number P_H of the feedback branch.

Then is the closed-loop control system stable if, and only if the inverse curve $N(s)$ of the closed-loop transfer function surrounds the origin in a positive direction as many times, as the sum of the number of right-half-plane zeros Z_G of the forward branch $G(s)$ and the number of right-half-plane poles P_H of the feedback branch $H(s)$ is.

d) The obtainable number Z of the right-half-plane roots of the characteristic equation is given by the number of the right-half-plane zeros Z_G of the forward branch, the number of the right-half-plane poles P_H of the feedback branch and the number of (positive directed) net rotations of the plot $M(s)$ around the origin.

Then is the closed-loop control system stable, if, and only if the origin is encircled in a *negative* direction by the curve $M(s)$ as many times as the sum of the number of the right-half-plane zeros Z_G of the forward branch $G(s)$ and the number of the right-half-plane poles P_H of the feedback branch $H(s)$ is.

As the plot $M(s)$ generally runs into the origin of the complex plane $M(s)$, determining $R_0\{M(s)\}$ encounters difficulties. While in the previous cases adoption of the quadrant theorems is not especially advantageous, lending itself rather for checking, now its application is advantageous and recommended.

Equations (3.5) and (3.6) yield now

$$Z = Z_G + P_H + \frac{Q_M}{2} - \frac{D_M}{2} \, . \tag{3.8}$$

Taking into consideration the quadrant theorem, the stability criterion for the closed-loop frequency response may be summerized as follows: Then, and only then is the closed-loop control system stable, if the main branch of the closed-loop frequency response $M(s)$ runs through as many quadrants into the origin in clock-wise, *negative* direction, as the difference $-D_M$ between the highest powers of polynomials $M_p(s)$ and $M_z(s)$ is, plus twice the number of the right-half-plane zeros of the forward branch and the right-half-plane poles of the feedback branch: $2(Z_G + P_H)$. Thus, the system is stable if, and only if:

$$-Q_M = -D_M + 2Z_G + 2P_H. \tag{3.9}$$

Naturally, a similar theorem may be established for the main branch of $N(s)$, nevertheless, the main branch of the plot $N(s)$ has to run in a positive direction, and accordingly for a stable system the number of quadrants is

$$Q_N = D_N + 2Z_G + 2P_H. \tag{3.10}$$

4. Complementary notes

If we want to determine the questionable number of the right-half-plane zeros (or poles) Z, based on the plot of the transfer function components $M_1(s)$, $N_1(s)$, or $M_2(s)$, $N_2(s)$, mentioned in *clause* 1., the following formulae offer themselves:

$$Z = Z_G + Z_H + R_0\{M_1(s)\} \tag{4.1}$$

or

$$Z = Z_G + Z_H - R_0\{N_1(s)\}, \tag{4.2}$$

where

$$R_0\{M_1(s)\} = \frac{Q_{M1}}{2} - \frac{D_{M1}}{2} = -\frac{Q_{N1}}{2} + \frac{D_{N1}}{2} = -R_0\{N_1(s)\}. \tag{4.3}$$

On the other hand

$$Z = P_G + P_H + R_0\{M_2(s)\} \tag{4.4}$$

or

$$Z = P_G + P_H - R_0\{N_2(s)\}, \tag{4.5}$$

where

$$R_0\{M_2(s)\} = \frac{Q_{M2}}{2} - \frac{D_{M2}}{2} = -\frac{Q_{N2}}{2} + \frac{D_{N2}}{2} = -R_0\{N_2(s)\}. \tag{4.6}$$

Starting out from the above formulae, the methods for determining the number of the right-half-plane roots of the characteristic equation, as well as the stability criteria can be stated without difficulty.

In some special cases the formulae may be brought to a more simple form. For example, if there is a unit feedback in the system: $H(s) = 1$, then the substitution $P_H = Z_H = 0$ must be made.

At that time, the formulae for determining the number Z of the right-half-plane roots of the characteristic equation are reduced to the following forms:

a) Based on the complex-plane plot of the open-loop transfer function

$$Z = P_G - R_{-1}\{G(s)\}. \tag{4.7}$$

b) Using the inverse complex-plane plot of the openloop transfer function

$$Z = Z_G - R_{-1}\left\{\frac{1}{G(s)}\right\}. \tag{4.8}$$

c) Considering the inverse complex-plane plot of the closed-loop system

$$Z = Z_G - R_0\{N(s)\} = Z_G - R_0\{N_1(s)\}. \tag{4.9}$$

d) Taking into account the complex-plane plot of the closed-loop system

$$Z = Z_G + R_0\{M(s)\} = Z_G + R_0\{M_1(s)\}. \tag{4.10}$$

Now the criteria *b)* and *c)* are of identical form, as $N(s) = 1 + \frac{1}{G(s)}$.

If the open-loop is of minimum phase and is stable in itself, then $P_G = Z_G = P_H = Z_H = 0$ is to be substituted.

Formulae for determining Z are now:

a) Considering the complex-plane plot of the openloop transfer function

$$Z = -R_{-1}\{G(s)\,H(s)\}. \tag{4.11}$$

b) On the basis of the inverse complex-plane plot of the open-loop transfer function

$$Z = -R_1\left\{\frac{1}{G(s)\,H(s)}\right\}. \tag{4.12}$$

c) Regarding the inverse complex-plane plot of the closed-loop system

$$Z = -R_0\{N(s)\} = -R_0\{N_1(s)\} = -R_0\{N_2(s)\}. \tag{4.13}$$

d) Considering the inverse complex-plane plot of the closed-loop system

$$Z = R_0\{M(s)\} = R_0\{M_1(s)\} = R_0\{M_2(s)\}. \tag{4.14}$$

In the latter case the stability criteria are as follows:

The closed-loop system is stable if, and only if *a)* the *main branch* of the plot $G(s)\,H(s)$ or *b)* of the plot $1/G(s)\,H(s)$ does not encircle the point —1 and if, and only if *c)* the *main branch* of the curve $N(s)$ does not surround the origin. (In all three cases, traversing the main branch in the direction of increasing angular frequencies, the reference points —1 and 0, respectively, are situated to the *left-side* of the curves).

Finally, if the main branch of the closed-loop frequency response $M(s)$ runs in a clock-wise, *negative* direction, the system is stable, while in the opposite case it is unstable.

It may also be added, that in the technical literature sometimes the plotting of the main branch of $N_z(s) = M_p(s)$ belonging to the characteristic equation can also be met with (*e. g.* [4]), though this method is, on account of the above-mentioned summation, rather laborious and naturally gives less data than the plot of $N(s)$ or $M(s)$. For the sake of completeness also the so-called МИХАЙЛОВ — LEONHARD criterion, relative to the main branch of $N_z(s)$ is given below:

$$2Z = D_{N_z} - Q_{N_z}, \tag{4.15}$$

where D_{N_z} now euqals the degree of the characteristic equation. If the closed-loop control system is stable, the main branch of $N_z(s)$ passes through exactly as many quadrants in the *positive* direction as the degree of the characteristic equation is. (Otherwise the quadrants deficiency gives 2Z, that is, twice the number of the right-half-plane zeros sought for).

5. Examples

For demonstrating the generalized stability criteria and illustrating the methods for determining the questionable number of the right-half-plane roots, Figs. 1 and 2 serve, respectively. In the partial figures both 1/a and 2/a, the loci of the open-loop poles and zeros, as well as the course of the path C enclosing the right-half-plane is shown. In the further partial figures mapping of contour C onto the planes $G(s)\,H(s)$ (Figs. 1/b and 2/b), $1/G(s)\,H(s)$ (Figs. 1/c and 2/c), as well as onto the planes $N(s)$ (Figs. 1/d and 2/d) and $M(s)$ (Figs. 1/e and 2/e) are demonstrated. All figures are of distorted scale, taking care of maintaining the angles rather. This is due to the fact, that when determining the net number of rotations, latter is the more important datum. As a consequence of this are the curves directed in case of $s \to +j0$ *e. g.* in figures 1/b and 2/b, towards the negative imaginary axis, while in reality a straight line parallel with the imaginary axis is the asymptote. The main branches are illustrated in all cases by a thicker line. From the curve of $M(s)$ only the main branch was plotted in Figs. 1/f and 2/f, while in the other cases the complete plots figure. Some correlated points have also been marked ($\pm j0$; $\pm j\infty$).

The transfer functions of the forward and feedback branch, as well as the number of poles and zeros of the open-loop are summarized in a separate table (Tables 1 and 2). In the same tables the net number of rotations to be determined on the basis of the different plots and the number of the quadrants in the main branches, respectively, may be found. Naturally, adopting whichever of the general stability criteria given, the results are the same. In our present example, if the system is unstable, $Z = 2$, that is, the characteristic equation has two right-half-plane roots. If $Z = 0$, the closed-loop system is stable.

As a completion it may be mentioned, that in both examples one pole of the forward branch $G(s)$ is located at $s = 0$. In figures 1/a and 2/a the by-

Table 1

Summary of the results

Transfer functions of the forward and feedback branch, resp.		$G(s)=K\dfrac{1}{s(1+s)(1+0{,}25s)}$ $H(s)=1$		Remarks	
Right-half-plane poles and zeros		$P_G=0$(or 1), $P_H=0$, $Z_G=0$, $Z_H=0$			
Range of gain factor		$K<5$	$5<K$		
Number of net encirclements or quadrants from the corresponding figures	$R_{-1}\{G(s)H(s)\}$	0(or 1)	—2(or —1)	From Fig. 1/*b*.	Use Eq. (3.1)
	$R_{-1}\left\{\dfrac{1}{G(s)H(s)}\right\}$	0	—2	From Fig. 1/*c*.	Use Eq. (3.2)
	$R_0\{N(s)\}$	0	—2	From Fig. 1/*d*.	Use Eq. (3.3) and/or Eq. (3.4)
	Q_N	+3	—1		
	D_N	+3	+3		
	$R_0\{M(s)\}$	0	+2	From Figs. 1/*e* or 1/*f*.	Use Eq. (3.5) and/or Eq. (3.6)
	Q_M	—3	+1		
	D_M	—3	—3		
Number of the right-half-plane roots of the characteristic equation	Z	0	2	From each of the above mentioned equations	
Conclusion: Closed-loop system is		stable	unstable		

pass of the pole by a semicircle of infinitely small radius may be seen by a full line (running from the right in a positive direction), as well as by a dashed line (traversing from the left in the negative direction). At that time, naturally, passing from one kind of by-pass to the other, the number of the right-half-plane poles is modified by one (see the two values P_G in Tables 1 and 2). In Figs. 1/b and 2/b the semicircles of radius tending to infinity, being oppositely directed and corresponding to the semicircles of infinitely small radius may be seen likewise as full and dashed lines.

In all cases the frequency-response curves have been plotted for two cases, in the first one the gain factor being so chosen that the closed-loop system should be stable, while in the second one so as it should be unstable.

Table 2

Summary of the results

Transfer functions of the forward and feedback branch, resp.		$G(s) =$ $-K\frac{1}{s}\cdot\frac{1-0.1s}{1-s}\cdot\frac{1+0.5s}{1+0.05s}$ $H(s) = 1$		Remarks	
Right-half-plane poles and zeros		$P_G = 1(\text{or}2), P_H = 0,$ $Z_G = 1, Z_H = 0$			
Range of gain factor		$2.15 < K$ $K < 190$	$K < 2.15$		
Number of net encirclements or quadrants from the corresponding figures	$R_{-1}\{G(s)H(s)\}$	1(or 2)	—1(or 0)	From Fig. 1/*b*.	Use Eq. (3.1)
	$R_{-1}\left\{\frac{1}{G(s)H(s)}\right\}$	+1	—1	From Fig. 1/*c*.	Use Eq. (3.2)
	$R_0\{N(s)\}$ Q_N D_N	+1 +3 +1	—1 —1 +1	From Fig. 1/*d*.	Use Eq. (3.3) and/or Eq. (3.4)
	$R_0\{M(s)\}$ Q_M D_M	—1 —3 —1	1 +1 —1	From Figs. 1/*e* or 1/*f*.	Use Eq. (3.5) and/or Eq. (3.6)
Number of the right-half-plane roots of the characteristic equation	Z	0	2	From each of the above mentioned equations	
Conclusion: Closed-loop system is		stable	unstable		

Finally mention must be made of the circumstance, that if as a result of the adjustment of the gain factor, some of the closed-loop system poles (in this case two of them) fall exactly onto the imaginary, axis, then curve $G(s)$ in Figs. 1/b and 2/b and curve $1/G(s)$ in Figs. 1/c and 2/c pass exactly through the point —1, the curve of $N(s)$ in Figs. 1/d and 2/d traverses the origin, and at that times the curve of $M(s)$ in Figs. 1/e and 1/f and 2/e and 2/f, respectively, pass through the infinite point of the complex plane.

Summary

In this paper, starting out from the argument principle and logarithmic integral theorem of complex variables and using some auxiliary theorems, the complex-plane methods and formulae for determining the number of right-half-plane roots of the characteristic equation and the stability criteria are summarized for the four main cases, that is, for the open-loop and closed-loop transfer functions (or frequency responses) and for their inverse functions (or plots).

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