SOME REMARKS CONCERNING THE STATISTICAL ANALYSIS AND SYNTHESIS OF CONTROL SYSTEMS

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1. Introduction

Analysis and synthesis of control systems used to be effected, for the sake of simplicity, assuming reference signals and disturbing signals, respectively, of predetermined regular course. Such typical signals may be, for instance, the impulse function, the step function, as well as the ramp function, or the sinusoidal signals. The input signals of control systems, however, show in reality irregular variations, and may be described statistically only. Also, for these so called stochastic signals the investigation of the system may be realized, though with certain limitations. The aim of present paper is to summarize the most important fundamental conceptions and to bring the mathematical theorems and relations close to the engineering practice, neglecting the deduction of the formulas.

2. Fundamental relations

In the following only stationary random processes are dealt with. A process is called stationary, if its statistical characteristics (probability-distribution, or probability-density functions) are independent of the time origin, that is, the statistical character of the random process does not change with time. For stationary processes the ergodic hypothesis is accepted; the time average and the ensemble average are equal to one another:

$$\overline{x(t)} = \widetilde{x(t)} \tag{1a}$$

where

$$\overline{x(t)} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, dt \tag{1b}$$

and

$$\widetilde{x(t)} = \frac{1}{N} \sum_{\nu=1}^{N} x_\nu(t_0) \tag{1c}$$

In other words, it is assumed, that when forming the average value, the same result is obtained, either by studying a single process at many different instants, or numerous processes are observed at the same instant t_0.

3. Function transforms

In control engineering describing the transient responses for typical signals as zero for negative time, that is, for determining the dynamic behaviour, generally, the Laplace transform (further on $\mathscr{L}$-transform) is applied first of all. The $\mathscr{L}$-transform and the inverse transform of a certain so-called positive-time function $f_+(t)$ yield

$$F_+(s) = \mathscr{L}[f_+(t)] = \int_0^\infty f_+(t)\, e^{-st}\, dt \tag{2a}$$

$$f_+(t) = \mathscr{L}^{-1}[F_+(s)] = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F_+(s)\, e^{ts}\, ds \tag{2b}$$

where

$$s = \sigma + j\omega\,; \qquad 0 < t \tag{2c}$$

and c must be so chosen, that in the complex plane the integration path should be on the right side of all of the poles. When $f_+(t) \to 0$ if $t \to \infty$, then function $F_+(s)$ has only left-half-plane poles.

To study the stochastic signals, the Fourier transform (in the following $\mathscr{F}$-transform) may be adopted:

$$F(j\omega) = \mathscr{F}[f(t)] = \int_{-\infty}^\infty f(t)\, e^{-j\omega t}\, dt \tag{3a}$$

$$f(t) = \mathscr{F}^{-1}[F(j\omega)] = \frac{1}{2\pi} \int_{-\infty}^\infty F(j\omega)\, e^{j\omega t}\, d\omega \tag{3b}$$

Recently also the two-sided Laplace transform (in the following $\mathscr{T}$-transform) is frequently applied.

$$F(s) = \mathscr{T}[f(t)] = \int_{-\infty}^\infty f(t)\, e^{-st}\, dt \tag{4a}$$

$$f(t) = \mathscr{T}^{-1}[F(s)] = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F(s)\, e^{ts}\, ds \tag{4b}$$

where

$$s = \sigma + j\omega\,; \qquad -\infty < t < \infty \tag{4c}$$

Here the function $f(t)$ consists of two components: of a positive-time function and of a negative-time function

$$f(t) = f_-(t) + f_+(t) \tag{5a}$$

where

$$f_-(t) = \begin{cases} f(t) & t < 0 \\ 0 & 0 < t \end{cases} ; \quad f_+(t) = \begin{cases} 0 & t < 0 \\ f(t) & 0 < t \end{cases} \tag{5b,c}$$

According to this also the $\mathcal{T}$-transform may be resolved into two parts:

$$F(s) = F_-(s) + F_+(s) \tag{6a}$$

where

$$F_-(s) = \int_{-\infty}^{0} f_-(t)\, e^{-st}\, dt \tag{6b}$$

$$F_+(s) = \int_{0}^{\infty} f_+(t)\, e^{-st}\, dt \tag{6c}$$

Here, in case of a limited $f(t)$ function $F_-(s)$ has only right-half-plane poles, while function $F_+(s)$ only left-half-plane ones (c must be chosen so that the integration path should fall between the left-half-plane and right-half-plane poles). As is to be seen, $F_+(s)$ is the common $\mathcal{L}$-transform, while calculation of $F_-(s)$ may be performed by substitution $t = -\vartheta$ (if $t < 0$, then $0 < \vartheta$):

$$F_-(s) = \int_{\infty}^{0} f_-(-\vartheta)\, e^{s\vartheta}\,(-d\vartheta) = \int_{0}^{\infty} f_-(-\vartheta)\, e^{-(-s)\vartheta}\, d\vartheta \tag{6d}$$

Thus, this latter function may be reduced to the common $\mathcal{L}$-transform. Its calculation is effected as follows: *a)* by reflexion (substituting $t = -\vartheta$), from the negative time function $f_-(t)$ a positive time function $f_-(-\vartheta)$ is established; *b)* determining the $\mathcal{L}$-transform; *c)* substituting the value $-s$ instead of s. The $\mathcal{T}$-transform is of the most general form among the above-mentioned three integral transforms. On the one hand, it turns in case of $f(t) = f_+(t)$ to the $\mathcal{L}$-transform, and, on the other hand, substituting $s = j\omega$ the $\mathcal{F}$-transform may be obtained. Consequently, the $\mathcal{T}$-transform may be adopted for representing also the $\mathcal{F}$-transform, with $c = 0$ and introducing $j\omega = s$.

Concerning the inverse transformations: the inverse of the $\mathcal{T}$-transform may be performed instead of calculating the complex integral generally *a)* by resolution into partial fractions and applying tables, *b)* by the expansion theorem, *c)* by aid of the CAUCHY residue theorem. From the latter, in case of single poles, the time functions are

$$f_+(t) = \sum_{\nu} \lim_{s \to p_\nu} \left\{(s - p_\nu)\, F(s)\, e^{ts}\right\}; \quad 0 < t \tag{7a}$$

$$f_-(t) = -\sum_{\mu} \lim_{s \to p_\mu} \left\{(s - p_\mu)\, F(s)\, e^{ts}\right\}; \quad t < 0 \tag{7b}$$

while in case of multiple poles

$$f_{+}(t) = \sum_{\nu} \lim_{s \to p_\nu} \left\{ \frac{1}{(n-1)!} \frac{d^{n-1}}{ds^{n-1}} \left[(s-p_\nu)^n F(s)\, e^{ts} \right] \right\}; \quad 0 < t \tag{8a}$$

$$f_{-}(t) = - \sum_{\mu} \lim_{s \to p_\mu} \left\{ \frac{1}{(n-1)!} \frac{d^{n-1}}{ds^{n-1}} \left[(s-p_\mu)^n F(s)\, e^{ts} \right] \right\}; \quad t < 0 \tag{8b}$$

and finally

$$f(t) = f_{-}(t) + f_{+}(t) \tag{5a}$$

Here p_ν means a left-half-plane-pole (of negative real part), while p_μ a right-half-plane pole (of positive real part)*

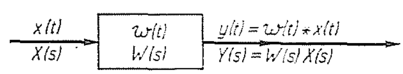

Fig. 1

The single-sided $\mathscr{L}$-transform may be calculated by the same formulas, at these times, however, always $f_{-}(t) = 0$ and $f(t) = f_{+}(t)$. If there is a right-half-plane pole, this must be arranged to the left-half-plane ones p_ν, nevertheless, at these times the time function increases beyond all limits. The inverse $\mathscr{F}$-transform may also be formed with the same formulae, after substituting $s = j\omega$.

An important feature of the $\mathscr{L}$-, $\mathscr{F}$-, $\mathscr{T}$-transforms enumerated above is, that they transform the differential equations into algebraic ones, while the superposition (convolution, faltung) integral referring to the time functions

$$y(t) = \int_{-\infty}^{\infty} w(\vartheta)\, x(t-\vartheta)\, d\vartheta = \int_{-\infty}^{\infty} w(t-\vartheta)\, x(\vartheta)\, d\vartheta \tag{9a}$$

is transformed into a simple multiplication (Fig. 1):

$$Y(s) = W(s)\, X(s) \tag{9b}$$

provided, that $X(s)$ and $Y(s)$ exist.

* If $f(t) = f(t+T)$, then $f_{(-)}(t) = f_{-}(t+T)$ is meant within the range $t < -T$, while $f_{(+)}(t) = f_{+}(t+T)$ in the range $-T < t$! The functions $f_{(-)}(t)$ and $f_{(+)}(t)$ are pseudo-negative-time functions and pseudo-positive-time function, respectivelly, because their point of separation is $-T$ instead of 0.

If the input signal is a Dirac unit impulse, $x(t) = \delta(t)$, then the output signal is the weighting function itself

$$y(t) = \int_{-\infty}^{\infty} w(t - \vartheta)\, \delta(\vartheta)\, d\vartheta = w(t) \tag{10a}$$

$$Y(s) = W(s) \cdot 1 = W(s) \tag{10b}$$

4. Correlation functions

Two kinds of correlation functions may be distinguished: cross-correlation and auto-correlation functions. For stationary processes, by means of time average, definition of the cross-correlation function of signals $x(t)$ and $y(t)$ is:

$$\varphi_{xy}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, y(t + \tau)\, dt \tag{11a}$$

and

$$\varphi_{yx}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} y(t)\, x(t + \tau)\, dt \tag{11b}$$

respectively.

The auto-correlation function of signal $x(t)$ may be obtained by substituting into the cross-correlation function $y(t) = x(t)$

$$\varphi_{xx}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, x(t + \tau)\, dt \tag{12}$$

Determination of the correlation function is illustrated by Fig. 2.

The main features of the auto-correlation function are as follows:

a) it is an even function $\varphi_{xx}(\tau) = \varphi_{xx}(-\tau)$;

b) its maximum is attained at zero shift parameter;

$$|\varphi_{xx}(\tau)| \leq \varphi_{xx}(0)$$

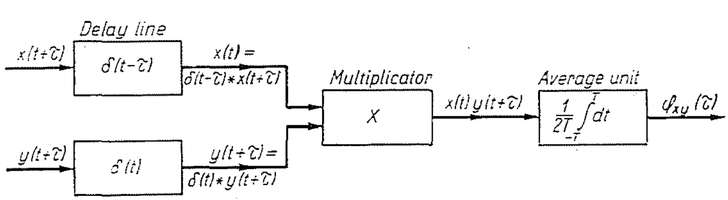

Fig. 2

c) latter expression furnishes the mean-square value of the signal $\varphi_{xx}(0) = \overline{x^2(t)}$;

d) for large shifting times $\varphi_{xx}(\tau) = \overline{x(t)}^2$ as generally $\overline{x(t)} = 0$, consequently $\lim\limits_{\tau\to\infty} \varphi_{xx}(\tau) = 0$.

If the stochastic signal includes a periodic component

$$x(t) = a(t) + b\cos(\omega t + \theta) \tag{13a}$$

then

$$\varphi_{xx}(\tau) = \varphi_{aa}(\tau) + \frac{b}{2}\cos\omega\tau \tag{13b}$$

Presence of the periodic sinusoidal signal-component may be gathered from the behaviour relating to large values of τ, for

$$\lim_{T\to\infty} \varphi_{aa}(\tau) = \overline{a^2(t)} = \text{const} \tag{13c}$$

while the second term of Eq. (13b) varies periodically also for large values of τ.

Features of the cross-correlation function are

a) it is not an even function, but

$$\varphi_{xy}(\tau) = \varphi_{yx}(-\tau)$$

so the two cross-correlation functions are reflexions of each other (index-changing rule);

b) $|\varphi_{xy}(\tau)| \leq \sqrt{\varphi_{xx}(0)\,\varphi_{yy}(0)}$;
c) $\varphi_{xy}(0) = \overline{x(t)\,y(t)}$;
d) $\lim\limits_{\tau\to\infty}\varphi_{xy}(\tau) = \overline{x(t)}\;\overline{y(t)}$ as a rule $= 0$.

Finally it must be mentioned, that no unique correspondence exists between the signals and correlation functions, the same correlation function may belong to different signals.

5. Power-density spectra

Though the stochastic signals have no $\mathcal{F}$-transform, their correlation functions have. Under the power-density spectra (sometimes briefly: power spectra) of the signal, $\mathcal{F}$-transforms of the correlation functions are understood. Consequently the relation between the auto-correlation function and the corresponding power-density spectrum is as follows

$$\Phi_{xx}(j\omega) = \int_{-\infty}^{\infty} \varphi_{xx}(\tau)\, e^{-j\omega\tau}\, d\tau \tag{14a}$$

$$\varphi_{xx}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_{xx}(j\omega)\, e^{j\omega\tau}\, d\omega \tag{14b}$$

the relation between the cross-correlation function and the power-density spectrum may be expressed by

$$\Phi_{xy}(j\omega) = \int_{-\infty}^{\infty} \varphi_{xy}(\tau)\, e^{-j\omega\tau}\, d\tau \tag{15a}$$

$$\varphi_{xy}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_{xy}(j\omega)\, e^{j\omega\tau}\, d\omega \tag{15b}$$

These are the so-called WIENER—HINTSHIN relations. Also the forms by the substitution $s = j\omega$ are in general use:

$$\Phi_{xx}(s) = \int_{-\infty}^{\infty} \varphi_{xx}(\tau)\, e^{-s\tau}\, d\tau \tag{16a}$$

$$\varphi_{xx}(\tau) = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \Phi_{xx}(s)\, e^{\tau s}\, ds \tag{16b}$$

and

$$\Phi_{xy}(s) = \int_{-\infty}^{\infty} \varphi_{xx}(\tau)\, e^{-s\tau}\, d\tau \tag{17a}$$

$$\varphi_{xx}(\tau) = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \Phi_{xy}(s)\, e^{\tau s}\, ds \tag{17b}$$

Features of the power-density spectra are:

a) $\Phi_{xx}(s) = \Phi_{xx}(j\omega)$ is an even function of s or ω, so it is a real function, consequently $\Phi_{xx}(s) = \Phi_{xx}(j\omega) = \Phi^{*}_{xx}(j\omega) = \Phi_{xx}(-j\omega) = \Phi_{xx}(-s)$;

b) $\Phi_{xy}(s) = \Phi_{xy}(j\omega)$ is generally not an even function, so it is a complex variable function, but $\Phi_{xy}(s) = \Phi_{xy}(j\omega) = \Phi^{*}_{yx}(j\omega) = \Phi_{yx}(-j\omega) = \Phi_{yx}(-s)$ (index changing rule).

On the basis of the former definitions, after substituting $\tau = 0$, the following significant relations may be obtained:

$$\overline{x^2(t)} = \varphi_{xx}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_{xx}(j\omega)\, d\omega = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \Phi_{xx}(s)\, ds \tag{18}$$

$$\overline{x(t)\, y(t)} = \varphi_{xy}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_{xy}(j\omega)\, d\omega = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \Phi_{xy}(s)\, ds \tag{19}$$

Consequently the mean-square value and the mean-product value may easily be evaluated from the integrals of the power-density spectra.

6. Examination of linear systems' behaviour

The relation between the input signal $x(t)$ and output signal $y(t)$ of a linear system with a given weighting function $w(t)$ is described by the superposition integral ($9a$). This is valid both for deterministic signals and for stochastic ones. The transformed form ($9b$) is valid for nothing but the deterministic signals, as the stochastic signals have no transforms.

With the aid of the correlation functions and the power-density spectra the behaviour of the linear systems may be characterised, however, not only in the time domain, but also in the frequency domain.

The following fundamental relations may be stated (Fig. 3). First:

$$\varphi_{xy}(\tau) = \int_{-\infty}^{\infty} w(\vartheta)\, \varphi_{xx}(\tau - \vartheta)\, d\vartheta \tag{20a}$$

$$\Phi_{xy}(s) = W(s)\, \Phi_{xx}(s) \tag{20b}$$

Second:

$$\varphi_{yy}(\tau) = \int_{-\infty}^{\infty} w(\vartheta)\, \varphi_{yx}(\tau - \vartheta)\, d\tau \tag{21a}$$

$$\Phi_{yy}(s) = W(s)\, \Phi_{yx}(s) \tag{21b}$$

Third, considering the features of the respective functions

$$\varphi_{yx}(\tau) = \int_{-\infty}^{\infty} w(\zeta)\, \varphi_{xx}(\tau + \zeta)\, d\zeta = \int_{-\infty}^{\infty} w(\zeta)\, \varphi_{xx}(-\tau - \zeta)\, d\zeta = \varphi_{xy}(-\tau) \tag{22a}$$

$$\Phi_{yx}(s) = W(-s)\, \Phi_{xx}(s) = W(-s)\, \Phi_{xx}(-s) = \Phi_{xy}(-s) \tag{22b}$$

Finally, in the fourth place, on the basis of the second and third relation-couples:

$$\varphi_{yy}(\tau) = \int_{-\infty}^{\infty} w(\vartheta) \int_{-\infty}^{\infty} w(\zeta)\, \varphi_{xx}(\tau - \vartheta + \zeta)\, d\zeta\, d\vartheta \tag{23a}$$

$$\Phi_{yy}(s) = W(s)\, W(-s)\, \Phi_{xx}(s) = |W(s)|^2\, \Phi_{xx}(s) \tag{23b}$$

or

$$\Phi_{yy}(j\omega) = W(j\omega)\, W^*(j\omega)\, \Phi_{xx}(j\omega) = |W(j\omega)|^2\, \Phi_{xx}(j\omega). \tag{23c}$$

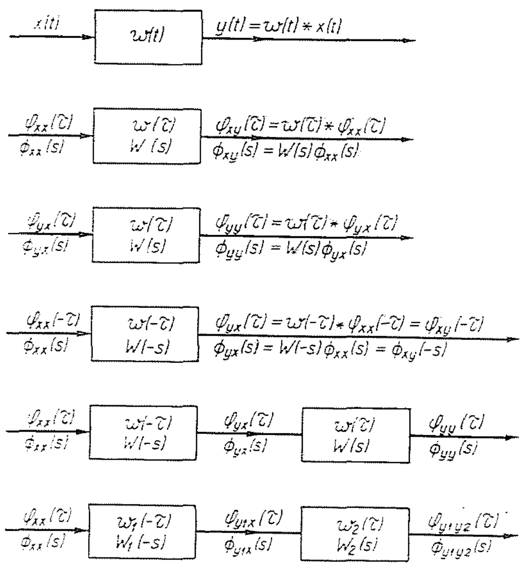

Fig. 3

As a general rule it may be stated, that the second index of a certain power-density spectrum may be changed by the multiplication of the transfer function, while its first index by the multiplication of the conjugate of the transfer function. So for example:

$$\Phi_{y_1 y_2}(s) = W_1(-s)\, W_2(s)\, \Phi_{xx}(s) \tag{24}$$

The same refers to the correlation function, where the second index may be changed by a common convolution, while the first by a reflected convolution.

7. Some practical remarks on the analysis of the systems

From a theoretical point of view the so-called "white" noise is of great importance. The stochastic signal is called a "white noise", if the components of arbitrary frequency are figuring with the same, *e. g.* unit amplitude. In other words, the power-density spectrum of the "white noise" is constant. The auto-correlation function of the "white noise" is the (unit) impulse function:

$$\varphi_{xx}(\tau) = \delta(\tau) \qquad \Phi_{xx}(s) = \Phi(j\omega) = 1 \tag{25}$$

The "white noise" cannot be physically realized, as an infinite power (or mean-square value) would belong to it (*see* Formula 18). In the practical realization of the noise generators an approximately constant power spectrum is to be aimed at, at least up to a certain critical frequency.

It must be noted, that from the "white noise" any desired power-density spectrum may be established with the aid of a filter of suitable transfer function $W(s)$ (see Eq. 23), consequently any required correlation function may also be obtained.

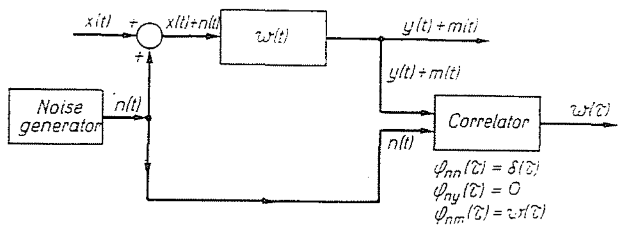

Fig. 4

On the other hand, the correlation functions and the power-density spectra permit the determination of the transfer function of complete control systems, or the transfer functions of some elements. For this purpose merely the auto-correlation function of the input (or output) signal, and one of the cross-correlation functions, (*i. e.* the respective power-density spectra) must be determined by measurement or calculation, starting from formulae (20), or (21), whereafter the desired weighting function (or transfer function) may easily be obtained. The great advantage of the above method is, that it may also be realized in working systems. [5] The task becomes especially simple, if the input signal is a "white noise", as in this case, according to Formula (20) the weighting function itself is provided by the cross-correlation function, while the transfer function is obtained by the power-density spectrum.

$$\varphi_{xy}(\tau) = \int_{-\infty}^{\infty} w(\vartheta)\, \delta(\tau - \vartheta)\, d\vartheta = w(\tau) \tag{26a}$$

$$\Phi_{xy}(s) = W(s) \cdot 1 = W(s) \tag{26b}$$

By measuring with an actual noise generator, efforts must be made in order that the noise generator critical frequency should be higher than that of the system to be investigated. The measurement may be carried out also in a living system (*see* Fig. 4).

The advantage of the above method is, that it is insensitive to disturbances appearing at different points of the system, further it may also be applied for multipole systems, that is, for many input and output signals and for clearing up the relations between the individual input and output signals.

8. Synthesis of control systems

The fundamental principles of the control system synthesis based on stochastic signals, in other words, the fundamental principles of statistical design are laid down in the KOLMOGOROF—WIENER theorem. In this theorem the question is raised, how to choose the dynamic characteristics (weighting function, transfer function) of the control system in such a way that the deviation, or error, respectively, between the actual output signal and the desired output signal should be a minimum. As a consequence of this the statistical design is not a unique task, its result depends on three factors: *a)* on the statistical characteristics of the input signal; *b)* on the statistical characteristics of the desired output signal; *c)* on the criterion of the minimum error. If whichever of the three factors changes, another result is ob-

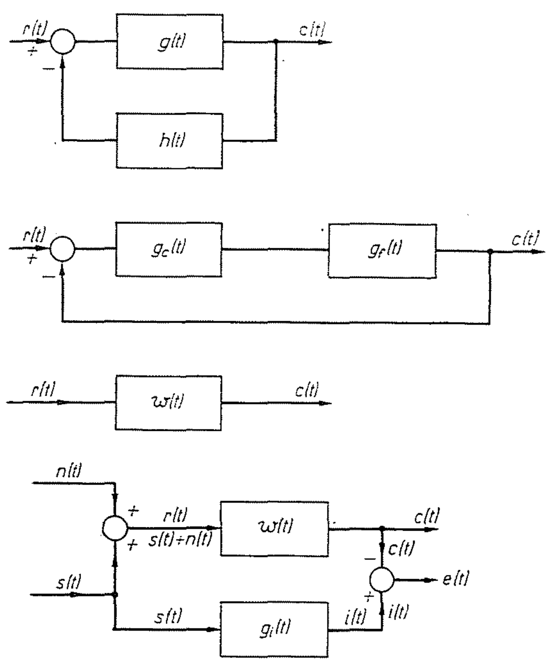

Fig. 5

tained for the optimum weighting function. The optimization is generally realized on the basis of the minimum mean-square error criterion, not that this would give the best result, but because this task may be solved the most easily mathematically.

The task of the statistical design for continuously operating linear control systems affected by stationary stochastic signals may be formulated as follows (Fig. 5). How to determine the resultant weighting function $w(t)$ and the resultant transfer function $W(s)$, respectively, of the closed-loop control system, in order that the $\varepsilon(t)$ mean-square value of the error signal, that is, the difference between the ideal output signal $i(t)$ and the actual output signal $c(t)$ should be a minimum:

$$w(t) = ? \qquad W(s) = ? \qquad \text{in order that} \qquad \overline{\varepsilon^2(t)} = \text{Min.}$$

It is worth mentioning, that the stochastic input signal $r(t)$ may include, in addition to the signal component $s(t)$ also the undesirable noise component $n(t)$. Finally relation may often be established between the ideal output signal $i(t)$ and the input signal component $s(t)$ through the ideal weighting function $g_i(t)$.

The task of optimization may obviously refer to the determination of the physically realizable weighting function $w(t)$ only. The condition of the physical realization is, that the weighting function must be zero for negative time (consequently, the weighting function may include a positive-time function component only), as the effect can never precede the cause. As the ideal output signal $i(t)$ does not exist in reality, it being merely a creature of our imagination, the ideal weighting function $g_i(t)$ must not be a physically realizable one.

In the knowledge of the weighting function $w(t)$ and the transfer function $W(s)$, respectively, design of the control system individual parts [Fig.5 $G_c(s)$ or $H(s)$] means no special problem.

9. The main relations of the statistical design

According to Fig. 5. the error signal is

$$\varepsilon(t) = i(t) - c(t) \tag{27}$$

consequently the mean-square error to be minimized is

$$\overline{\varepsilon^2(t)} = \overline{[i(t) - c(t)]^2} \tag{28}$$

Adopting the definitions of the correlation functions, latter may be formulated as follows:

$$\overline{\varepsilon^2(t)} = \varphi_{ii}(0) - 2\int\limits_{-\infty}^{\infty} w(\vartheta)\,\varphi_{ri}(\vartheta)\,d\vartheta + \int\limits_{-\infty}^{\infty} w(\vartheta)\int\limits_{-\infty}^{\infty} w(\zeta)\,\varphi_{rr}(\vartheta-\zeta)\,d\zeta\,d\vartheta \qquad (29)$$

By the calculus of variations, substituting $w(t) = w_\sigma(t) + \lambda\, w_\lambda(t)$ and differentiating according to λ and by the substitution $d\overline{\varepsilon^2}/d\lambda = 0$, for the minimalizing optimum weighting function $w_\sigma(t)$ the following integral equation may be obtained:

$$\int\limits_{-\infty}^{\infty} w_\sigma(\zeta)\,\varphi_{rr}(\vartheta-\zeta)\,d\zeta - \varphi_{ri}(\vartheta) = 0\,; \qquad 0 \leqq \vartheta \qquad (30)$$

This is the WIENER-HOPF integral equation. If it were valid not only for $0 \leqq \vartheta$, but for all values, then it could be solved by $\mathscr{F}$-transforms.

For negative times the left-side term is not zero, but equals a certain unknown negative-time function $f_-(\vartheta)$ ($f_-(\vartheta) = 0,\ 0 < \vartheta$).

$$\int\limits_{-\infty}^{\infty} w_\sigma(\zeta)\,\varphi_{rr}(\vartheta-\zeta)\,d\zeta - \varphi_{ri}(\vartheta) = f_-(\vartheta)\,; \qquad \vartheta < 0 \qquad (31)$$

Summing equations (30) and (31)

$$\int\limits_{-\infty}^{\infty} w_\sigma(\zeta)\,\varphi_{rr}(\vartheta-\zeta)\,d\zeta - \varphi_{ri}(\vartheta) = f_-(\vartheta)\,; \qquad -\infty < \vartheta < \infty \qquad (32)$$

This integral equation can already be solved with the $\mathscr{F}$-transform (provided the correlation functions have power-density spectra):

$$W_\sigma(s)\,\Phi_{rr}(s) - \Phi_{ri}(s) = F_-(s) \qquad (33)$$

Here function $F_-(s)$ has only right-half-plane poles, while the other functions have poles in the entire complex plane.

Let us resolve the power-density spectrum $\Phi_{rr}(s)$ into two factors:

$$\Phi_{rr}(s) = \Phi_{rr}^+(s)\,\Phi_{rr}^-(s) \qquad (34)$$

where $\Phi_{rr}^+(s)$ contains all left-half-plane zeros and poles of $\Phi_{rr}(s)$, while $\Phi_{rr}^-(s)$ all of those of the right-half-plane.

To avoid misunderstandings it must be noted, that

$$\varphi_{rr}(\tau) = \varphi_{rr-}(\tau) + \varphi_{rr+}(\tau)$$

its $\mathcal{F}$-transform would be

$$\Phi_{rr}(s) = \Phi_{rr-}(s) + \Phi_{rr+}(s)$$

and generally

$$\Phi_{rr}^{+}(s) \neq \Phi_{rr+}(s); \quad \Phi_{rr}^{-}(s) \neq \Phi_{rr-}(s).$$

Substituting relation (34) into equation (33), after division and re-arrangement

$$W_{\sigma}(s)\,\Phi_{rr}^{+}(s) = \frac{\Phi_{ri}(s)}{\Phi_{rr}^{-}(s)} + \frac{F_{-}(s)}{\Phi_{rr}^{-}(s)} \tag{35}$$

At the right side the second term has exclusively right-half-plane poles, as $F_{-}(s)$ is a transform belonging to a negative-time function, while the component $\Phi_{rr}^{-}(s)$ has only right-half-plane zeros. Consequently

$$\frac{F_{-}(s)}{\Phi_{rr}^{-}(s)} = \left[\frac{F_{-}(s)}{\Phi_{rr}^{-}(s)}\right]_{-} \tag{36}$$

Also the first term may be resolved into components:

$$\frac{\Phi_{ri}(s)}{\Phi_{rr}^{-}(s)} = \left[\frac{\Phi_{ri}(s)}{\Phi_{rr}^{-}(s)}\right]_{-} + \left[\frac{\Phi_{ri}(s)}{\Phi_{rr}^{-}(s)}\right]_{+} \tag{37}$$

Thus, taking Eqs. (36) and (37) into consideration, starting from Eq. (35)

$$W_{\sigma}(s)\,\Phi_{rr}^{+}(s) = \left[\frac{\Phi_{ri}(s)}{\Phi_{rr}^{-}(s)} + \frac{F_{-}(s)}{\Phi_{rr}^{-}(s)}\right]_{-} + \left[\frac{\Phi_{ri}(s)}{\Phi_{rr}^{-}(s)}\right]_{+} \tag{38}$$

It is evident from Eq. (38), that the optimalizing transfer function may not be a physically realizable one, as a rule, as at the right side of (38) there are generally figuring left-half-plane, as well as right-half-plane poles. If we restrict ourselves to the transfer function $W_m(s)$ that may physically be realized, then at the right side of Eq. (38) only the part furnishing the left-half-plane poles must be considered, so

$$W_m(s)\,\Phi_{rr}^{+}(s) = \left[\frac{\Phi_{ri}(s)}{\Phi_{rr}^{-}(s)}\right]_{+} \tag{39}$$

According to this, the physically realizable optimum transfer function is

$$W_m(s) = \frac{1}{\Phi_{rr}^{+}(s)}\left[\frac{\Phi_{ri}(s)}{\Phi_{rr}^{-}(s)}\right]_{+} \tag{40}$$

Calculation of the right-side terms of equation (39) may be effected generally on the basis of the instructions given below:

$$\left[\frac{\Phi_{ri}(s)}{\Phi_{rr}^{-}(s)}\right]_{+} = \mathcal{L}\left\{\mathcal{F}^{-1}\left\{\frac{\Phi_{ri}(s)}{\Phi_{rr}^{-}(s)}\right\}\right\} \tag{41}$$

Often, when the part of Eq. (39) in square brackets contains a rational fractional function only, our aim may be achieved more quickly by resolving into simple partial fractions and omitting the terms containing the right-half-plane poles.

The course of determining the physically realizable optimum transfer function $W_m(s)$ is illustrated in Fig. 6. for a general case.

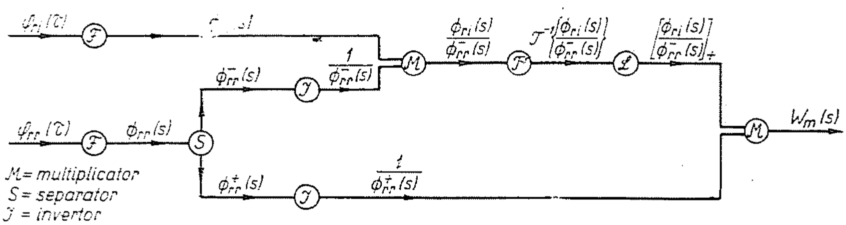

Fig. 6

In Table 1 the formulae, that may be derived from the above general formula and those serving for the determination of the optimum transfer function $W_m(s)$ are summarized for some special cases. It is to be remarked that in the formulae the power-density spectra of the input signal and of the noise, as well as the ideal transfer function $G_i(s)$ are figuring only.

10. The value of the mean-square error

Knowing the optimum weighting function and the transfer function, respectively, the mean-square value of the error may also be determined. Again restricting ourselves to the system shown in Fig. 5, the mean-square error is

$$\overline{\varepsilon^2(t)} = \varphi_{\varepsilon\varepsilon}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi_{\varepsilon\varepsilon}(j\omega)\,d\omega = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\Phi_{\varepsilon\varepsilon}(s)\,ds \tag{42}$$

where

$$\varphi_{\varepsilon\varepsilon}(\tau) = \varphi_{ii}(\tau) - \varphi_{ic}(\tau) - \varphi_{ci}(\tau) + \varphi_{cc}(\tau), \tag{43}$$

and

$$\Phi_{\varepsilon\varepsilon}(s) = \Phi_{ii}(s) - \Phi_{ic}(s) - \Phi_{ci}(s) + \Phi_{cc}(s) \tag{44}$$

respectively.

Latter may be written, considering the index changing rules, as follows

$$\Phi_{\varepsilon\varepsilon}(s) = G_i(-s)\, G_i(s)\, \Phi_{ss}(s) - G_i(-s)\, W_m(s)\, \Phi_{sr}(s) - \\ - W_m(-s)\, G_i(s)\, \Phi_{rs}(s) + W_m(-s)\, W_m(s)\, \Phi_{rr}(s) \tag{45}$$

which may be formulated, taking into account

$$\Phi_{rs}(s) = \Phi_{ss}(s) + \Phi_{ns}(s)$$

also in this way:

$$\Phi_{\varepsilon\varepsilon}(s) = [G_i(-s)\, G(s) - G_i(-s)\, W_m(s) - W_m(-s)\, G_i(s) + \\ + W_m(-s)\, W_m(s)]\, \Phi_{ss}(s) + [W_m(-s)\, W_m(s) - \\ - W_m(-s)\, G_i(s)]\, \Phi_{ns}(s) + [W_m(-s)\, W_m(s) - \\ - G_i(-s)\, W_m(s)]\, \Phi_{sn}(s) + [W_m(-s)]\, W_m(s)\, \Phi_{nn}(s) \tag{46}$$

Introducing the notation

$$W_\varepsilon(s) = G_i(s) - W_m(s) \tag{47}$$

we get

$$\Phi_{\varepsilon\varepsilon}(s) = W_\varepsilon(-s)\, W_\varepsilon(s)\, \Phi_{ss}(s) - \\ - W_m(-s)\, W_\varepsilon(s)\, \Phi_{ns}(s) - \\ - W_\varepsilon(-s)\, W_m(s)\, \Phi_{sn}(s) + \\ + W_m(-s)\, W_m(s)\, \Phi_{nn}(s) \tag{48}$$

In case of uncorrelated signal and noise component

$$\Phi_{\varepsilon\varepsilon}(s) = |W_{\varepsilon\varepsilon}(s)|^2\, \Phi_{ss}(s) + |W_m(s)|^2\, \Phi_{nn}(s) = \\ = |G_i(s) - W_m(s)|^2\, \Phi_{ss}(s) + |W_m(s)|^2\, \Phi_{nn}(s) \tag{49}$$

For a noise-free system, however,

$$\Phi_{\varepsilon\varepsilon}(s) = |W_\varepsilon(s)|^2\, \Phi_{ss}(s) = |G_i(s) - W_m(s)|^2\, \Phi_{ss}(s) \tag{50}$$

Knowing the power-density spectrum of the error, the mean-square error may relatively easily be determined, as $\Phi_{\varepsilon\varepsilon}(s)$ being an even function, can be formulated in the following way

$$\overline{\varepsilon^2(t)} = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \Phi_{\varepsilon\varepsilon}(s)\, ds = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \frac{C(s)\, C(-s)}{D(s)\, D(-s)}\, ds \tag{51}$$

and for evaluating the latter integral, ready formulae are at our disposal *e. g.* [7,9].

Table I

Special formulae* for determining $W_m(s)$

	Correlated signal and noise $\Phi_{sn}(s) = \Phi_{ns}(s) \neq 0$	Un-correlated signal and noise $\Phi_{ns}(s) = \Phi_{sn}(-s) = 0$	Noise-free system $n(t) = 0\ ;\ \Phi_{nn}(s) = 0$
General case $g_i(t)$ $G_i(s)$	$\dfrac{\left[\dfrac{(\Phi_{ss} + \Phi_{ns})\, G_i}{(\Phi_{ss} + \Phi_{sn} + \Phi_{ns} + \Phi_{nn})^-}\right]_+}{(\Phi_{ss} + \Phi_{sn} + \Phi_{ns} + \Phi_{nn})^+}$	$\dfrac{\left[\dfrac{\Phi_{ss}\, G_i}{(\Phi_{ss} + \Phi_{nn})^-}\right]_+}{(\Phi_{ss} + \Phi_{nn})^+}$	$\dfrac{\left[\dfrac{\Phi_{ss}\, G_i}{\Phi_{ss}^-}\right]_+}{\Phi_{ss}^+} = \dfrac{[\Phi_{ss}^+\, G_i]_+}{\Phi_{ss}^+}$
Prediction $g_i(t) = \delta\,(t + T)$ $G_i(s) = e^{sT}$	$\dfrac{\left[\dfrac{(\Phi_{ss} + \Phi_{ns})\, e^{sT}}{(\Phi_{ss} + \Phi_{sn} + \Phi_{ns} + \Phi_{nn})^-}\right]_+}{(\Phi_{ss} + \Phi_{sn} + \Phi_{ns} + \Phi_{nn})^+}$	$\dfrac{\left[\dfrac{\Phi_{ss}\, e^{sT}}{(\Phi_{ss} + \Phi_{nn})^-}\right]_+}{(\Phi_{ss} + \Phi_{nn})^+}$	$\dfrac{[\Phi_{ss}^+\, e^{sT}]_+}{\Phi_{ss}^+}$
High-fidelity reconstruction $g_i(t) = \delta(t)$ $G_i(s) = 1$	$\dfrac{\left[\dfrac{\Phi_{ss} + \Phi_{ns}}{(\Phi_{ss} + \Phi_{sn} + \Phi_{ns} + \Phi_{nn})^-}\right]_+}{(\Phi_{ss} + \Phi_{sn} + \Phi_{ns} + \Phi_{nn})^+}$	$\dfrac{\left[\dfrac{\Phi_{ss}}{(\Phi_{ss} + \Phi_{nn})^-}\right]_+}{(\Phi_{ss} + \Phi_{nn})^+}$	1
Signal reconstruction with delay time $g_i(t) = \delta(t - T)$ $G_i(s) = e^{-sT}$	$\dfrac{\left[\dfrac{(\Phi_{ss} + \Phi_{ns})\, e^{-sT}}{(\Phi_{ss} + \Phi_{sn} + \Phi_{ns} + \Phi_{nn})^-}\right]_+}{(\Phi_{ss} + \Phi_{sn} + \Phi_{ns} + \Phi_{nn})^+}$	$\dfrac{\left[\dfrac{\Phi_{ss}\, e^{-sT}}{(\Phi_{ss} + \Phi_{nn})^-}\right]_+}{(\Phi_{ss} + \Phi_{nn})^+}$	e^{-sT}

For the sake of simplicity in the power-density spectra and in the transfer functions the independent variable s is omitted.

11. Trends for the further development of the theory

In the foregoing the fundamental relations of the control system analysis and synthesis exposed to the effect of stochastic signals were summarized. We restricted ourselves to single-pole linear systems of continuous operation, with stationary signals, the base of design was the minimization of the mean-square error. The theory introduced here will be extended to sampled data systems and (or) to nonlinear and (or) to multipole systems, moreover, to take non-stationary signals into consideration, as well as to adopt other optimization criteria, and so on.

Summary

This paper of reviewing character summarizes the main concepts of the stochastic signal theory, characterizes the correlation functions and the power-density spectra, and in it is briefly shown, how to apply these functions to the analysis and synthesis of single-pole control systems. Finally some possibilities of further development are given too.

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