SIMPLIFIED DERIVATION OF OPTIMUM TRANSFER FUNCTIONS IN THE WIENER–NEWTON SENSE*

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(Received July 25, 1962)

In continuous linear communication and control systems, with stochastic input and output signals, an important problem is the optimal filtering or prediction. In the most simple cases, the signals are assumed to be stationary and the ergodic hypothesis is adopted. Further, as a basis of optimalization the least mean square error criterion is taken.

In the time domain determination of the optimum weighting function is performed with the aid of convolution integrals and variational calculus. This method leads to the well-known WIENER—HOPF integral equation. NEWTON extended this method to cases of semi-free configuration and also to those of semi-free configuration with constrains. Unfortunately, the integral equations of the WIENER—HOPF type cannot be easily solved in the time domain. Therefore in most of the cases, when the correlation functions are Fourier transformable, it is necessary to transform the integral equation into the frequency domain and to use WIENER's spectrum factorization procedure. Thus, explicit formulae of the physically realizable transfer function can be obtained.

The following question arises: if, in the most practical cases it is necessary to use the frequency domain, why should this not be done at the beginning? Some years ago this question was positively answered by BODE and SHANNON. In their original paper, they derived formulas only for the simple case of completely free configuration with uncorrelated signal and noise components.

In this paper a similar but much simpler method will be given for somewhat more complicated configurations with correlated noise and signal. The proposed method seems to be more suitable than that of the time domain, because by using the frequency domain technique, convolution integrals and variational calculus can be avoided.

* This lecture was delivered at the Third Prague Conference of Information Theory, Statistical Decision Functions and Random Processes, organized 1962 on the occasion of the tenth anniversary of the foundation of the Czechoslovak Academy of Sciences.

1. Completely free configuration

The problem will be demonstrated by Fig. 1. The reference signal $r(t)$ assumed as being a stationary stochastic process, contains a signal component $s(t)$ and a noise component $n(t)$. Further $c(t)$ is the actual output, otherwise named the controlled signal, while $i(t)$ is the idealized or desired output.

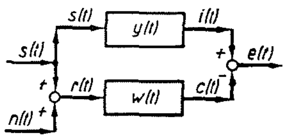

Fig. 1

The question is: which is the weighting function $w(t)$ minimizing the mean square value

$$\overline{e^2(t)} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^2(t)\, dt \tag{1}$$

of the error

$$e(t) = i(t) - c(t). \tag{2}$$

The mean square error can be expressed with the aid of the auto-correlation function or the power-density spectrum, as follows:

$$\overline{e^2(t)} = \varphi_{ee}(0) = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \Phi_{ee}(s)\, ds \tag{3}$$

where

$$\varphi_{ee}(\tau) = \overline{e(t)\, e(t+\tau)} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e(t)\, e(t+\tau)\, dt \tag{4}$$

is the auto-correlation function of the error and $\Phi_{ee}(s)$ is its Fourier transform, that is, the corresponding power-density spectrum. (Here and in the following the notation $s = j\,\omega$ is used.) According to the WIENER—KHINTCHIN relations:

$$\Phi_{ee}(s) = \int_{-j\infty}^{j\infty} \varphi_{ee}(\tau)\, e^{-s\tau}\, d\tau = \mathcal{F}\left[\varphi_{ee}(\tau)\right] \tag{5}$$

$$\varphi_{ee}(\tau) = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \Phi_{ee}(s)\, e^{\tau s}\, ds = \mathcal{F}^{-1}\left[\Phi_{ee}(s)\right]. \tag{6}$$

Taking into consideration Fig. 1 or Eq. (2) and Eq. (4) the auto-correlation function of the error can be expressed as follows:

$$\varphi_{ee}(\tau) = \overline{[i(t) - c(t)][i(t+\tau) - c(t+\tau)]} \tag{7}$$

or in an expanded form:

$$\varphi_{ee}(\tau) = \varphi_{ii}(\tau) - \varphi_{ic}(\tau) - \varphi_{ci}(\tau) + \varphi_{cc}(\tau). \tag{8}$$

Thus, the power-density spectrum of the error is:

$$\Phi_{ee}(s) = \Phi_{ii}(s) - \Phi_{ic}(s) - \Phi_{ci}(s) + \Phi_{cc}(s) \tag{9}$$

which, with the aid of the well-known index-changing rule, can also be written as

$$\Phi_{ee}(s) = \Phi_{ii}(s) - W(s)\,\Phi_{ir}(s) - W(-s)\,\Phi_{ri}(s) + W(-s)\,W(s)\,\Phi_{rr}(s). \tag{10}$$

Here $W(s) = \mathcal{F}[w(\tau)]$. Let us define the following auxiliary transfer function:

$$G(s) = \frac{\Phi_{ri}(s)}{\Phi_{rr}(s)}. \tag{11}$$

As the power density spectra are known functions, $G(s)$ is also known. With the aid of function $G(s)$ defined above, the power-density spectrum of the error $\Phi_{ee}(s)$ can be written in the following form:

$$\begin{aligned}\Phi_{ee}(s) = \Phi_{ii}(s) - G(-s)\,G(s)\,\Phi_{rr}(s) &+ \\ [G(-s) - W(-s)][G(s) - W(s)]\,\Phi_{rr}(s).&\end{aligned} \tag{12}$$

It is worthwhile to mention that only the last term of Eq. (12) contains the minimizing transfer function $W(s)$. Evidently the mean square error will be minimum, if and only if the last term is zero*. In this case the optimum transfer function must be

$$W_o(s) = G(s) \tag{13}$$

or according to Eq. (11):

$$W_o(s) = \frac{\Phi_{ri}(s)}{\Phi_{rr}(s)}. \tag{14}$$

It must be emphasized that $W_0(s)$ is, in general, not physically realizable. The physically realizable optimum transfer function $W_m(s)$ minimizing the

* Naturally, the difference of the first and second term on the right side of Eq. (12) must be a non-negative function of ω^2 ($s = j\,\omega$).

mean square error, can be obtained by the well-known spectrum factorization procedure (see: Appendix) giving

$$W_m(s) = \frac{\left[\dfrac{\Phi_{ri}(s)}{\Phi_{rr}^-(s)}\right]_+}{\Phi_{rr}^+(s)} \tag{15}$$

where

$$\Phi_{rr}(s) = \Phi_{rr}^-(s)\,\Phi_{rr}^+(s) \tag{16}$$

and the factor $\Phi_{rr}^+(s)$ contains all the left-half-plane poles and zeros of $\Phi_{rr}(s)$ accordingly $\Phi_{rr}^-(s)$ contains all the right-half-plane poles and zeros of $\Phi_{rr}(s)$; while

$$\frac{\Phi_{ri}(s)}{\Phi_{rr}(s)} = \left[\frac{\Phi_{ri}(s)}{\Phi_{rr}^-(s)}\right]_- + \left[\frac{\Phi_{ri}(s)}{\Phi_{rr}^-(s)}\right]_+ \tag{17}$$

The second component can be obtained as follows:

$$\left[\frac{\Phi_{ri}(s)}{\Phi_{rr}(s)}\right]_+ = \mathscr{L}\left[\mathscr{F}^{-1}\left[\frac{\Phi_{ri}(s)}{\Phi_{rr}^-(s)}\right]\right] \tag{18}$$

where $\mathscr{L}$ denotes the Laplace transform and $\mathscr{F}^{-1}$ the inverse Fourier transform. Particularly if the power-density spectra figuring here are rational fractional functions of s, instead of the twofold transformation the partial fraction expansion gives the result desired.

2. Semi-free configuration

Let us study the problem of the semi-free configuration. This case is illustrated in Fig. 2. Everything is the same as in Fig. 1, only the link of $w(t)$ is replaced by the cascade connection of the fixed elements with weighting function $w_f(t)$ and the compensating elements with weighting function $w_c(t)$. (See Fig. 2a.) Theoretically in the link in question the sequence of elements may be changed (see Fig. 2b), thus, following relations hold:

$$W_f(s)\,\Phi_{ir}(s) = \Phi_{if}(s) \tag{19}$$

$$W_f(-s)\,W_f(s)\,\Phi_{rr}(s) = \Phi_{ff}(s) \tag{20}$$

and in view of Fig. 2 the power-density spectrum of the error can be expressed as:

$$\Phi_{ee}(s) = \Phi_{ii}(s) - W_c(s)\,\Phi_{if}(s) - W_c(-s)\,\Phi_{fi}(s) + W_c(-s)\,W_c(s)\,\Phi_{ff}(s). \tag{21}$$

It is worthwhile mentioning that latter expression has the same form as in the previous case Eq. (10), but other functions figure here. By the same method as the previous one or by the analogy of the corresponding expressions the

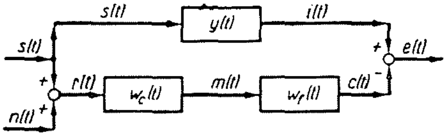

Fig. 2a

Fig. 2b

optimum transfer function of the compensating elements can immediately be obtained

$$W_{co} = \frac{\Phi_{fi}(s)}{\Phi_{ff}(s)} \tag{22}$$

or

$$W_{co}(s) = \frac{W_j(-s)\,\Phi_{ri}(s)}{W_j(-s)\,W_j(s)\,\Phi_{rr}(s)}\,. \tag{23}$$

Finally, the physically realizable optimum transfer function of the compensating elements is:

$$W_{cm}(s) = \frac{\left[\dfrac{\Phi_{fi}(s)}{\Phi_{ff}^{-}(s)}\right]_{-}}{\Phi_{ff}^{+}(s)} \tag{24}$$

or

$$W_{cm}(s) = \frac{\left[\dfrac{W_j(-s)\,\Phi_{ri}(s)}{[W_j(-s)\,W_j(s)]^{-}\,\Phi_{rr}^{-}(s)}\right]_{+}}{[W_j(-s)\,W_j(s)]^{+}\,\Phi_{rr}^{+}(s)}\,. \tag{25}$$

3. Semi-free configuration with constraints

In practical control systems some signals are limited or saturated, consequently constraints arise in the optimization procedure. For the sake of simplicity, only one constraint is assumed as immediately succeeding the compensating element. The problem is illustrated in Fig. 3.

Let us assume the constraint as being expressed in the following form:

$$\overline{l^2(t)} = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \Phi_{ll}(s)\,ds \leq \sigma_m^2\,. \tag{26}$$

This condition will be called the unequality of constraint.

The problem in question can be solved by the Lagrangean conditional extremum technique. The function to be minimized is now:

$$\overline{x^2(t)} = \overline{e^2(t)} + \lambda \overline{l^2(t)} \tag{27}$$

or

$$\overline{x^2(t)} = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} [\Phi_{ee}(s) + \lambda \Phi_{ll}(s)] \, ds . \tag{28}$$

The power-density spectra figuring here, can be expressed as follows:

$$\Phi_{ee}(s) = \Phi_{ii}(s) - W_c(-s)\, \Phi_{fi}(s) - W_c(s)\, \Phi_{if}(s) + W_c(-s)\, W_c(s)\, \Phi_{ff}(s) \tag{29}$$

where

$$\Phi_{ff}(s) = W_f(-s)\, W_f(s)\, \Phi_{rr}(s) \tag{30}$$

$$\Phi_{fi}(s) = W_f(-s)\, \Phi_{ri}(s) \tag{31}$$

and

$$\Phi_{ll}(s) = W_c(-s)\, W_k(-s)\, W_c(s)\, W_k(s)\, \Phi_{rr}(s) \tag{32}$$

where

$$W_k(\;) = \mathscr{F}\left[w_k(\tau)\right].$$

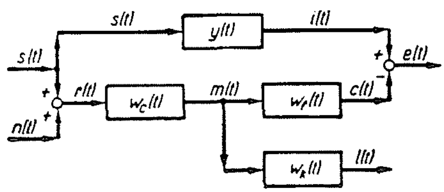

Fig. 3

Let us define the following auxiliary power-density spectrum:

$$\Phi_{aa}(s) = [W_f(-s)\, W_f(s) + \lambda\, W_k(-s)\, W_k(s)]\, \Phi_{rr}(s) . \tag{33}$$

In the present case the auxiliary transfer function needed is

$$G_a(s) = \frac{\Phi_{fi}(s)}{\Phi_{aa}(s)} \tag{34}$$

Thus, with the above notations and notions:

$$\begin{aligned} \Phi_{ee}(s) + \lambda\, \Phi_{ll}(s) = \Phi_{ii}(s) - G_a(-s)\, G_a(s)\, \Phi_{aa}(s) + \\ + [G_a(-s) - W_c(-s)][G_a(s) - W_c(s)]\, \Phi_{aa}(s) . \end{aligned} \tag{35}$$

The necessary and sufficient condition for $\overline{x^2(t)}$ = Min, is the following:

$$W_{co}(s) = G_a(s) \tag{36}$$

that is

$$W_{co}(s) = \frac{\Phi_{fi}(s)}{\Phi_{aa}(s)} \tag{37}$$

or in an expanded form:

$$W_{co}(s) = \frac{W_j(-s)\,\Phi_{ri}(s)}{[W_j(-s)\,W_j(s) + \lambda W_k(-s)\,W_k(s)]\,\Phi_{rr}(s)}\,. \tag{38}$$

Thus, by the spectrum factorization procedure, the physically realizable optimum transfer function of the compensating element is

$$W_{cm}(s) = \frac{\left[\dfrac{\Phi_{fi}(s)}{\Phi_{aa}^{-}(s)}\right]_{+}}{\Phi_{aa}^{+}(s)} \tag{39}$$

or in an expanded form:

$$W_{cm}(s) = \frac{\left[\dfrac{W_f(-s)\,\Phi_{ri}(s)}{[W_f(-s)\,W_f(s) + \lambda W_k(-s)\,W_k(s)]^{-}\,\Phi_{rr}^{-}(s)}\right]_{+}}{[W_f(-s)\,W_f(s) + \lambda W_k(-s)\,W_k(s)]^{+}\,\Phi_{rr}^{+}(s)}\,. \tag{40}$$

It must be emphasized that in this case the undetermined Lagrangean multiplier λ figures in the expressions of the physically realizable optimum transfer function. It can be eliminated with the aid of the unequality of the constraint: Eq. (26). If in the expression of $\Phi_{ll}(s)$ the transfer function $W_c(s)$ is replaced by the physically realizable optimum transfer function $W_{cm}(s)$ and the mean square value $\overline{l^2(t)}$ of the output signal of constraint $l(t)$ is determined by CAUCHY residue theorem, then the multiplier λ can be so adjusted that the equation of constraint should be fulfilled. After this procedure, we have the desired physically realizable optimum transfer function.

Appendix

The derivation of Eq. (15) can be effected in the following way. According to Eq (14):

$$\Phi_{rr}(s)\ W_o(s) - \Phi_{ri}(s) = 0$$

If $W_0(s)$ is substituted by the physically realizable transfer function $W_m(s)$ then the above relation assume the following form:

$$\Phi_{rr}(s)\ W_m(s) - \Phi_r^{i}(s) = F_-(s)$$

where $F_-(s)$ is some function not known at present and having no poles and zeros in the left-half-plane. Taking into consideration Eq. (16) and Eq. (17) the latter equation can be written as

$$\Phi_{rr}^{+}(s)\ W_m(s) = \left[\frac{\Phi_{ri}(s)}{\Phi_{rr}^{-}(s)}\right]_{+} + \left[\frac{\Phi_{ri}(s)}{\Phi_{rr}^{-}(s)}\right]_{-} + \frac{F_-(s)}{\Phi_{rr}^{-}(s)} .$$

The left side of this equation contains only left-half-plane poles and zeros, consequently the right-half-plane poles and zeros must be cancelled on the right side of the above mentioned equation, that is

$$F_-(s) = -\Phi_{rr}^{-}(s)\left[\frac{\Phi_{ri}(s)}{\Phi_{rr}^{-}(s)}\right]_{-} .$$

Thus, the expression of the physically realizable optimum transfer function figuring in Eq. (15) becomes evident:

$$W_m(s) = \frac{1}{\Phi_{rr}^{+}(s)}\left[\frac{\Phi_{ri}(s)}{\Phi_{rr}^{-}(s)}\right]_{+} .$$

Example

For the sake of illustration, only a very simple example is given. Let it be

$$\Phi_{ss}(s) = \frac{3}{1-s^2}\ ;\ \Phi_{nn}(s) = 1\ ;\ \Phi_{ns} = \Phi_{sn}(s) = 0$$

that is

$$\Phi_{rr}(s) = \frac{4-s^2}{1-s^2} .$$

If $Y_i(s) = 1$, then

$$\Phi_{ri}(s) = Y_i(s)\ \Phi_{rs}(s) = \Phi_{rs}(s) = \Phi_{ss}(s) + \Phi_{ns}(s) = \Phi_{ss} = \frac{3}{1-s^2}$$

and

$$\Phi_{ii}(s) = Y_i(-s)\ Y_i(s)\ \Phi_{ss}(s) = \frac{3}{1-s^2} .$$

According to Eq. (16):

$$\Phi_{rr}^{+}(s) = \frac{2+s}{1+s}\ ;\ \ \Phi_{rr}^{-}(s) = \frac{2-s}{1-s} .$$

Thus,

$$\frac{\Phi_{ri}(s)}{\Phi_{rr}^{-}(s)} = \frac{3}{(1+s)(2-s)} = \frac{1}{1+s} + \frac{1}{2-s}$$

and

$$\left[\frac{\Phi_{ri}(s)}{\Phi_{rr}^{-}(s)}\right]_{+} = \frac{1}{1+s} .$$

Taking into consideration Eq. (15) the physically realizable optimum transfer function is:

$$W_m(s) = \frac{1}{2+s}$$

while according to Eq. (14) the optimum transfer function without physical realizability is:

$$W_0(s) = \frac{3}{4 - s^2} = G(s).$$

By the way

$$F_-(s) = -\frac{1}{1 - s}.$$

From Eq. (12) with $W(s) = W_0(s)$

$$\Phi_{ee}(s) = \frac{3}{4 - s^2}$$

and

$$\operatorname*{Min}_{W = W_0} \overline{e^2(t)} = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \frac{3}{4 - s^2}\, ds = \frac{3}{4}$$

This would be the minimum mean square error, if $W_0(s)$ had been realizable. In fact, only $W(s) = W_m(s)$ may be assumed, thus from Eq. (12):

$$\Phi_{ee}(s) = \frac{3}{4 - s^2} + \frac{1}{4 - s^2} = \frac{4}{4 - s^2}$$

and

$$\operatorname*{Min}_{W = W_m} \overline{e^2(t)} = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \frac{4}{4 - s^2}\, ds = 1 > \frac{3}{4}.$$

This latter is the physically realizable minimum mean square error.

Summary

In the foregoing treatment a very simple method was presented for the determination of optimum transfer functions for cases leading in the time domain to integral equations of the WIENER—HOPF type. Using, from the beginning, the frequency domain technique the explicit solution formulae can be obtained in a relatively simple way. The proposed method can be generalized also to other cases of continuous linear systems, for example, to multipole systems with several stochastic input and output signals, and also to digital or sampled data control systems i. e. to discret systems.

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